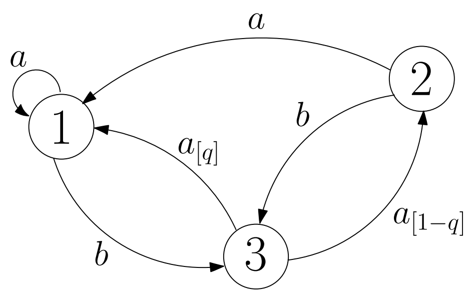
Test of Discrete Event Systems - 29.10.2020

Exercise 1

Consider the stochastic timed automaton in the figure, where $q = 2/5$.



The initial state is uncertain, with pmf $p_{X_0}(1) = \frac{2}{3}$, $p_{X_0}(2) = 0$ and $p_{X_0}(3) = \frac{1}{3}$. Lifetimes of event a have a uniform distribution over the interval $[6, 9]$ min, while lifetimes of event b have an exponential distribution with expected value 5 min.

1. Compute $P(X_2 = 3)$.
2. Compute $P(E_2 = a)$.
3. Compute the probability that event b occurs before $T = 15$ min.
4. Compute the cdf of the state holding time in $x = 2$. And in $x = 1$?

Exercise 1

Stochastic clock structure:

$V_a \sim U(6,9)$, $V_b \sim Exp(5)$

All the lifetimes are expressed in minutes.

pdf of V_a:

$$f_a(u) = \begin{cases} \frac{1}{3} & \text{if } 6 \leq u \leq 9 \\ 0 & \text{otherwise} \end{cases}$$

pdf of V_b:

$$f_b(v) = \begin{cases} \frac{1}{5} e^{-\frac{v}{5}} & \text{if } v \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

1. Compute $P(X_2=3)$

First, we have to identify all the sample paths such that $\{X_2=3\}$. Notice that we can exclude sample paths starting with $\{X_0=2\}$, because $p_{X_0}(2)=0$.

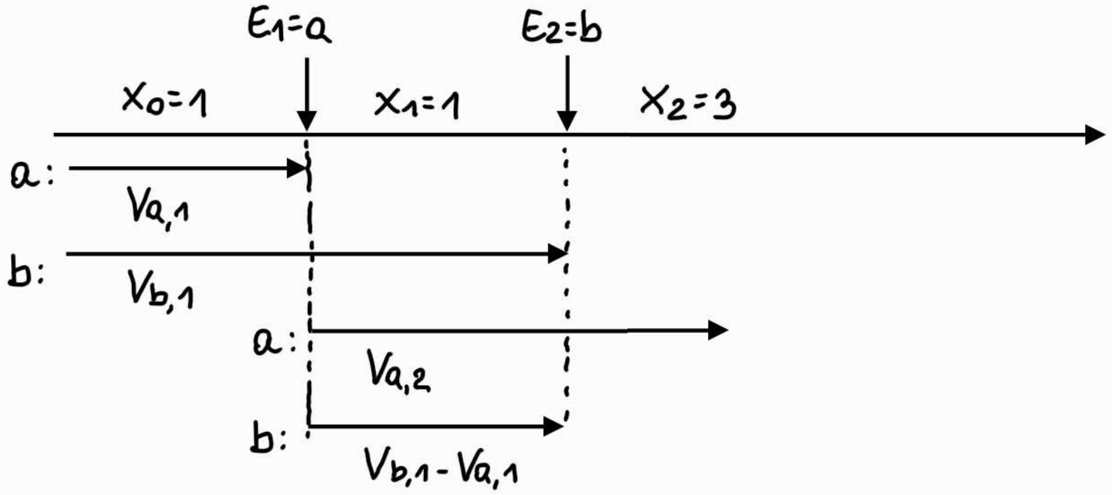
1) $X_0=1 \xrightarrow{E_1=a} X_1=1 \xrightarrow{E_2=b} X_2=3$

