Test of Discrete Event Systems - 05.12.2019

Exercise 1

A low-cost hotel has a small room for fitness activities. Guests willing to have physical activity, have to book in before 10AM. Their number is a random variable uniformly distributed over $\{0, 1, 2, 3\}$. The room opens at 5PM and can be used by one guest at a time. If a guest arrives and finds the room busy, he/she returns later.

1. Model the system through a state automaton $(\mathcal{E}, \mathcal{X}, \Gamma, f, p_0)$.

Assume that the first time when a guest shows up at the fitness room is uniformly distributed between 5PM and 8PM. The duration of the physical activity is uniformly distributed between 20 and 40 minutes. If a guest arrives and finds the room busy, he/she returns after 45 minutes.

2. Define the stochastic clock structure F of the corresponding stochastic timed automaton.
3. Assume that the number of guests who booked in for the fitness room is 2. Compute the probability that both of them can use the room the first time they arrive.
4. Assume that the number of guests who booked in for the fitness room is 2. Compute the cumulative distribution function of the state holding time when the room is busy and one guest has still to show up for the first time.
5. Assume that the number of guests who booked in for the fitness room is 2. Compute the probability that both of them terminate the use of the room before dinner, which is served at 7:30PM.

Assume that the time after 5PM when a guest shows up at the fitness room for the first time, is exponentially distributed with expected value 1 h. The duration of the physical activity is exponentially distributed with expected value 30 min. If a guest arrives and finds the room busy, he/she returns after a time following an exponential distribution with expected value 45 minutes.

6. Repeat questions from 2 to 5 in this case.

1)

state $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$x_1 \to$ room: 0 (empty), 1 (busy)

$x_2 \to$ #guests who have not tried the access to the room $\in \{0,1,2,3\}$

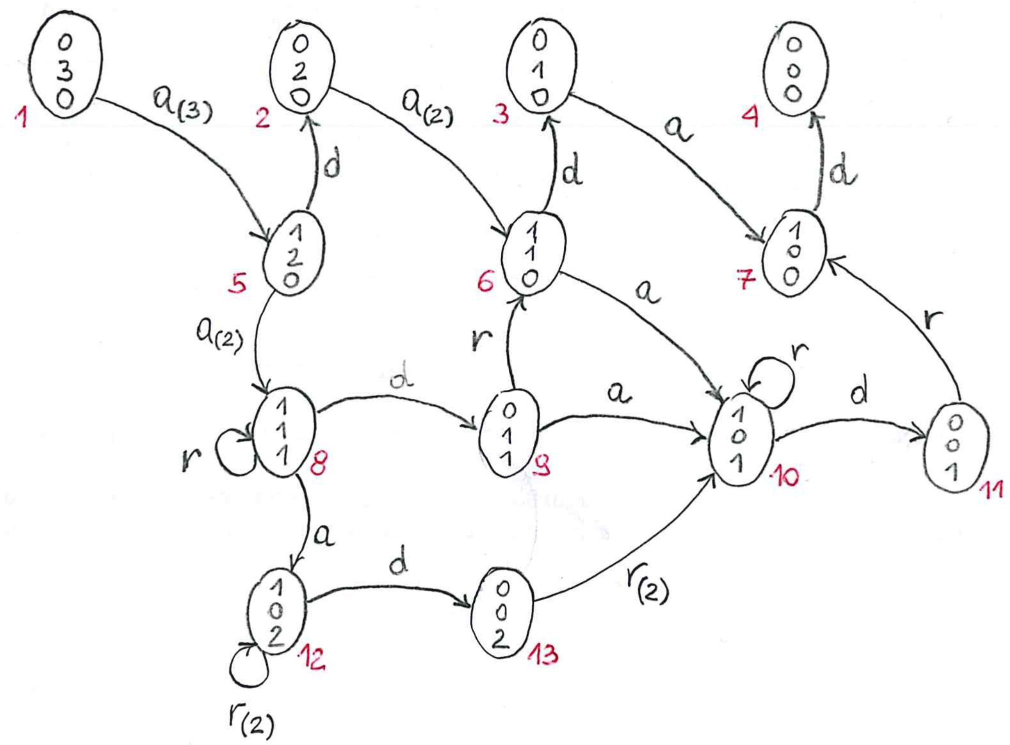
$x_3 \to$ #guests who have to repeat the access to the room $\in \{0,1,2\}$

events $\mathcal{E} = \{a, d, r\}$

$a \to$ arrival of a guest

$d \to$ termination of the use of the room

$r \to$ return of a guest



$p_0(1) = p_0(2) = p_0(3) = p_0(4) = 1/4$

$p_0(5) = \ldots = p_0(13) = 0$

2: $(\mathcal{E}, \mathcal{X}, \Gamma, f, p_0, F)$

with:

