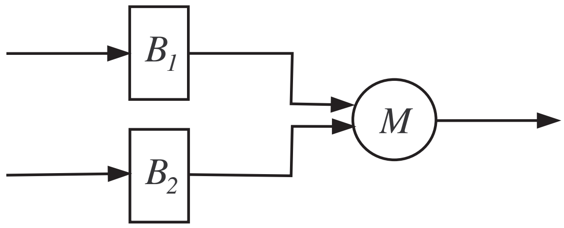
Test of Discrete Event Systems - 09.11.2020

Exercise 1

A manufacturing cell is composed of two one-place buffers B_1 and B_2 and one assembling machine M, as shown in the figure.

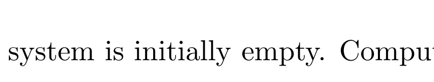


Arrivals of raw parts are generated by a Poisson process with rate 10 arrivals/hour. Arriving parts are of type 1 with probability $p = 1/2$ and of type 2 otherwise. Type 1 parts are stored in buffer B_1, whereas type 2 parts are stored in buffer B_2. An arriving part is rejected if the corresponding buffer is full. Machine M assembles one type 1 part and one type 2 part to make a finished product. Assembling starts instantaneously as soon as parts of both types are available in the buffers and M is ready. Assembling times have an exponential distribution with expected value 5 minutes. The manufacturing cell is initially empty.

1. Model the manufacturing cell through a stochastic timed automaton $(\mathcal{E}, \mathcal{X}, \Gamma, p, x_0, F)$.
2. Assume that M is working and both buffers are full. Compute the probability that the manufacturing cell is empty
   (a) when a new part arrives;
   (b) when a new part is accepted.
3. Compute the average state holding time when B_1 is full, B_2 is empty and M is working.
4. Assume that M is working and both buffers are full. Compute the probability that two products are finished within $T = 10$ minutes, and no arrival of type 1 parts occurs.

Exercise 2

Consider the queueing system in the figure, characterized by deterministic interarrival times equal to $T = 20$ min, and service times having an exponential distribution with expected value 15 min.



1. Assume that the queueing system is initially empty. Compute the probability that the third customer has not to wait for service.
2. Assume that one customer is initially in the queueing system. Compute the cdf of the waiting time of the next customer.
3. Assume that the initial number of customers in the system has a uniform distribution over the set $\{0, 1, 2, 3\}$. Compute the probability that at least two places are available in the system when the next arrival occurs.

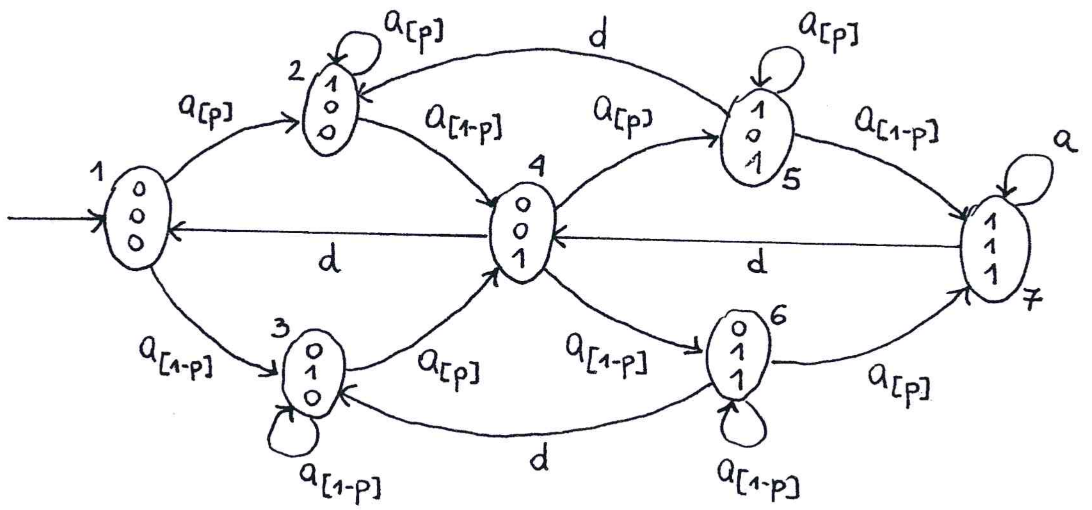
Exercise 1

1. state $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ $\begin{array}{l} \to B_1: \text{ 0 (empty), 1 (full)} \\ \to B_2: \text{ " , "} \\ \to M: \text{ 0 (idle), 1 (working)} \end{array}$

