## Test of Discrete Event Systems - 21.11.2019

### Exercise 1

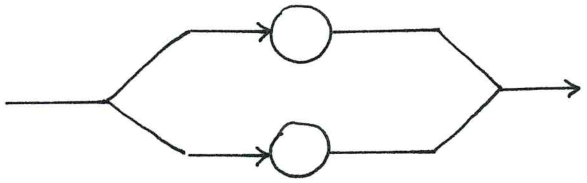
A small hair salon has two chairs and two hairdressers. Customers arrive according to a Poisson process with rate 3 arrivals/hour. A customer is male with probability $p = 1/3$. The duration of a hair-cut is independent of the hairdresser, but depends on the gender of the customer. It is exponentially distributed with expected value 20 minutes for men, and 45 minutes for women. Since the hair salon does not have a waiting room, customers arriving when both chairs are busy, decide to give up hair cutting. The hair salon is empty at the opening.

1. Model the hair salon through a stochastic timed automaton $(\mathcal{E}, \mathcal{X}, \Gamma, p, x_0, F)$.
2. Assume that one hairdresser is serving a man and the other is serving a woman. Compute the probability that the next event is the arrival of a new customer.
3. Assume that both hairdressers are busy with male customers. Compute the probability that the next event is the termination of a hair cut.
4. Assume that both hairdressers are busy with male customers of different age. Compute the probability that the next event is the termination of the hair cut of the youngest man.
5. Assume that one hairdresser is serving a man and the other is serving a woman. Compute the probability that the hair cut of the man terminates before the hair cut of the woman.
6. Compute the probability that the third customer arriving after the opening has to give up hair cutting.
7. Assume that one hairdresser is serving a man and the other is serving a woman. Compute the probability that both hair cuts terminate:
   (a) before another customer arrives;
   (b) before another customer sits for a hair cut.
8. Assume that one hairdresser is serving a man and the other is serving a woman. Compute the probability that, in the next hour, both hair cuts are terminated and no other customer arrives.
9. Compute the probability that at least three customers arrive in the next hour.
10. Compute the average state holding time when:
    (a) one hairdresser is serving a man and the other is idle;
    (b) one hairdresser is serving a man and the other is serving a woman.

# Exercise 1



1. The system can be represented as a queueing system with two servers and no queue:



REMARK: the two servers are equal, therefore there is no need to distinguish them in the definition of the state.

Definition of state: $x = \begin{bmatrix} x_M \\ x_F \end{bmatrix}$ $x_M \to$ number of male customers $\in \{0,1,2\}$, $x_F \to$ number of female customers $\in \{0,1,2\}$

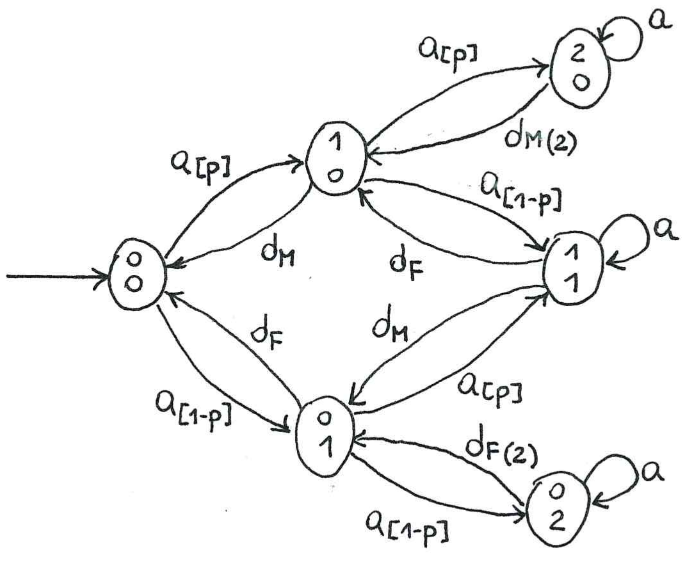
State space: $\mathcal{X} = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \end{bmatrix} \right\} \Rightarrow$ 6 states

Definition of events: $\mathcal{E} = \{a, d_M, d_F\}$

$a$ → arrival of a customer

$d_M$ → termination of the hair cut of a male customer

$d_F$ → termination of the haircut of a female customer




$F_a(t) = 1 - e^{-\lambda t}, \ t \geq 0$

where $\lambda = 3$ arrivals/hour

$F_{d_M}(t) = 1 - e^{-\mu_M t}, \ t \geq 0$

where $\frac{1}{\mu_M} = 20$ minutes $= \frac{1}{3}$ hours

$\Rightarrow \mu_M = 3$ services/hour

$F_{d_F}(t) = 1 - e^{-\mu_F t}, \ t \geq 0$

where $\frac{1}{\mu_F} = 45$ minutes $= \frac{3}{4}$ hours

$\Rightarrow \mu_F = \frac{4}{3}$ services/hour

2. The current state is $X_k = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. The probability we are asked to compute is:

$$P\left(E_{k+1} = a \,\middle|\, X_k = \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right) = \frac{\lambda}{\lambda + \mu_M + \mu_F} \simeq 0.4091$$

