Test of Discrete Event Systems - 12.11.2020

Exercise 1

A minibus has a maximum capacity of 9 passengers. A bus stop has a waiting room with a maximum capacity of 4 seats. Passengers arriving at the waiting room and finding it full, are routed to another line. When the minibus arrives at the bus stop, the number of passengers in the minibus is a random variable following a discrete uniform distribution over $\{0, 1, \ldots, 9\}$. None of the passengers gets off the minibus at the bus stop. Waiting passengers are admitted on the minibus so as to not exceed its maximum capacity. Non-admitted passengers keep waiting for the next minibus. Assume that the duration of a stop is negligible.

1. Model the system above through a state automaton $(\mathcal{E}, \mathcal{X}, \Gamma, p, x_0)$, assuming that the waiting room is initially empty.

Assume that the minibus arrives at the bus stop every $T = 15$ min, and arrivals of passengers at the waiting room are generated by a Poisson process with average interarrival time of 5 min.

2. Compute the average number of passengers arriving at the waiting room between a transit of the minibus and the next.
3. Compute the probability that the waiting room is empty after the first transit of the minibus.
4. Compute the probability that, the first time the minibus arrives, exactly one waiting passenger gets on the minibus.

Then, assume that arrivals of the minibus at the bus stop are now generated by a Poisson process with rate 4 arrivals/hour.

5. Assuming that two passengers are sitting in the waiting room, compute the probability that the waiting room is empty when the next passenger arrives.
6. Compute the probability that there are at least three arrivals of any type (both passengers and minibus) in one hour.

1. events $\mathcal{E} = \{a, d\}$

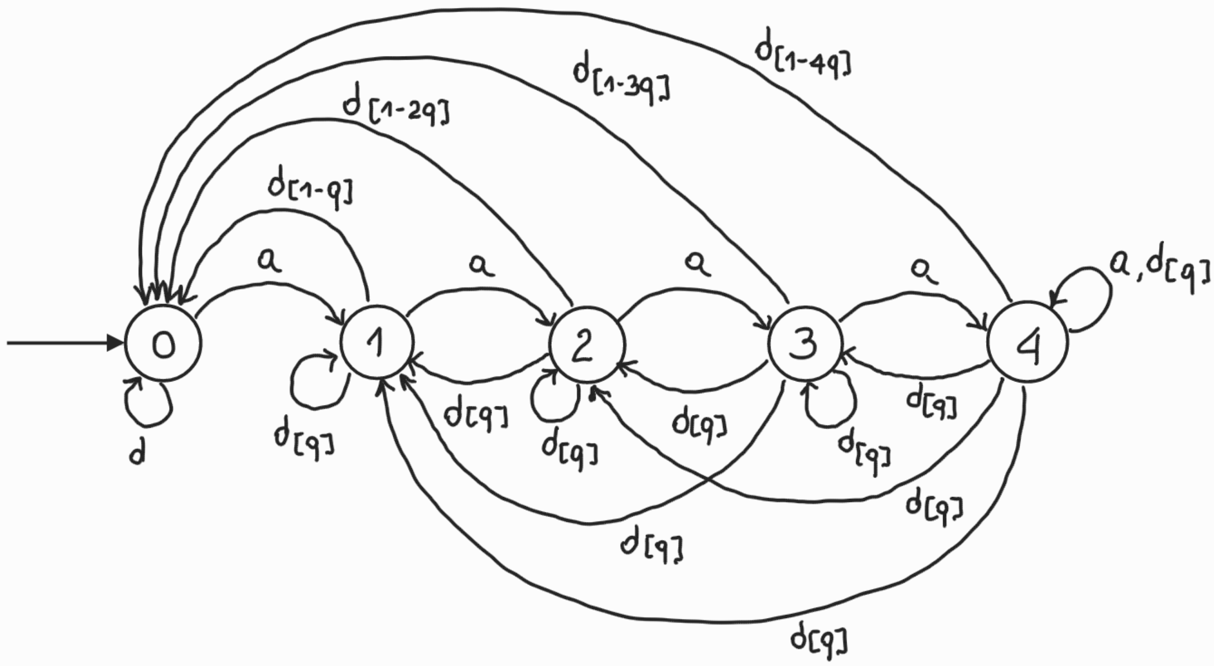
a: arrival of a passenger at the waiting room

d: transit of the minibus

state x = # waiting passengers $\in \{0,1,2,3,4\}$

Let M be the number of passengers <u>in the minibus</u> when the minibus arrives at the bus stop.

