

Test of Discrete Event Systems - 05.11.2020

Exercise 1

A small hair salon has two chairs and two hairdressers. Customers arrive according to a Poisson process with rate 3 arrivals/hour. A customer is male with probability $p = 1/3$. The duration of a hair-cut is independent of the hairdresser, but depends on the gender of the customer. It is exponentially distributed with expected value 20 minutes for men, and 45 minutes for women. Since the hair salon does not have a waiting room, customers arriving when both chairs are busy, decide to give up hair cutting. The hair salon is empty at the opening.

1. Model the hair salon through a stochastic timed automaton $(\mathcal{E}, \mathcal{X}, \Gamma, p, x_0, F)$.
2. Assume that one hairdresser is serving a man and the other is serving a woman. Compute the probability that the next event is the arrival of a new customer.
3. Assume that both hairdressers are busy with male customers. Compute the probability that the next event is the termination of a hair cut.
4. Assume that both hairdressers are busy with male customers of different age. Compute the probability that the next event is the termination of the hair cut of the youngest man.
5. Assume that one hairdresser is serving a man and the other is serving a woman. Compute the probability that the hair cut of the man terminates before the hair cut of the woman.
6. Compute the probability that the third customer arriving after the opening has to give up hair cutting.
7. Assume that one hairdresser is serving a man and the other is serving a woman. Compute the probability that both hair cuts terminate:
 - (a) before another customer arrives;
 - (b) before another customer sits for a hair cut.
8. Assume that one hairdresser is serving a man and the other is serving a woman. Compute the probability that, in the next hour, both hair cuts are terminated and no other customer arrives.
9. Compute the probability that at least three customers arrive in the next hour.
10. Compute the average state holding time when:
 - (a) one hairdresser is serving a man and the other is idle;
 - (b) one hairdresser is serving a man and the other is serving a woman.

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Definition of state: $x = \begin{bmatrix} x_M \\ x_F \end{bmatrix} \rightarrow \begin{matrix} \text{number of male customers} \in \{0, 1, 2\} \\ \text{number of female customers} \in \{0, 1, 2\} \end{matrix}$

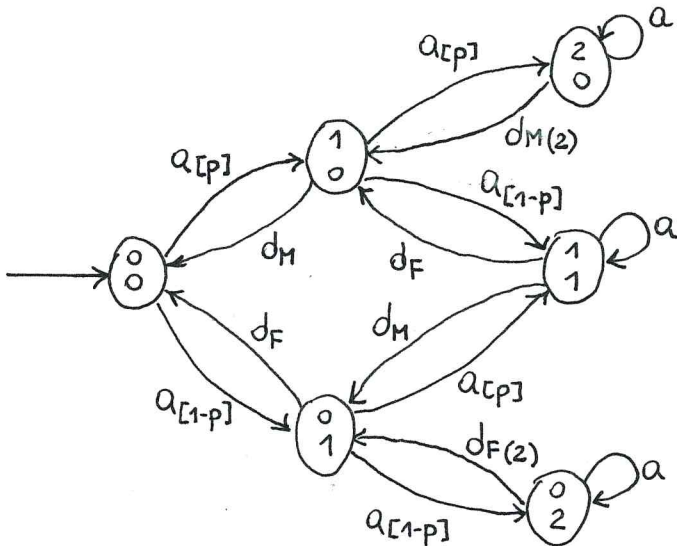
State space: $\mathcal{X} = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \end{bmatrix} \right\} \Rightarrow 6 \text{ states}$