2) $X_0=3 \xrightarrow{E_1=a} X_1=1 \xrightarrow{E_2=b} X_2=3$

3) $X_0=3 \xrightarrow{E_1=a} X_1=2 \xrightarrow{E_2=b} X_2=3$

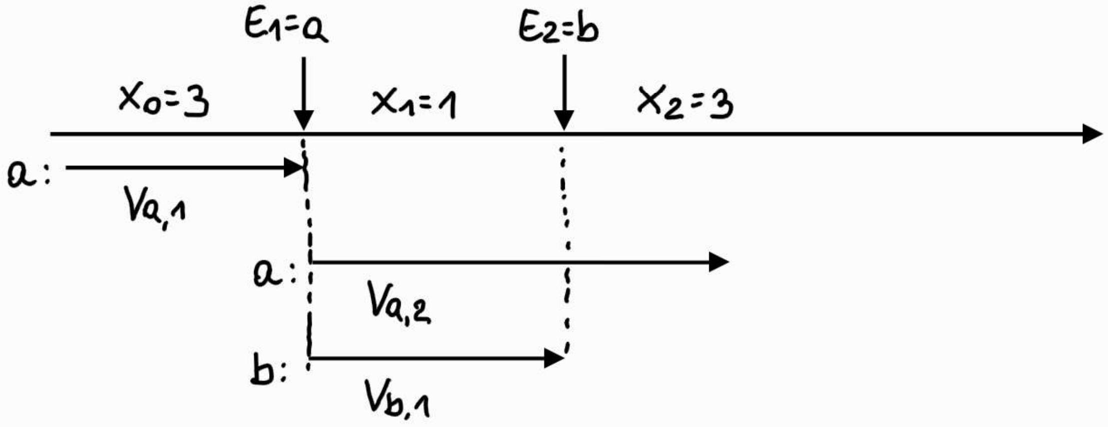
Then, we have to compute the probability of each sample path.

$\boxed{1}$



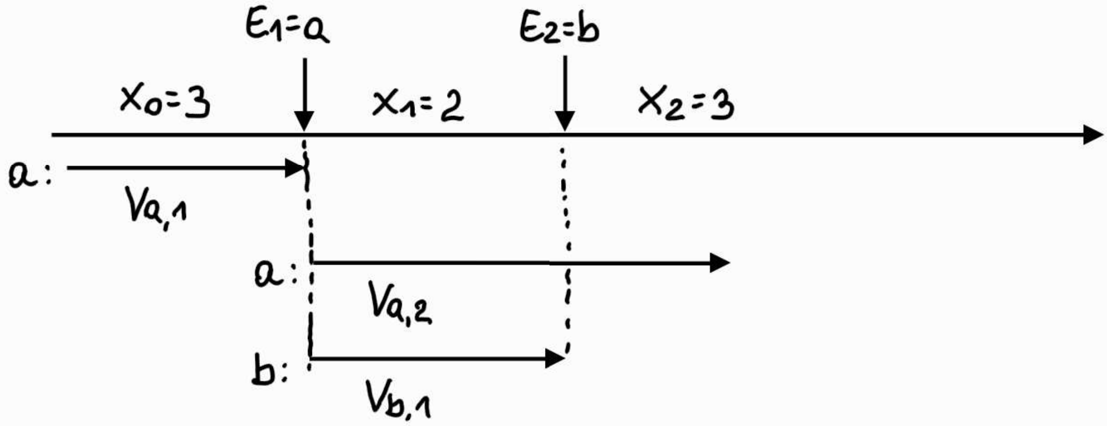
$$\Rightarrow P(\boxed{1}) = p_{x_0}(1)\cdot \underbrace{p(1|1,a)}_{1}\cdot \underbrace{p(3|1,b)}_{1}\cdot P(V_{a,1}<V_{b,1},\ V_{b,1}-V_{a,1}<V_{a,2})$$