- $V_a \sim U(0,3)$ [h] $\rightsquigarrow$ Notice that $t=0$ is 5PM.
- $V_d \sim U(\frac{1}{3}, \frac{2}{3})$ [h]
- $V_r = \frac{3}{4}$ [h]

3. $X_0 = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$

At initialization, there are two guests who are willing to use the fitness room, say Mick and Lucy. Hence, we have two cases, depending whether Mick or Lucy arrives first.

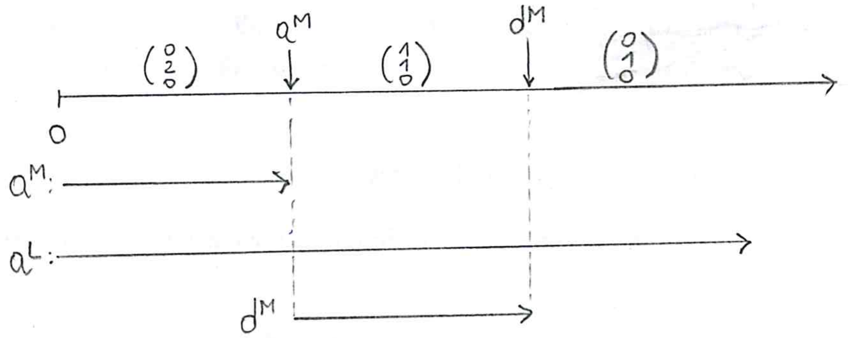
$\boxed{1}$ $\begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} \xrightarrow{a^M} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \xrightarrow{d^M} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$\boxed{2}$ $\begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} \xrightarrow{a^L} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \xrightarrow{d^L} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

Notice that, for the sake of clarity, we use a superscript M or L to distinguish the guest that the event refers to.

The two cases correspond to the following sample paths:

$\boxed{1}$



$$\Rightarrow P(\boxed{1}) = P\left(V_{a^M} < V_{a^L},\ V_{a^M} + V_{d^M} < V_{a^L}\right) = P\left(V_{a^M} + V_{d^M} < V_{a^L}\right) \simeq 0.3477$$

estimated with Matlab

$\boxed{2}$ The sample path is the same as in case 1, but with M and L interchanged.

$$\Rightarrow P(\boxed{2}) = P\left(V_{a^L} + V_{d^L} < V_{a^M}\right) \simeq 0.3477$$

Notice that $P(\boxed{1}) = P(\boxed{2})$, because V_{a^M} and V_{a^L} are identically distributed, and the same holds for V_{d^M} and V_{d^L}.

$$\Rightarrow P(\dots) = P(\boxed{1}) + P(\boxed{2}) \simeq 0.6354$$

Notice that $P(\dots)$ can be also rewritten as

$$P(\dots) = P\left(\min\{V_{a^M}, V_{a^L}\} + V_d < \max\{V_{a^M}, V_{a^L}\}\right) \simeq 0.6354$$

estimated with Matlab.

4. $X_0 = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$

We want to compute $P\left(V\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \leq t\right)$.

First, we notice that $V\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \min\{Y_a, V_d\}$, where Y_a is a residual lifetime of event a and V_d is a total lifetime of event d. Hence,

$$P\left(V\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \leq t\right) = 1 - P\left(V\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} > t\right) = 1 - P\left(\min\{Y_a, V_d\} > t\right)$$

$$= 1 - P(Y_a > t, V_d > t) = 1 - P(Y_a > t)P(V_d > t)$$

$$= 1 - [1 - P(Y_a \leq t)][1 - P(V_d \leq t)]$$

this cdf is unknown $\Downarrow$ to be computed

this cdf is known: $P(V_d \leq t) = \begin{cases} 0 & \text{if } t < \frac{1}{3} \\ 3t-1 & \text{if } \frac{1}{3} \leq t \leq \frac{2}{3} \\ 1 & \text{if } t > \frac{2}{3} \end{cases}$

Notice that, coming from state $X_0 = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$,

$$Y_a = \max\{V_{a^L} - V_{a^M}, V_{a^M} - V_{a^L}\}$$

where V_{a^L} and V_{a^M} are independent and their distributions are known.