Definition of events: $\mathcal{E} = \{a, d_m, d_f\}$

a : arrival of a customer

d_m : termination of the hair cut of a male customer

d_f : termination of the hair cut of a female customer



$$F_a(t) = 1 - e^{-\lambda t}, t \geq 0$$

where $\lambda = 3$ arrivals/hour

$$F_{dm}(t) = 1 - e^{-\mu_m t}, \quad t \geq 0$$

where $\frac{1}{M_M} = 20 \text{ minutes} = \frac{1}{3} \text{ hours}$

$$\Rightarrow \mu_M = 3 \text{ services/hour}$$

$$F_{DF}(t) = 1 - e^{-M_F t}, \quad t \geq 0$$

where $\frac{1}{\mu_F} = 45 \text{ minutes} = \frac{3}{4} \text{ hours}$

$$\Rightarrow MF = \frac{4}{3} \text{ services/hour}$$

2. The current state is $X_k = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. The probability we are asked to compute is:

$$P(E_{k+1}=a \mid X_k = \begin{pmatrix} 1 \\ 1 \end{pmatrix}) = \frac{\lambda}{\lambda + \mu_M + \mu_F} \approx 0.4091$$

3. The current state is $X_k = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$.

The probability we are asked to compute is:

$$P(E_{k+1} = d_M | X_k = \begin{pmatrix} 2 \\ 0 \end{pmatrix}) = \frac{2\mu_M}{\lambda + 2\mu_M} \approx 0.6667$$

we are not interested in which one of the two events d_M will occur first

two different events d_M are possible

4. The current state is $X_k = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$.

Denote by d_M^Y the termination of the hair cut of the youngest man, and by d_M^O the termination of the hair cut of the oldest man.

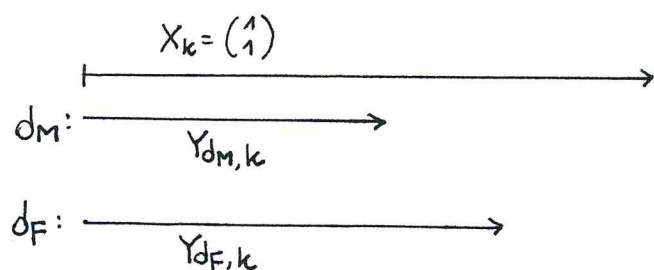
The probability we are asked to compute is:

$$P(E_{k+1} = d_M^Y | X_k = \begin{pmatrix} 2 \\ 0 \end{pmatrix}) = \frac{\mu_M}{\lambda + 2\mu_M} \approx 0.3333$$

events d_M^Y and d_M^O have the same rate μ_M

5. The current state is $X_k = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

The situation can be represented as follows:



\Rightarrow The residual lifetime $Y_{d_M, k}$ of event d_M must be smaller than the lifetime $Y_{d_F, k}$ of event d_F .

$$\Rightarrow P(Y_{d_M, k} \leq Y_{d_F, k} | X_k = \begin{pmatrix} 1 \\ 1 \end{pmatrix}) = \frac{\mu_M}{\mu_M + \mu_F} \approx 0.6923$$

6. When the third customer arrives, the state must be one of $\begin{pmatrix} 3 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$

\Rightarrow Starting from the initial state $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$, the first three events must be arrivals.

We identify four favorable cases, corresponding to the following paths on the state transition diagram:

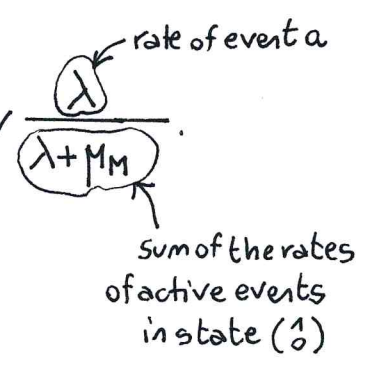
- ① $\begin{pmatrix} 0 \\ 0 \end{pmatrix} \xrightarrow{a} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \xrightarrow{a} \begin{pmatrix} 2 \\ 0 \end{pmatrix} \xrightarrow{a}$
- ② $\begin{pmatrix} 0 \\ 0 \end{pmatrix} \xrightarrow{a} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \xrightarrow{a} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \xrightarrow{a}$
- ③ $\begin{pmatrix} 0 \\ 0 \end{pmatrix} \xrightarrow{a} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \xrightarrow{a} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \xrightarrow{a}$
- ④ $\begin{pmatrix} 0 \\ 0 \end{pmatrix} \xrightarrow{a} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \xrightarrow{a} \begin{pmatrix} 0 \\ 2 \end{pmatrix} \xrightarrow{a}$

The probability we are looking for, is the sum of the probabilities of these four cases.