events $\mathcal{E} = \{a, d\}$

a: arrival of a new part

d: termination of assembling



$p = \frac{1}{2}$

$F = \{F_a, F_d\}$

$F_a(t) = 1 - e^{-\lambda t}, \; t \geq 0$ where $\lambda = 10$ arrivals/hour

$F_d(t) = 1 - e^{-\mu t}, \; t \geq 0$ where $\frac{1}{\mu} = 5$ minutes $= \frac{1}{12}$ hours $\Rightarrow \mu = 12$ services/hour

2. $X_k = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

(a) There is only one favorable case:

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \xrightarrow{d} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \xrightarrow{d} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$P(\ldots) = \frac{\mu}{\lambda+\mu} \cdot \frac{\mu}{\lambda+\mu} = \left(\frac{\mu}{\lambda+\mu}\right)^2 \simeq 0.2975$$

(b) There is only one favorable case:

$$\begin{pmatrix}1\\1\\1\end{pmatrix} \xrightarrow{d} \begin{pmatrix}0\\0\\1\end{pmatrix} \xrightarrow{d} \begin{pmatrix}0\\0\\0\end{pmatrix}$$

(ignore a)

because arrivals are not accepted in state $\begin{pmatrix}1\\1\\1\end{pmatrix}$!!!

$$P(\ldots) = 1 \cdot \frac{\mu}{\lambda+\mu} = \frac{\mu}{\lambda+\mu} \simeq 0.5455$$

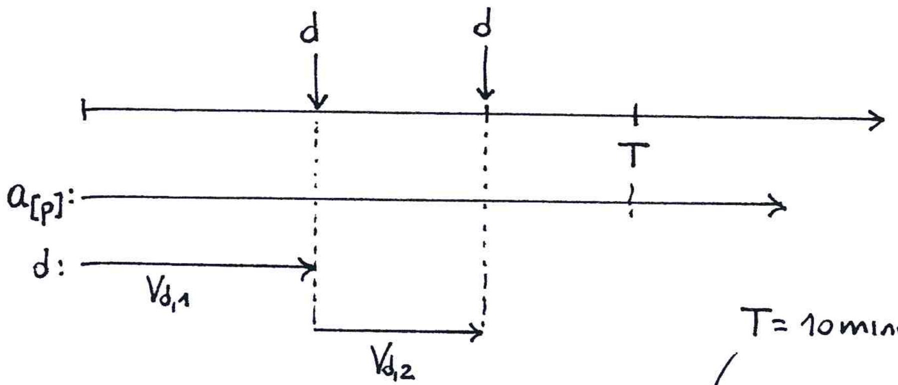
3. $X_k = \begin{pmatrix}1\\0\\1\end{pmatrix}$

The state holding time $V\begin{pmatrix}1\\0\\1\end{pmatrix}$ is exponentially distributed with rate $\lambda(1-p)+\mu$.

$$\Rightarrow E\left[V\begin{pmatrix}1\\0\\1\end{pmatrix}\right] = \frac{1}{\lambda(1-p)+\mu} \simeq 0.0588 \text{ hours} \simeq 3 \text{ min } 32 \text{ sec}$$

4. $X_k = \begin{pmatrix}1\\1\\1\end{pmatrix}$

There is only one favorable case:



$T = 10 \text{ minutes} = \frac{1}{6} \text{ hours}$

$$\Rightarrow P(N_{a_{[p]}}(T)=0)\, P(V_{d,1}+V_{d,2} \leq T) = e^{-\lambda p \cdot T} \cdot \left[1 - e^{-\mu T}(1+\mu T)\right] \simeq 0.2581$$

If 'd' were generated by a Poisson process, this would correspond to $P(N_d(T) \geq 2)$

EXERCISE 2

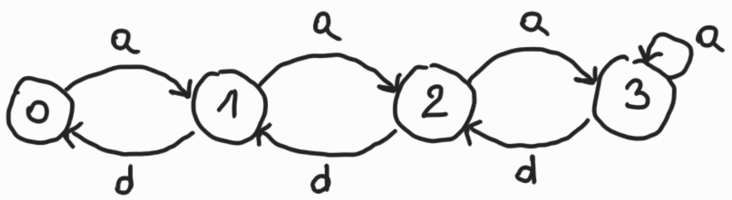
model

events $\mathcal{E} = \{a, d\}$

arrival of a new customer

termination of a service

state x = # customers in the system $\in \{0,1,2,3\}$



stochastic clock structure:

(all the lifetimes are expressed in min)

$V_a = T = 20$

$V_d \sim Exp(15) \quad \Rightarrow \quad \mu = \frac{1}{15}$ services/min

1. We have to compute the probability that, when the third customer arrives, the system is empty, so that the customer has not to wait for service.