$\boxed{2}$



$$\Rightarrow P(\boxed{2}) = p_{x_0}(3)\cdot \underbrace{p(1|3,a)}_{q}\cdot \underbrace{p(3|1,b)}_{1}\cdot P(V_{b,1}<V_{a,2})$$

$\boxed{3}$



$$\Rightarrow P(\boxed{3}) = p_{x_0}(3)\cdot \underbrace{p(2|3,a)}_{1-q}\cdot \underbrace{p(3|2,b)}_{1}\cdot P(V_{b,1}<V_{a,2})$$

Finally:

$$P(x_2=3) = \sum_{i=1}^{3} P(\boxed{i})$$

$$= p_{x_0}(1)\,P(V_{a,1}<V_{b,1}<V_{a,1}+V_{a,2}) + p_{x_0}(3)\,P(V_{b,1}<V_{a,2})$$

$$\simeq 0.3746$$

computed numerically with Matlab

REMARK

Computing a probability involving two or more random variables requires the evaluation of a multiple integral.

In the exam this computation is normally not requested.

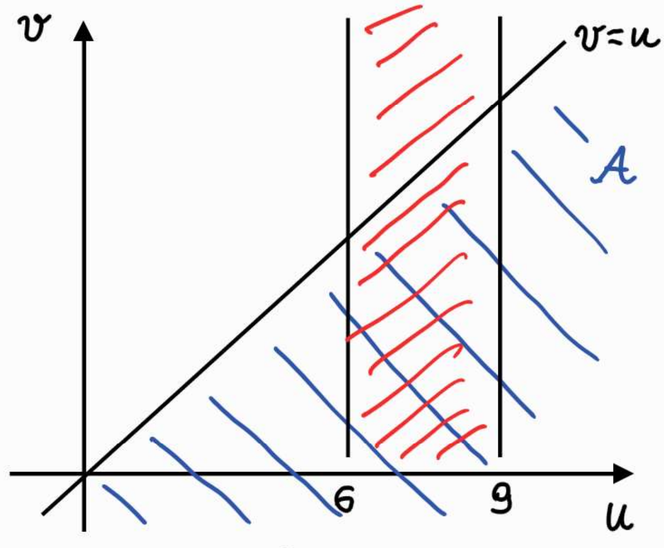
In the second part of the course we will see how to compute these probabilities numerically.

For illustrative purposes only, the computation of $P(V_{b,1} < V_{a,2})$ is shown next.

$$P(\underset{\downarrow \atop v}{V_{b,1}} < \underset{\downarrow \atop u}{V_{a,2}}) = \iint_A f_a(u) f_b(v)\, du\, dv$$

$V_{a,2}$ and $V_{b,1}$ are independent, therefore their joint pdf is the product of the marginal pdfs.

$$A = \{(u,v) \in \mathbb{R}^2 : v < u\}$$



The product $f_a(u) f_b(v)$ is nonzero only in the red region.

Hence, the integral is to be computed only over the blue-and-red region.

$$P(V_{b,1} < V_{a,2}) = \int_6^9 \int_0^u \frac{1}{3} \cdot \frac{1}{5} e^{-\frac{v}{5}}\, du\, dv = \int_6^9 \frac{1}{3} \left[-e^{-\frac{v}{5}} \right]_0^u du$$

$$= \int_6^9 \frac{1}{3} \left(1 - e^{-\frac{u}{5}}\right) du = \frac{1}{3} \left[u + 5e^{-\frac{u}{5}} \right]_6^9 = \frac{1}{3} \left(9 + 5e^{-\frac{9}{5}} - 6 - 5e^{-\frac{6}{5}} \right)$$

$$= 1 - \frac{5}{3} \left(e^{-\frac{6}{5}} - e^{-\frac{9}{5}} \right) \simeq 0.7735$$

2. Compute $P(E_2 = a)$.

First, we have to identify all the sample paths such that $\{E_2 = a\}$.
As before, we can exclude sample paths starting with $\{X_0 = 2\}$.