3. The current state is $X_k = \binom{2}{0}$.

The probability we are asked to compute is:

$$P\left(E_{k+1} = d_M \,\middle|\, X_k = \binom{2}{0}\right) = \frac{2\mu_M}{\lambda + 2\mu_M} \simeq 0.6667$$

we are not interested in which one of the two events $d_M$ will occur first

two different events $d_M$ are possible

4. The current state is $X_k = \binom{2}{0}$.

Denote by $d_M^Y$ the termination of the haircut of the youngest man, and by $d_M^O$ the termination of the haircut of the oldest man.

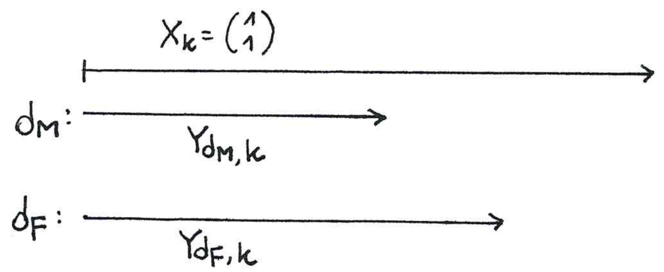
The probability we are asked to compute is:

$$P\left(E_{k+1} = d_M^Y \,\middle|\, X_k = \binom{2}{0}\right) = \frac{\mu_M}{\lambda + 2\mu_M} \simeq 0.3333$$

events $d_M^Y$ and $d_M^O$ have the same rate $\mu_M$

5. The current state is $X_k = \binom{1}{1}$.

The situation can be represented as follows:




$\Rightarrow$ The residual lifetime $Y_{d_M,k}$ of event $d_M$ must be smaller than the lifetime $Y_{d_F,k}$ of event $d_F$.

$$\Rightarrow P\left(Y_{d_M,k} \leq Y_{d_F,k} \,\middle|\, X_k = \binom{1}{1}\right) = \frac{\mu_M}{\mu_M + \mu_F} \simeq 0.6923$$

6. When the third customer arrives, the state must be one of $\binom{3}{0}$, $\binom{1}{1}$ and $\binom{0}{2}$

$\Rightarrow$ Starting from the initial state $\binom{0}{0}$, the first three events must be arrivals.

We identify four favorable cases, corresponding to the following paths on the state transition diagram:

1. $\binom{0}{0} \xrightarrow{a} \binom{1}{0} \xrightarrow{a} \binom{2}{0} \xrightarrow{a}$

2. $\binom{0}{0} \xrightarrow{a} \binom{1}{0} \xrightarrow{a} \binom{1}{1} \xrightarrow{a}$

3. $\binom{0}{0} \xrightarrow{a} \binom{0}{1} \xrightarrow{a} \binom{1}{1} \xrightarrow{a}$

4. $\binom{0}{0} \xrightarrow{a} \binom{0}{1} \xrightarrow{a} \binom{0}{2} \xrightarrow{a}$

The probability we are looking for, is the sum of the probabilities of these four cases.

To compute the probability of case 1, we proceed as follows.

- In state $\binom{0}{0}$, the only active event is $a$, Therefore, the probability that the next state is $\binom{1}{0}$, corresponds to the probability that the arrival is a man. This probability is $p$.

- In state $\binom{1}{0}$, the next event is $a$ with probability $\dfrac{\lambda}{\lambda+\mu_M}$.

  rate of event a

  sum of the rates of active events in state $\binom{1}{0}$

  The arrival is a man, and therefore the next state is $\binom{2}{0}$, with probability $p$.