$\Rightarrow P(M=n) \triangleq q = \frac{1}{10}$, $n = 0,1,2,\dots,9$ (uniform distribution)

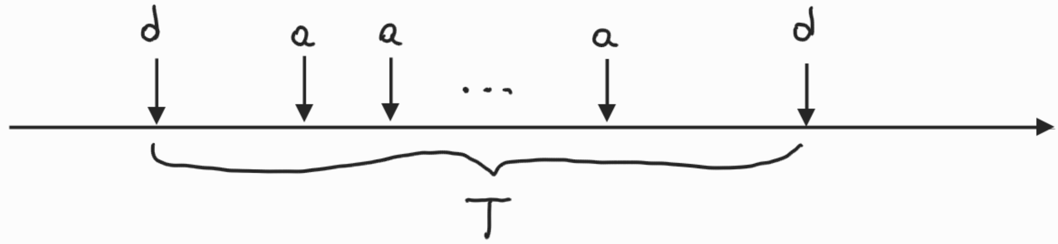


For the questions 2, 3 and 4, the stochastic clock structure is as follows:

$V_a \sim Exp\left(\frac{1}{\lambda}\right)$ where $\frac{1}{\lambda} = 5\,min \Rightarrow \lambda = \frac{1}{5}$ arrivals/min

$V_d = T = 15\,min$

2.

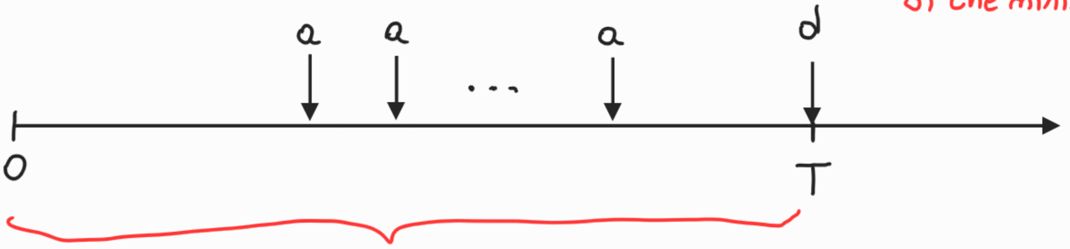


Event a is generated by a Poisson process:

$$\Rightarrow E[N_a(T)] = \lambda T = 3 \text{ passengers}$$

3.

First transit of the minibus



Over this interval there may occur $N_a(T) = 0, 1, 2, \ldots$ arrivals of passengers at the waiting room.

$$\Rightarrow P(\text{waiting room empty after the first transit of the minibus} \mid X_0 = 0)$$

$$= P(N_a(T)=0)$$

$$+ P(N_a(T)=1) \cdot P(M \leq 8)$$ ← 1 waiting customer, at least 1 place in the minibus

$$+ P(N_a(T)=2) \cdot P(M \leq 7)$$ ← 2 waiting customers, at least 2 places in the minibus

$$+ P(N_a(T)=3) \cdot P(M \leq 6)$$ ← 3 waiting customers, at least 3 places in the minibus

$$+ P(N_a(T)\geq 4) \cdot P(M \leq 5)$$ ← 4 waiting customers (waiting room full), at least 4 places in the minibus