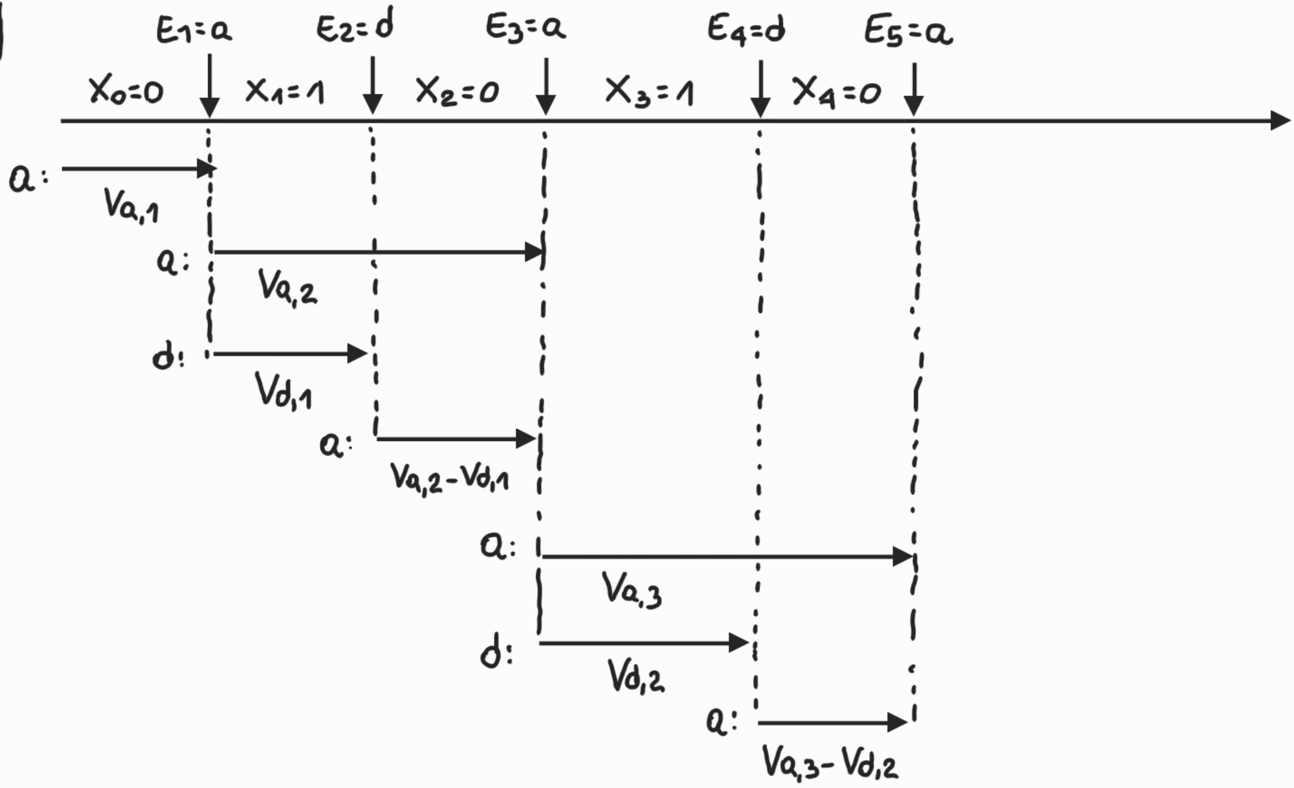
   The possible cases with initial state $X_0=0$ are the Following:

   1) $X_0=0 \xrightarrow{E_1=a} X_1=1 \xrightarrow{E_2=d} X_2=0 \xrightarrow{E_3=a} X_3=1 \xrightarrow{E_4=d} X_4=0 \xrightarrow{E_5=a}$

   2) $X_0=0 \xrightarrow{E_1=a} X_1=1 \xrightarrow{E_2=a} X_2=2 \xrightarrow{E_3=d} X_3=1 \xrightarrow{E_4=d} X_4=0 \xrightarrow{E_5=a}$

We have to compute the probability of each sample path.