With some effort one can obtain:

$$P(Y_a \leq t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 - \frac{(3-t)^2}{9} & \text{if } 0 \leq t \leq 3 \\ 1 & \text{if } t > 3 \end{cases}$$

Hence,

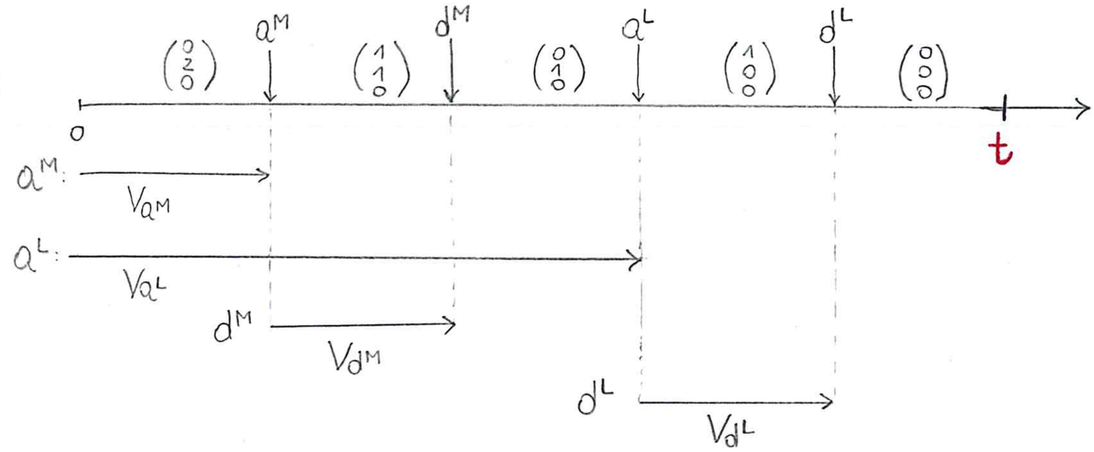
$$P\left(V\begin{pmatrix}1\\1\\0\end{pmatrix}\leq t\right)=\begin{cases}0 & \text{if } t<0\\ 1-\dfrac{(3-t)^2}{9} & \text{if } 0\leq t<\frac{1}{3}\\ 1-\dfrac{(2-3t)(3-t)^2}{9} & \text{if } \frac{1}{3}\leq t\leq\frac{2}{3}\\ 1 & \text{if } t>\frac{2}{3}\end{cases}$$

5. $X_0=\begin{pmatrix}0\\2\\0\end{pmatrix}$

We want to compute $P\left(X(t)=\begin{pmatrix}0\\0\\0\end{pmatrix}\right)$ with $t=2.5$ h.

With the given clock structure, there are only four cases:

$\boxed{1}$



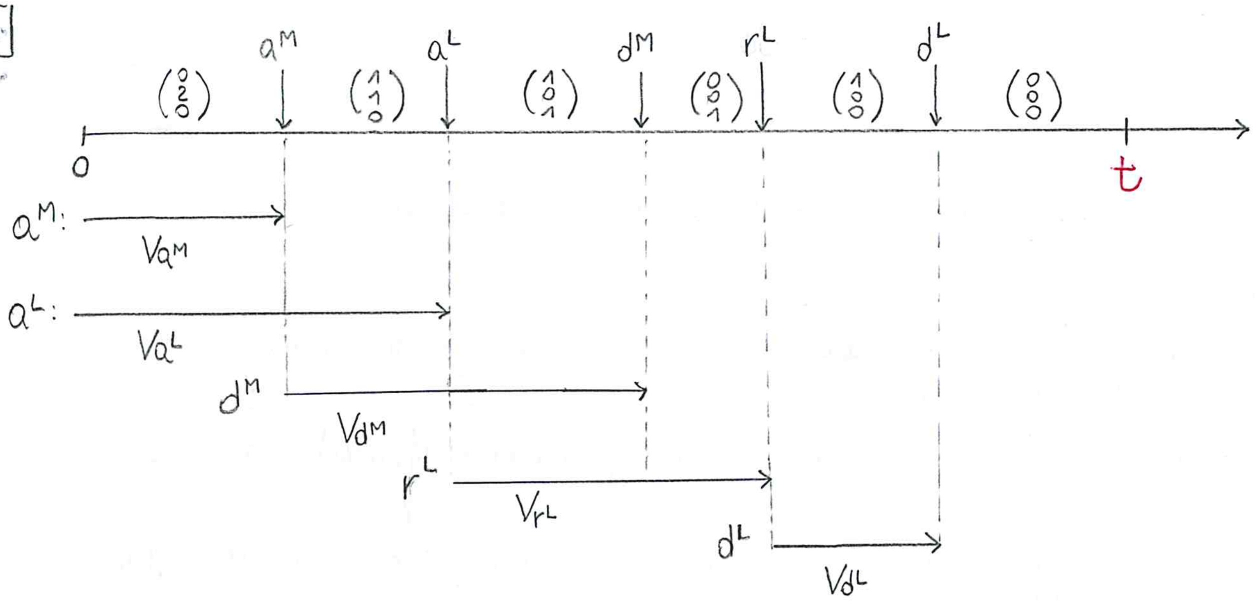
$$P(\boxed{1})=P\left(V_{a^M}<V_{a^L},\ V_{a^M}+V_{d^M}<V_{a^L},\ V_{a^L}+V_{d^L}<t\right)$$
$$=P\left(V_{a^M}+V_{d^M}<V_{a^L},\ V_{a^L}+V_{d^L}<t\right)\simeq 0.1260 \leftarrow \text{estimated with Matlab}$$

$\boxed{2}$ The sample path is the same as in case 1, but with M and L interchanged.

$$P(\boxed{2})=P\left(V_{a^L}+V_{d^L}<V_{a^M},\ V_{a^M}+V_{d^M}<t\right)\simeq 0.1260$$

Notice that $P(\boxed{1})=P(\boxed{2})$.

$\boxed{3}$



$$P(\boxed{3}) = P\left(V_{a^M} < V_{a^L},\ V_{a^L} < V_{a^M} + V_{d^M},\ V_{a^M} + V_{d^M} < V_{a^L} + V_{r^L},\ V_{a^L} + V_{r^L} + V_{d^L} < t\right)$$

$\simeq 0.0550 \leftarrow$ estimated with Matlab

$\boxed{4}$ The sample path is the same as in case 3, but with M and L interchanged.

$$P(\boxed{4}) = P\left(V_{a^L} < V_{a^M},\ V_{a^M} < V_{a^L} + V_{d^L},\ V_{a^L} + V_{d^L} < V_{a^M} + V_{r^M},\ V_{a^M} + V_{r^M} + V_{d^M} < t\right)$$

$\simeq 0.0550$

Notice that $P(\boxed{3}) = P(\boxed{4})$.

Moreover, notice that there are no other cases, because in state $\begin{pmatrix}1\\0\\1\end{pmatrix}$ we have that

$$Y_d < \frac{2}{3} < \frac{3}{4} = V_r.$$

$\uparrow$
residual lifetime of event d

Hence,

$$P\left(X(t) = \begin{pmatrix}0\\0\\0\end{pmatrix}\right) = P(\boxed{1}) + P(\boxed{2}) + P(\boxed{3}) + P(\boxed{4}) \simeq 0.3620$$

6: We repeat questions from 2 to 5 in this case.

6.2. $(\mathcal{E}, \mathcal{X}, \Gamma, f, p_0, F)$

with $F = \{F_a, F_d, F_r\}$

$F_a(t) = 1 - e^{-\lambda t}, \; t \geq 0 \qquad \frac{1}{\lambda} = 1\,h \Rightarrow \lambda = 1$ arrival/h

$F_d(t) = 1 - e^{-\mu t}, \; t \geq 0 \qquad \frac{1}{\mu} = 30\,min = \frac{1}{2}\,h \Rightarrow \mu = 2$ terminations/h

$F_r(t) = 1 - e^{-\gamma t}, \; t \geq 0 \qquad \frac{1}{\gamma} = 45\,min = \frac{3}{4}\,h \Rightarrow \gamma = \frac{4}{3}$ returns/h.