1) $X_0 = 1 \xrightarrow{E_1 = a} X_1 = 1 \xrightarrow{E_2 = a}$

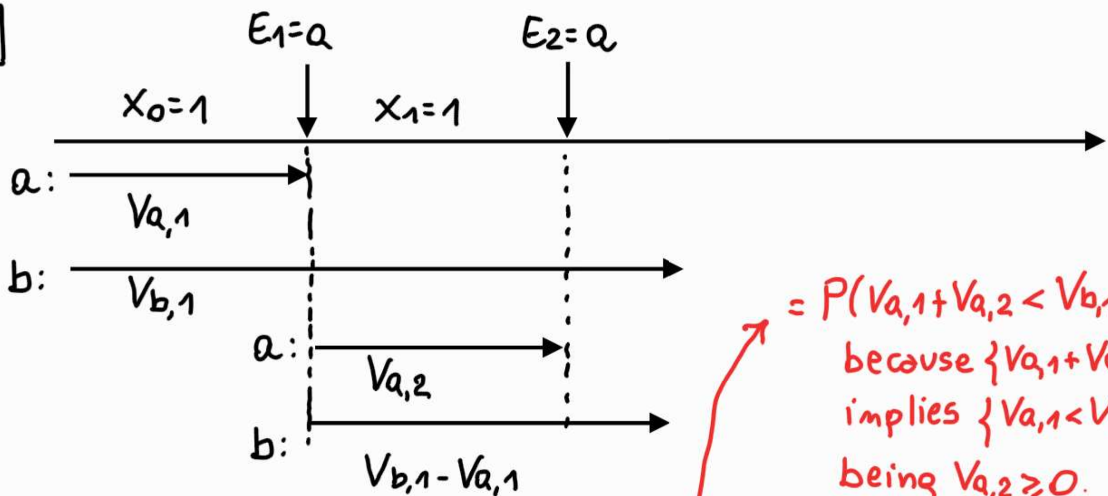
2) $X_0 = 1 \xrightarrow{E_1 = b} X_1 = 3 \xrightarrow{E_2 = a}$

3) $X_0 = 3 \xrightarrow{E_1 = a} X_1 = 1 \xrightarrow{E_2 = a}$

4) $X_0 = 3 \xrightarrow{E_1 = a} X_1 = 2 \xrightarrow{E_2 = a}$

Then, we have to compute the probability of each sample path.

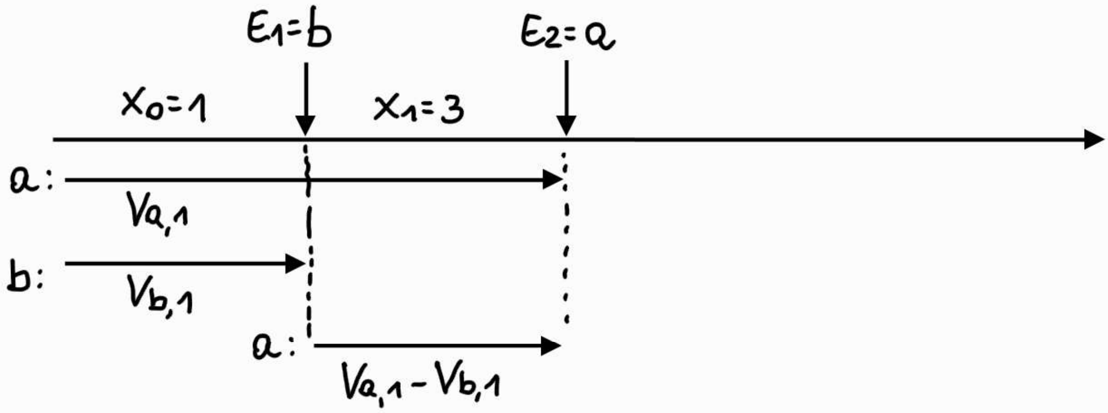
$\boxed{1}$



$= P(V_{a,1}+V_{a,2} < V_{b,1})$
because $\{V_{a,1}+V_{a,2} < V_{b,1}\}$
implies $\{V_{a,1} < V_{b,1}\}$,
being $V_{a,2} \geq 0$.