$\boxed{1}$

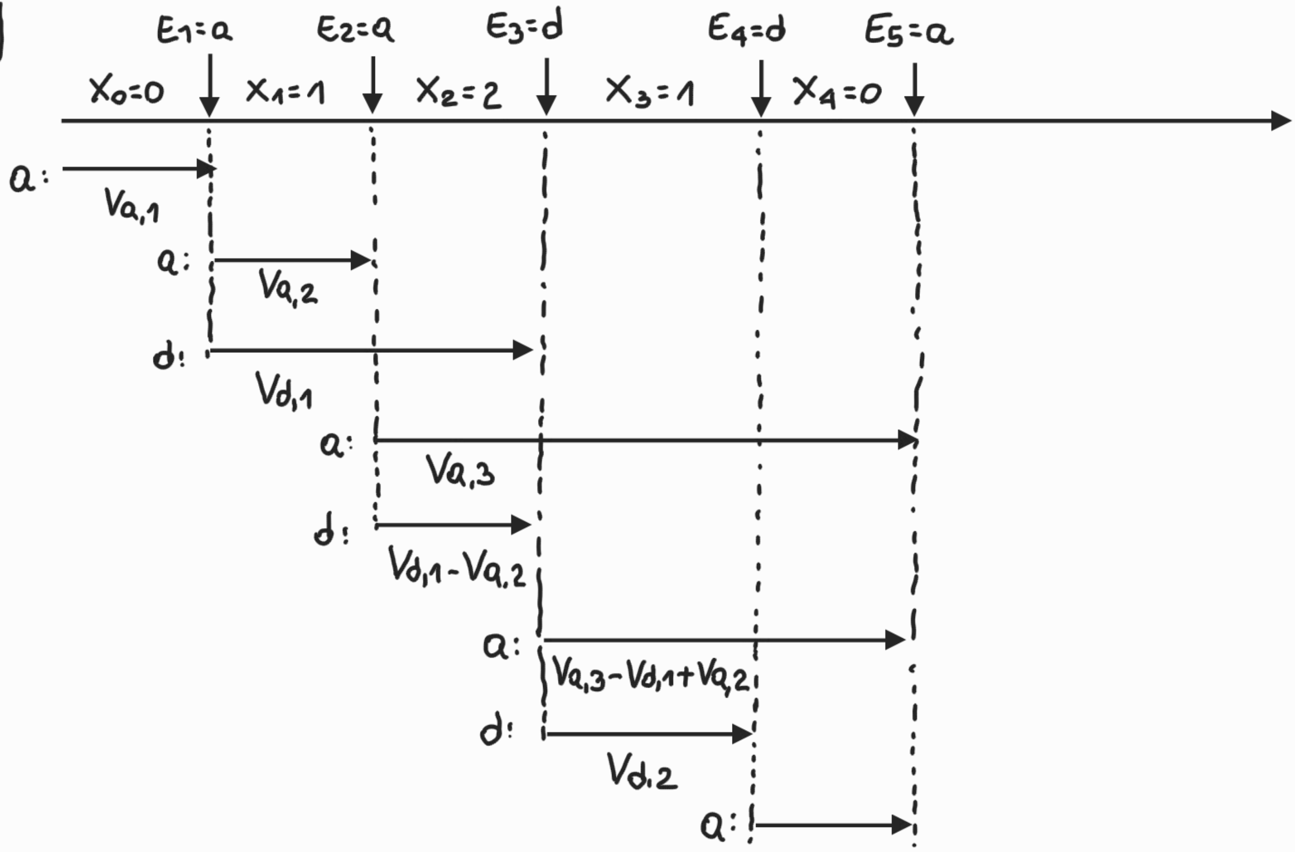


$$\Rightarrow P(\boxed{1}) = P(V_{d,1} < V_{a,2},\ V_{d,2} < V_{a,3})$$

$$= P(V_{d,1}<T,\ V_{d,2}<T) = P(V_{d,1}<T)\,P(V_{d,2}<T) = \left(1-e^{-\mu T}\right)^2$$

independent

$\boxed{2}$



$$\Rightarrow P(\boxed{2}) = P\left(V_{a,2} < V_{d,1},\ V_{d,1} - V_{a,2} < V_{a,3},\ V_{d,2} < V_{a,3} - V_{d,1} + V_{a,2}\right)$$