To compute the probability of case ①, we proceed as follows.

- In state $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$, the only active event is a . Therefore, the probability that the next state is $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, corresponds to the probability that the arrival is a man. This probability is p .

- In state $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, the next event is a with probability $\frac{\lambda}{\lambda + \mu_M}$.
The arrival is a man, and therefore the next state is $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$, with probability p .



- In state $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$, the next event is a with probability $\frac{\lambda}{\lambda + 2\mu_M}$

two male customers are being served, therefore we have scheduled two distinct events μ_M

By multiplying all these probabilities, we obtain the probability of case ①:

$$P(\textcircled{1}) = p \cdot \frac{\lambda}{\lambda + \mu_M} \cdot p \cdot \frac{\lambda}{\lambda + 2\mu_M} = \frac{(\lambda p)^2}{(\lambda + \mu_M)(\lambda + 2\mu_M)}$$

In the same fashion, we can compute the probabilities of all other cases:

$$P(\textcircled{2}) = p \cdot \frac{\lambda}{\lambda + \mu_M} \cdot (1-p) \cdot \frac{\lambda}{\lambda + \mu_M + \mu_F}$$

$$P(\textcircled{3}) = (1-p) \cdot \frac{\lambda}{\lambda + \mu_F} \cdot p \cdot \frac{\lambda}{\lambda + \mu_M + \mu_F}$$

$$P(\textcircled{4}) = (1-p) \cdot \frac{\lambda}{\lambda + \mu_F} \cdot (1-p) \cdot \frac{\lambda}{\lambda + 2\mu_F}$$

Finally:

$$\begin{aligned} P(\dots) &= P(\textcircled{1}) + P(\textcircled{2}) + P(\textcircled{3}) + P(\textcircled{4}) = \\ &= \frac{(\lambda p)^2}{(\lambda + \mu_M)(\lambda + 2\mu_M)} + \frac{\lambda^2 p(1-p)}{(\lambda + \mu_M)(\lambda + \mu_M + \mu_F)} + \\ &\quad + \frac{\lambda^2 p(1-p)}{(\lambda + \mu_F)(\lambda + \mu_M + \mu_F)} + \frac{[\lambda(1-p)]^2}{(\lambda + \mu_F)(\lambda + 2\mu_F)} \simeq 0.2898 \end{aligned}$$

7.a. The current state is $X_k = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Event a must occur after both event d_M and event d_F . We identify two favorable cases, corresponding to the following paths on the state transition diagram:

$$\textcircled{1} \quad \begin{pmatrix} 1 \\ 1 \end{pmatrix} \xrightarrow{d_M} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \xrightarrow{d_F} \rightarrow$$

$$\textcircled{2} \quad \begin{pmatrix} 1 \\ 1 \end{pmatrix} \xrightarrow{d_F} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \xrightarrow{d_M} \rightarrow$$

The probability we are looking for, is the sum of the probabilities of these two cases. We have:

$$P(\textcircled{1}) = \frac{\mu_M}{\lambda + \mu_M + \mu_F} \frac{\mu_F}{\lambda + \mu_F} \quad ; \quad P(\textcircled{2}) = \frac{\mu_F}{\lambda + \mu_M + \mu_F} \frac{\mu_M}{\lambda + \mu_M}$$

(5)

Therefore:

$$P(\dots) = P(①) + P(②) = \frac{\mu_M \mu_F}{(\lambda + \mu_M + \mu_F)(\lambda + \mu_F)} + \frac{\mu_M \mu_F}{(\lambda + \mu_M + \mu_F)(\lambda + \mu_M)} \approx 0.2168$$

7.b. Notice that a customer sits for the hair cut if and only if the hair salon is not full when he/she arrives. Hence, in this question, we ignore arrivals occurring when the system is full, since the arriving customer cannot sit for the hair cut.