6.3 We have to compute the probability of the path

$$\begin{pmatrix}0\\2\\0\end{pmatrix} \xrightarrow{a(2)} \begin{pmatrix}1\\1\\0\end{pmatrix} \xrightarrow{d} \begin{pmatrix}0\\1\\0\end{pmatrix}$$

$$\Rightarrow P(\ldots) = \frac{2\lambda}{2\lambda} \cdot 1 \cdot \frac{\mu}{\lambda+\mu} \cdot 1 = \frac{\mu}{\lambda+\mu} = \frac{2}{3} \simeq 0.6666$$

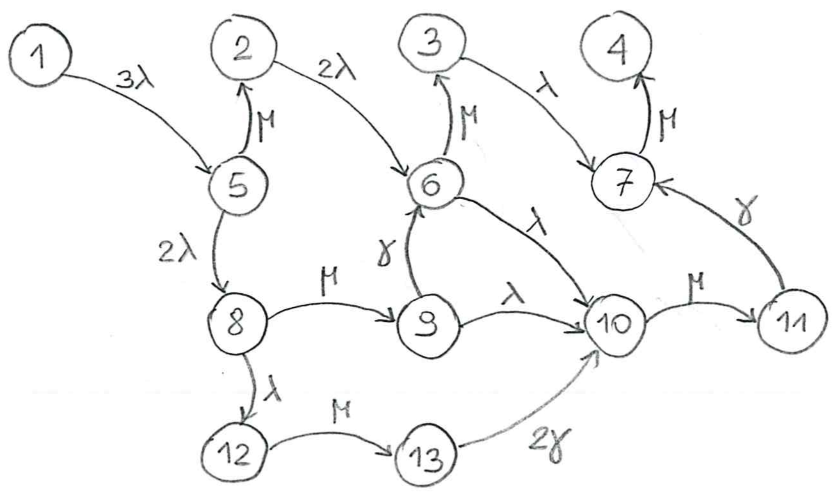
6.4 We have to compute $P\left(V\begin{pmatrix}1\\1\\0\end{pmatrix} \leq t\right)$. We have

$$P\left(V\begin{pmatrix}1\\1\\0\end{pmatrix} \leq t\right) = 1 - e^{-(\lambda+\mu)t} = 1 - e^{-3t}, \; t \geq 0.$$

6.5. We want to compute $P(X(t) = 4)$ with $t = 2.5h$.

according to the numbering of the states

We transform the stochastic timed automaton with Poisson clock structure into an equivalent continuous-time homogeneous Markov chain:



Since starting from state $X(0)=2$ it is possible to reach only the states in the subset $\{2,3,4,6,7,10,11\}$, we write the transition rate matrix Q restricted to this subset of states:

$$Q = \begin{array}{c} \\ 2 \\ 3 \\ 4 \\ 6 \\ 7 \\ 10 \\ 11 \end{array} \begin{array}{c} \begin{array}{ccccccc} 2 & 3 & 4 & 6 & 7 & 10 & 11 \end{array} \\ \begin{bmatrix} -2\lambda & 0 & 0 & 2\lambda & 0 & 0 & 0 \\ 0 & -\lambda & 0 & 0 & \lambda & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mu & 0 & -(\lambda+\mu) & 0 & \lambda & 0 \\ 0 & 0 & \mu & 0 & -\mu & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\mu & \mu \\ 0 & 0 & 0 & 0 & \gamma & 0 & -\gamma \end{bmatrix} \end{array}$$

Let $\pi(t) = [\pi_2(t)\ \pi_3(t)\ \pi_4(t)\ \pi_6(t)\ \pi_7(t)\ \pi_{10}(t)\ \pi_{11}(t)]$

and $\pi(0) = [1\ 0\ 0\ 0\ 0\ 0\ 0]$.

We have:

$$P(X(t)=4) = \pi_4(t) = \pi(0)\, e^{Qt} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \simeq 0.5986$$

NOTICE THAT, COMPARED TO THE FIRST STOCHASTIC CLOCK STRUCTURE, IN THE SECOND CASE (POISSON CLOCK STRUCTURE) WE CAN EXPLOIT FORMULAS, AND THEREFORE COMPUTATIONS ARE EASIER.