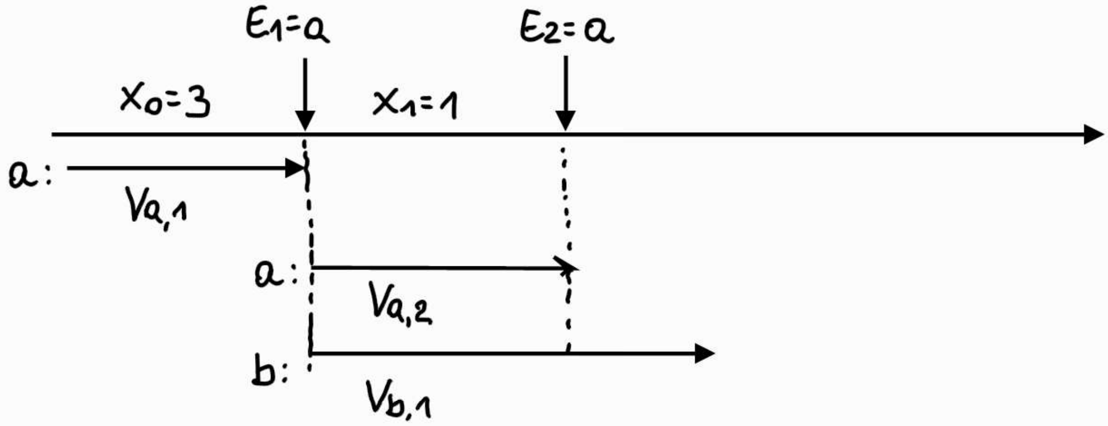
$$\Rightarrow P(\boxed{1}) = p_{x_0}(1) \cdot \underbrace{p(1|1,a)}_{1} \cdot P(V_{a,1} < V_{b,1},\ V_{a,2} < V_{b,1} - V_{a,1})$$

$\boxed{2}$



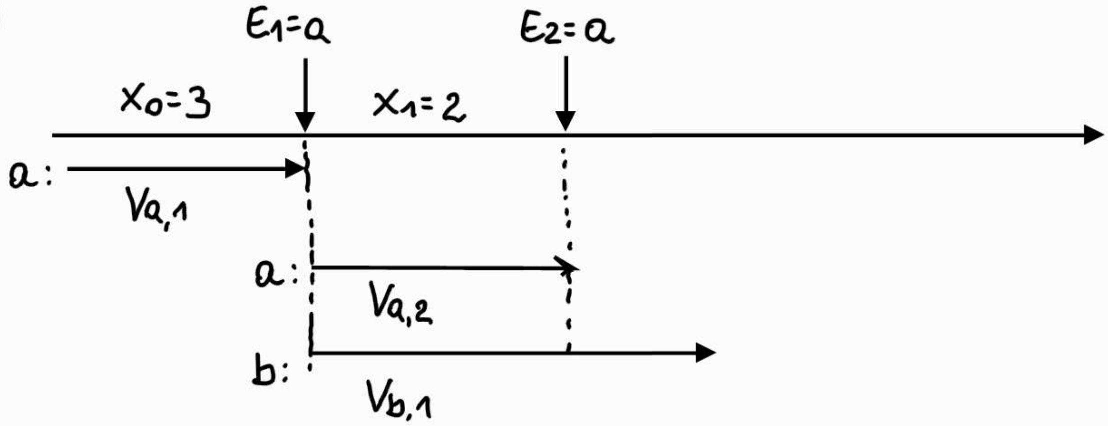
$$\Rightarrow P(\boxed{2}) = p_{x_0}(1) \cdot \underbrace{p(3|1,b)}_{1} \cdot P(V_{b,1} < V_{a,1})$$