$$= P\left(V_{a,2} < V_{d,1},\ V_{d,1} + V_{d,2} < V_{a,2} + V_{a,3}\right)$$

we remove redundant conditions

$$= P\left(V_{d,1} > T,\ V_{d,1} + V_{d,2} < 2T\right)$$

$$= e^{-\mu T} \cdot \left[1 - e^{-\mu T}(1 + \mu T)\right]$$

Computed exploiting the fact that, if "d" were generated by a Poisson process, then $P(V_{d,1} > T, V_{d,1} + V_{d,2} < 2T) =$

$= P(N_d(T) = 0)\, P(N_d(T) \geq 2)$

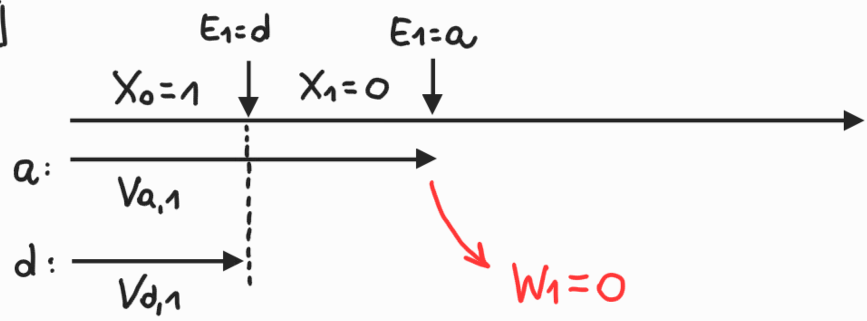
$= e^{-\mu T} \cdot [1 - e^{-\mu T}(1 + \mu T)]$

Finally,

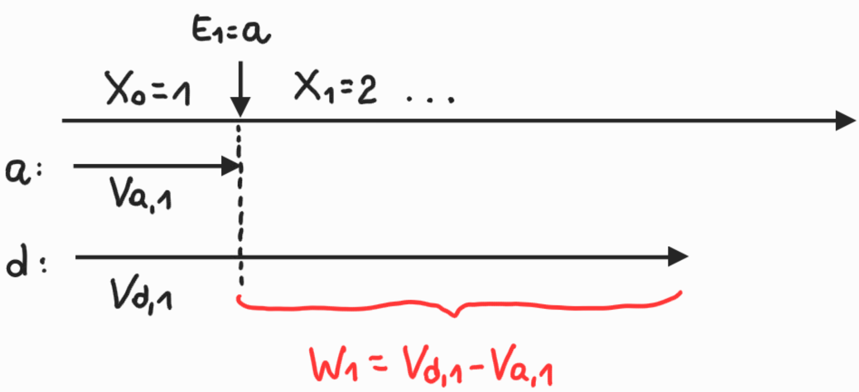
$$P(\ldots) = P(\boxed{1}) + P(\boxed{2}) \simeq 0.6438$$

2. The waiting time is the time that a customer spends in the queue. Let W_1 be the waiting time of the first customer. We have two possible cases with initial state $X_0 = 1$:

$\boxed{1}$



$\boxed{2}$



Hence,

$$W_1 = \begin{cases} 0 & \text{if } V_{d,1} < V_{a,1} \\ V_{d,1} - V_{a,1} & \text{if } V_{d,1} \geq V_{a,1} \end{cases} \Rightarrow W_1 = \max\{0, V_{d,1} - V_{a,1}\}$$

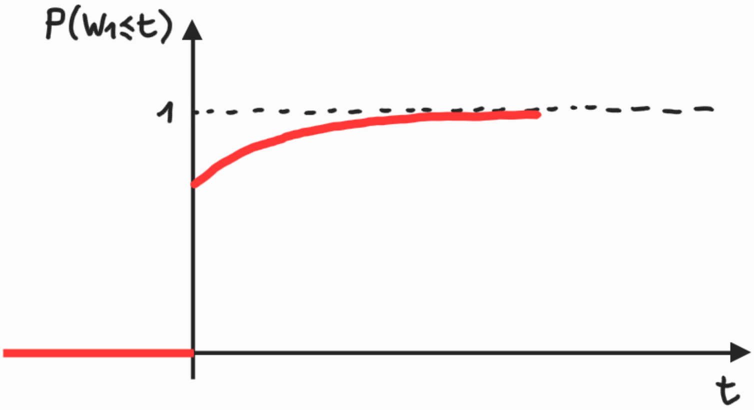
<u>Cdf of W_1</u>

certain event

Let $t \geq 0$.