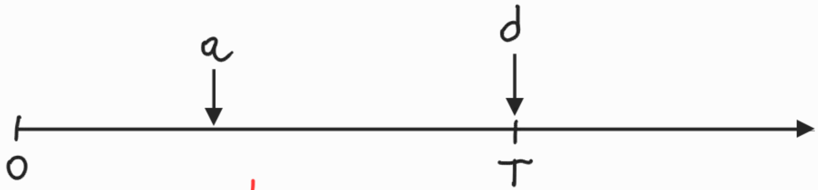
$$= e^{-\lambda T} + (\lambda T) e^{-\lambda T} \cdot (1-q) + \frac{(\lambda T)^2}{2} e^{-\lambda T} (1-2q) + \frac{(\lambda T)^3}{6} e^{-\lambda T} (1-3q)$$

$$+ \left[1 - e^{-\lambda T} - (\lambda T) e^{-\lambda T} - \frac{(\lambda T)^2}{2} e^{-\lambda T} - \frac{(\lambda T)^3}{6} e^{-\lambda T} \right] (1-4q)$$

$$\simeq 0.7319$$

4. There are two cases:

$\boxed{1}$



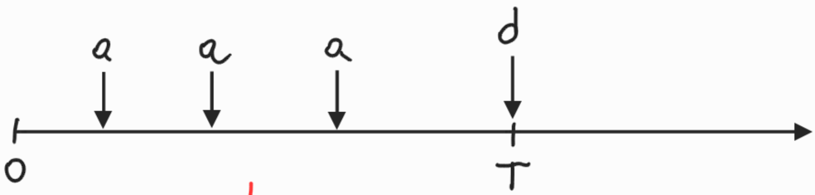
$N_a(T) = 1$

there is at least one place available in the minibus

$$\Rightarrow P(\boxed{1}) = P(N_a(T)=1) \cdot P(M \leq 8)$$

$$= (\lambda T) e^{-\lambda T} \cdot (1-q)$$

$\boxed{2}$



$N_a(T) > 1$

there is only one place available in the minibus

$$\Rightarrow P(\boxed{2}) = P(N_a(T) > 1) \cdot P(M=8)$$

$$= \left[1 - e^{-\lambda T} - (\lambda T) e^{-\lambda T} \right] \cdot q$$

$$\Rightarrow P(\ldots) = P(\boxed{1}) + P(\boxed{2}) \simeq 0.2145$$

For the questions 5 and 6, the stochastic clock structure is as follows:

$V_a \sim Exp\left(\frac{1}{\lambda}\right)$ where $\lambda = \frac{1}{5}$ arrivals/min

$V_d \sim Exp\left(\frac{1}{\mu}\right)$ where $\mu = 4$ arrivals/hour $= \frac{1}{15}$ arrivals/min

$\Downarrow$

Poisson clock structure

5. The current state is $X_k = 2$.

There are two cases:

$\boxed{1}$

$$2 \xrightarrow[\substack{\text{(ignore } d_{[q]} \\ \text{on the loop)}}]{d_{[q]}} 1 \xrightarrow[\substack{\text{(ignore } d_{[q]} \\ \text{on the loop)}}]{d_{[1-q]}} 0$$

$$\Rightarrow P(\boxed{1}) = \frac{\mu q}{\lambda + \mu(1-q)} \cdot \frac{\mu(1-q)}{\lambda + \mu(1-q)}$$

$\boxed{2}$

$$2 \xrightarrow[\substack{\text{(ignore } d_{[q]} \\ \text{on the loop)}}]{d_{[1-2q]}} 0$$

$$\Rightarrow P(\boxed{2}) = \frac{\mu(1-2q)}{\lambda + \mu(1-q)}$$

$$\Rightarrow P(\ldots) = P(\boxed{1}) + P(\boxed{2}) \simeq 0.2110$$

6. We define a new event, say r, which is the superposition of events a and d, and represents a generic arrival.
It follows that r is generated by a Poisson process with rate $\rho = \lambda + \mu$.

$$\Rightarrow P(\underset{\substack{\uparrow \\ t = 1\,\text{hour} \\ = 60\,\text{min}}}{N_r(t)} \geq 3) = 1 - P(N_r(t)=0) - P(N_r(t)=1) - P(N_r(t)=2)$$

$$= 1 - e^{-\rho t} - (\rho t)e^{-\rho t} - \frac{(\rho t)^2}{2}e^{-\rho t}$$

$$= 1 - \underbrace{e^{-\rho t}\left[1 + \rho t + \frac{(\rho t)^2}{2}\right]}_{\simeq 1.6318 \cdot 10^{-5}} \simeq 1$$