$\boxed{3}$



$$\Rightarrow P(\boxed{3}) = p_{x_0}(3) \cdot \underbrace{p(1|3,a)}_{q} \cdot P(V_{a,2} < V_{b,1})$$

$\boxed{4}$



$$\Rightarrow P(\boxed{4}) = p_{x_0}(3) \cdot \underbrace{p(2|3,a)}_{1-q} \cdot P(V_{a,2} < V_{b,1})$$

Finally:

$$P(E_2=a) = \sum_{i=1}^{4} P(\boxed{i})$$

$$= p_{x_0}(1) P(V_{a,1} + V_{a,2} < V_{b,1}) + p_{x_0}(1) P(V_{b,1} < V_{a,1}) + p_{x_0}(3) P(V_{a,2} < V_{b,1})$$

$\simeq 0.6254$ computed numerically with Matlab

3. Compute the probability that event b occurs before $T = 15$ min.

If b were always possible, the answer would be trivial:

$P(V_{b,1} \leq T) = F_b(T) = 1 - e^{-T/5}$. Unfortunately, this is not the case, because event b is not possible in state 3.

On the other hand, event a is always possible.

Therefore, we have in principle to consider all the possible sample paths with event b preceded by an undetermined number of events a, while imposing that event b occurs before time T.

Luckily enough, since $V_a \sim U(6,9)$, we can exploit the fact that event a occurs no more than two times before $T = 15$ min (the third occurrence cannot be before $6+6+6 = 18$ min).

The possible cases are thus the following:

1) $X_0 = 1 \xrightarrow{E_1 = b}$

2) $X_0 = 1 \xrightarrow{E_1 = a} X_1 = 1 \xrightarrow{E_2 = b}$

3) $X_0 = 1 \xrightarrow{E_1 = a} X_1 = 1 \xrightarrow{E_2 = a} X_2 = 1 \xrightarrow{E_3 = b}$

4) $X_0 = 3 \xrightarrow{E_1 = a} X_1 = 1 \xrightarrow{E_2 = b}$

5) $X_0 = 3 \xrightarrow{E_1 = a} X_1 = 2 \xrightarrow{E_2 = b}$

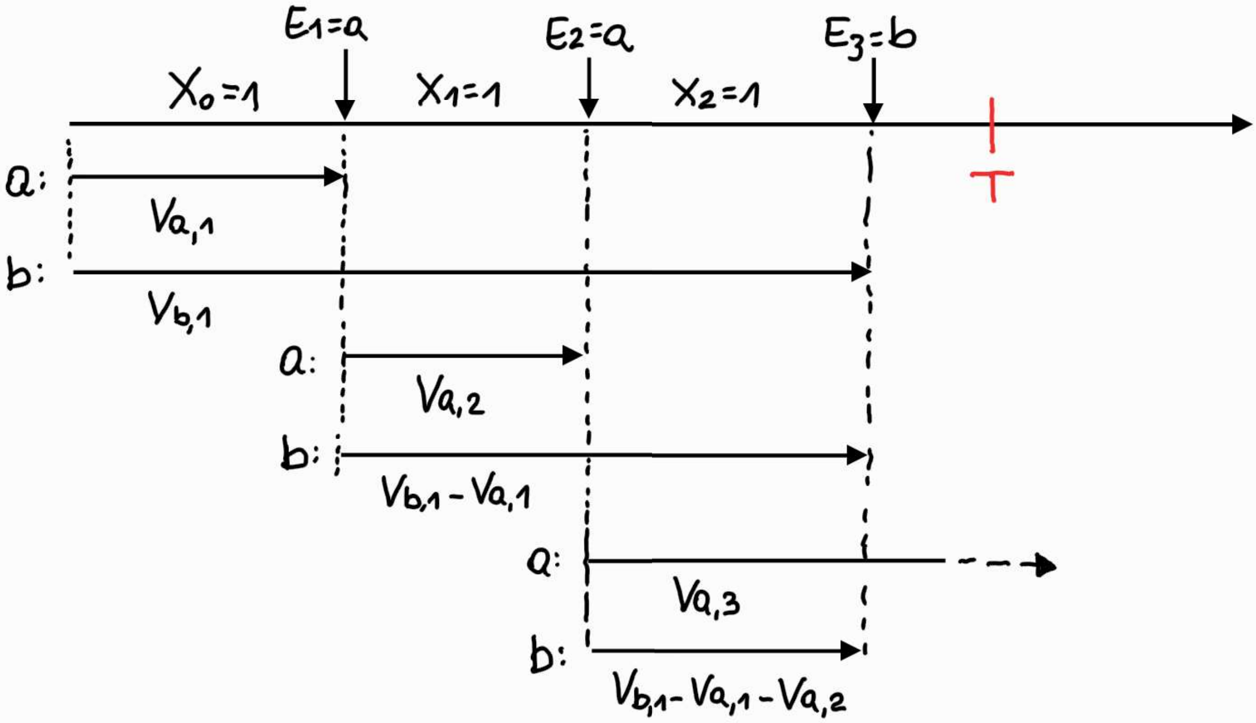
6) $X_0 = 3 \xrightarrow{E_1 = a} X_1 = 1 \xrightarrow{E_2 = a} X_2 = 1 \xrightarrow{E_3 = b}$