The favorable cases are:

$$① \quad \begin{pmatrix} 1 \\ 1 \end{pmatrix} \xrightarrow[\text{(ignore } a)]{d_M} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \xrightarrow{d_F}$$

$$② \quad \begin{pmatrix} 1 \\ 1 \end{pmatrix} \xrightarrow[\text{(ignore } a)]{d_F} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \xrightarrow{d_M}$$

The probabilities of these two cases are:

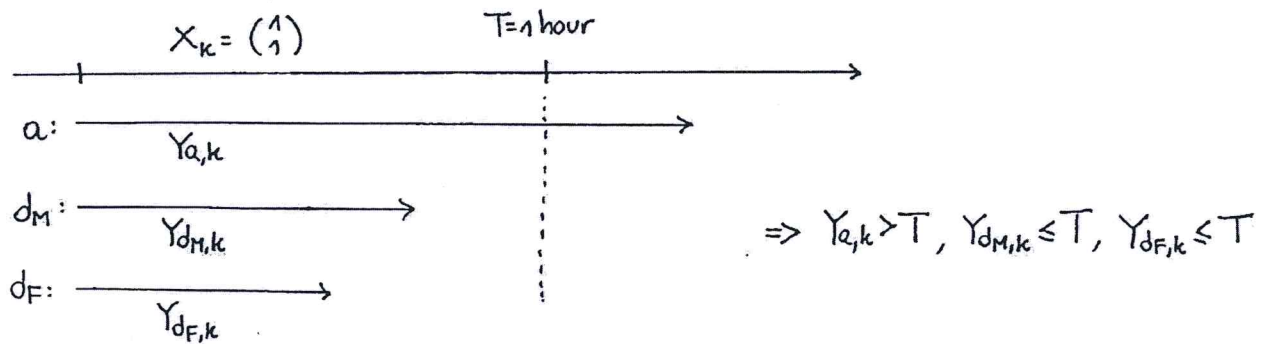
$$P(①) = \frac{\mu_M}{\mu_M + \mu_F} \cdot \frac{\mu_F}{\lambda + \mu_F} \quad ; \quad P(②) = \frac{\mu_F}{\mu_M + \mu_F} \cdot \frac{\mu_M}{\lambda + \mu_M}$$

We do not have λ here,
because we ignore a in state $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Hence,

$$P(\dots) = P(①) + P(②) = \frac{\mu_M \mu_F}{(\mu_M + \mu_F)(\lambda + \mu_F)} + \frac{\mu_M \mu_F}{(\mu_M + \mu_F)(\lambda + \mu_M)} \approx 0.3669$$

8. The current state is $X_k = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. We require that the residual lifetimes of both events d_M and d_F are smaller than one hour, and the residual lifetime of event a is larger than one hour.



$$\Rightarrow P(\dots) = P(Y_{a,k} > T, Y_{d_M,k} \leq T, Y_{d_F,k} \leq T) = P(Y_{a,k} > T) P(Y_{d_M,k} \leq T) P(Y_{d_F,k} \leq T)$$

independent random variables

$$= e^{-\lambda T} \cdot (1 - e^{-\mu_M T}) (1 - e^{-\mu_F T}) \approx 0.0348$$

9. Arrivals are generated by a Poisson process. We apply the Poisson distribution with $T = 1$ hour.

$$P(N_a(T) \geq 3) = 1 - P(N_a(T) = 0) - P(N_a(T) = 1) - P(N_a(T) = 2)$$

$$= 1 - e^{-\lambda T} - (\lambda T) e^{-\lambda T} - \frac{(\lambda T)^2}{2} e^{-\lambda T} = 1 - \left[1 + (\lambda T) + \frac{(\lambda T)^2}{2} \right] e^{-\lambda T} \approx 0.5768$$

- 10.a. The current state is $X_k = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

$$E[V(\begin{pmatrix} 1 \\ 0 \end{pmatrix})] = \frac{1}{\lambda + \mu_M}$$

sum of the rates of the events which take the system away from state $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

- 10.b. The current state is $X_k = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

$$E[V(\begin{pmatrix} 1 \\ 1 \end{pmatrix})] = \frac{1}{\mu_M + \mu_F}$$

The rate λ does not appear because event a leaves state $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ unchanged.