$$P(W_1 \leq t) = P(\max\{0, V_{d,1} - V_{a,1}\} \leq t) = P(0 \leq t, V_{d,1} - V_{a,1} \leq t)$$

$$= P(V_{d,1} - V_{a,1} \leq t) = P(V_{d,1} \leq T + t) = 1 - e^{-\mu(T+t)}, \quad t \geq 0$$



3. The initial state is uncertain, with pmf:

$$p_{X_0}(0) = p_{X_0}(1) = p_{X_0}(2) = p_{X_0}(3) = 1/4.$$

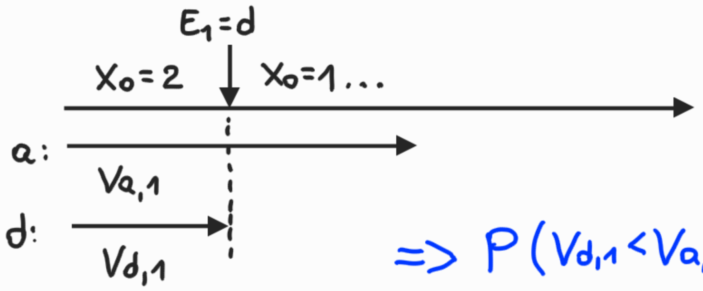
When the first arrival occurs, the state must be either 0 or 1.

There are the following cases:

$\boxed{1}$ If $X_0 = 0$, the state is 0 with probability 1, when the first arrival occurs.

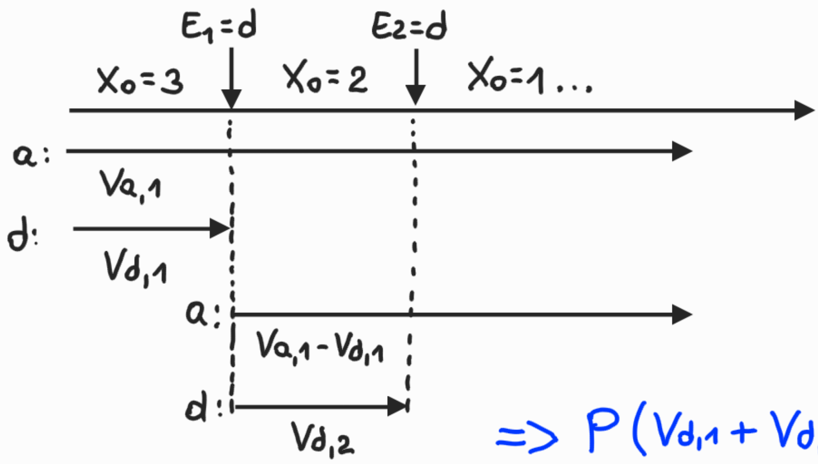
$\boxed{2}$ If $X_0 = 1$, the state is either 0 or 1 with probability 1, when the first arrival occurs.

$\boxed{3}$ If $X_0=2$, the state is either 0 or 1, when the first arrival occurs, if there is at least one termination of a service before the arrival.



$\Rightarrow P(V_{d,1}<V_{a,1})$

$\boxed{4}$ If $X_0=3$, the state is either 0 or 1, when the first arrival occurs, if there are at least two terminations of a service before the arrival.



$\Rightarrow P(V_{d,1}+V_{d,2}<V_{a,1})$

Finally:

$$P(\ldots)=p_{X_0}(0)+p_{X_0}(1)+p_{X_0}(2)\cdot P(V_{d,1}<V_{a,1})$$

$$+\,p_{X_0}(3)\cdot P(V_{d,1}+V_{d,2}<V_{a,1})$$

$$=p_{X_0}(0)+p_{X_0}(1)+p_{X_0}(2)\cdot P(V_{d,1}<T)+p_{X_0}(3)\cdot P(V_{d,1}+V_{d,2}<T)$$

$$=\frac{1}{4}+\frac{1}{4}+\frac{1}{4}\cdot(1-e^{-\mu T})+\frac{1}{4}\cdot\left[1-e^{-\mu T}(1+\mu T)\right]$$

$$\simeq 0.7803$$