7) $X_0 = 3 \xrightarrow{E_1 = a} X_1 = 2 \xrightarrow{E_2 = a} X_2 = 1 \xrightarrow{E_3 = b}$

We only show how to compute $P(\boxed{3})$ and $P(\boxed{4})$.

The computation of the other probabilities is similar.

$\boxed{3}$

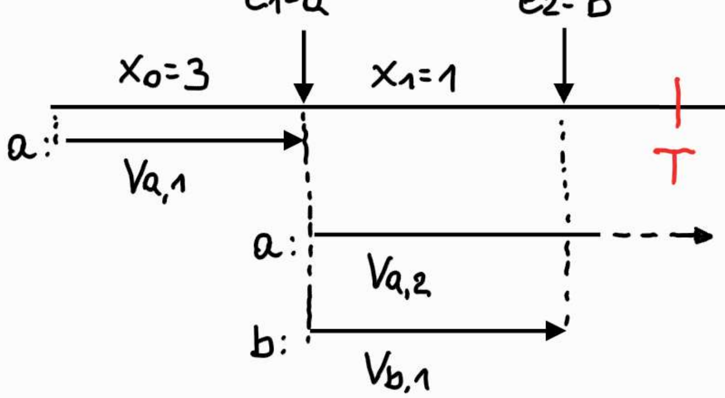


$$\Rightarrow P(\boxed{3}) = p_{x_0}(1) \cdot P\left(V_{a,1} + V_{a,2} < V_{b,1} < V_{a,1} + V_{a,2} + V_{a,3},\ V_{b,1} < T\right)$$

Here we omitted transition probabilities that are equal to 1, and conditions that are redundant (e.g. $\{V_{a,1} < V_{b,1}\}$ is implied by $\{V_{a,1} + V_{a,2} < V_{b,1}\}$, because $V_{a,2} \geq 0$).

time instant when event b occurs

$\boxed{4}$



time instant when event b occurs

$$\Rightarrow P(\boxed{4}) = p_{x_0}(3) \cdot q \cdot P\left(V_{b,1} < V_{a,2},\ \overbrace{V_{a,1} + V_{b,1} < T}\right)$$

The probabilities of the other cases are:

$$P(\boxed{1}) = p_{x_0}(1) \cdot P(V_{b,1} < V_{a,1},\ V_{b,1} < T)$$

$$P(\boxed{2}) = p_{x_0}(1) \cdot P(V_{a,1} < V_{b,1} < V_{a,1} + V_{a,2},\ V_{b,1} < T)$$

$$P(\boxed{5}) = p_{x_0}(3) \cdot (1-q) \cdot P(V_{b,1} < V_{a,2},\ V_{a,1} + V_{b,1} < T)$$

$$P(\boxed{6}) = p_{x_0}(3) \cdot q \cdot P(V_{a,2} < V_{b,1} < V_{a,2} + V_{a,3},\ V_{a,1} + V_{b,1} < T)$$

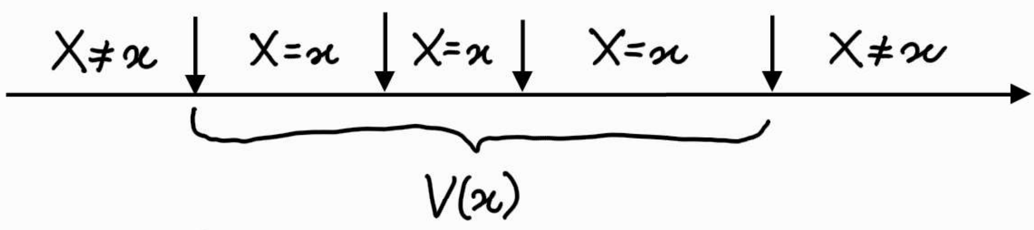
$$P(\boxed{7}) = p_{x_0}(3) \cdot (1-q) \cdot P(V_{a,2} < V_{b,1} < V_{a,2} + V_{a,3},\ V_{a,1} + V_{b,1} < T)$$

Finally,

$$P(\text{event } b \text{ occurs before } T = 15) = \sum_{i=1}^{7} P(\boxed{i})$$

$$\simeq 0.8913$$ computed numerically with Matlab

4. The state holding time is the time that the system remains in a given state.



(state holding time in state x)

We have to compute the cdf of $V(2)$.

We observe that:

- the system enters state 2 only from state 3 with event a, and in state 3 event b is not possible
  $\Rightarrow$ in state 2 the lifetimes of both events are total lifetimes.

- the system leaves state 2 when either of the two events occurs.

Hence, we have:

$$V(2) = \min\{V_a, V_b\}$$

total lifetime of event a   total lifetime of event b

To compute the cdf of $V(2)$, we start by computing

$$P(V(2)>t) = P(\min\{V_a, V_b\}>t)$$

$$= P(V_a>t, V_b>t) = P(V_a>t)P(V_b>t)$$

$\{\min\{V_a,V_b\}>t\}$ iff $\{V_a>t, V_b>t\}$ independent

$$= [1-F_a(t)][1-F_b(t)]$$

$$\Rightarrow P(V(2)\leq t) = 1-P(V(2)>t) = 1-[1-F_a(t)][1-F_b(t)]$$

Since

$$F_a(t) = \begin{cases} 0 & \text{if } t<6 \\ \frac{t-6}{3} & \text{if } 6\leq t\leq 9 \\ 1 & \text{if } t>9 \end{cases} \qquad F_b(t) = \begin{cases} 1-e^{-\frac{t}{5}} & \text{if } t\geq 0 \\ 0 & \text{if } t<0 \end{cases}$$

we have:

$$P(V(2)\leq t) = \begin{cases} 0 & \text{if } t<0 \\ 1-e^{-\frac{t}{5}} & \text{if } 0\leq t<6 \\ 1-\frac{9-t}{3}e^{-t/5} & \text{if } 6\leq t<9 \\ 1 & \text{if } t>9 \end{cases}$$

In state 1 we have:

$$V(1) = Y_b \leftarrow \text{residual lifetime of event b}$$

because the system leaves state 1 only with event b, but when the system enters state 1, event b may have a residual lifetime from the previous state. Indeed,

- $Y_b = V_b - V_a$ if the system arrives from state 2,
- $Y_b = V_b$ if the system arrives from state 3.

Computing $P(V(1) \leq t)$ is therefore a non-trivial task.