- In state $\binom{2}{0}$, the next event is $a$ with probability $\dfrac{\lambda}{\lambda+2\mu_M}$

  two male customers are being served, therefore we have scheduled two distinct events $d_M$

By multiplying all these probabilities, we obtain the probability of case 1:

$$P(1) = p \cdot \frac{\lambda}{\lambda+\mu_M} \cdot p \cdot \frac{\lambda}{\lambda+2\mu_M} = \frac{(\lambda p)^2}{(\lambda+\mu_M)(\lambda+2\mu_M)}$$

In the same fashion, we can compute the probabilities of all other cases:

$$P(2) = p \cdot \frac{\lambda}{\lambda+\mu_M} \cdot (1-p) \cdot \frac{\lambda}{\lambda+\mu_M+\mu_F}$$

$$P(3) = (1-p) \cdot \frac{\lambda}{\lambda+\mu_F} \cdot p \cdot \frac{\lambda}{\lambda+\mu_M+\mu_F}$$

$$P(4) = (1-p) \cdot \frac{\lambda}{\lambda+\mu_F} \cdot (1-p) \cdot \frac{\lambda}{\lambda+2\mu_F}$$

Finally:

$$P(\ldots) = P(1) + P(2) + P(3) + P(4) =$$

$$= \frac{(\lambda p)^2}{(\lambda+\mu_M)(\lambda+2\mu_M)} + \frac{\lambda^2 p(1-p)}{(\lambda+\mu_M)(\lambda+\mu_M+\mu_F)} +$$

$$+ \frac{\lambda^2 p(1-p)}{(\lambda+\mu_F)(\lambda+\mu_M+\mu_F)} + \frac{[\lambda(1-p)]^2}{(\lambda+\mu_F)(\lambda+2\mu_F)} \simeq 0.2898$$

7.a. The current state is $X_k = \binom{1}{1}$. Event $a$ must occur after both event $d_M$ and event $d_F$. We identify two favorable cases, corresponding to the following paths on the state transition diagram:

1 $\quad \binom{1}{1} \xrightarrow{d_M} \binom{0}{1} \xrightarrow{d_F}$

2 $\quad \binom{1}{1} \xrightarrow{d_F} \binom{1}{0} \xrightarrow{d_M}$

The probability we are looking for, is the sum of the probabilities of these two cases. We have:

$$P(1) = \frac{\mu_M}{\lambda+\mu_M+\mu_F} \frac{\mu_F}{\lambda+\mu_F} \quad ; \quad P(2) = \frac{\mu_F}{\lambda+\mu_M+\mu_F} \frac{\mu_M}{\lambda+\mu_M}$$

Therefore:

$$P(\ldots) = P(1) + P(2) = \frac{\mu_M \mu_F}{(\lambda+\mu_M+\mu_F)(\lambda+\mu_F)} + \frac{\mu_M \mu_F}{(\lambda+\mu_M+\mu_F)(\lambda+\mu_M)} \simeq 0.2168$$

7.b. Notice that a customer sits for the hair cut if and only if the hair salon is not full when he/she arrives. Hence, in this question, we ignore arrivals occurring when the system is full, since the arriving customer cannot sit for the hair cut.

The favorable cases are:

$$(1) \quad \begin{pmatrix}1\\1\end{pmatrix} \xrightarrow[\text{(ignore } a)]{d_M} \begin{pmatrix}0\\1\end{pmatrix} \xrightarrow{d_F}$$

$$(2) \quad \begin{pmatrix}1\\1\end{pmatrix} \xrightarrow[\text{(ignore } a)]{d_F} \begin{pmatrix}1\\0\end{pmatrix} \xrightarrow{d_M}$$

The probabilities of these two cases are:

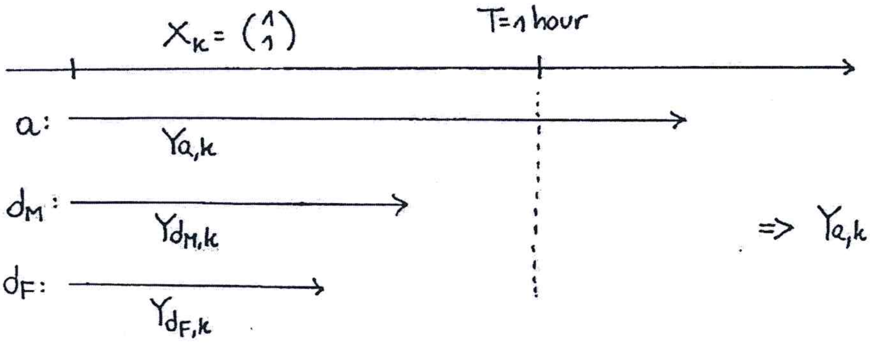
$$P(1) = \frac{\mu_M}{\mu_M+\mu_F}\,\frac{\mu_F}{\lambda+\mu_F} \quad ; \quad P(2) = \frac{\mu_F}{\mu_M+\mu_F}\,\frac{\mu_M}{\lambda+\mu_M}$$

We do not have $\lambda$ here,
because we ignore $a$ in state $\begin{pmatrix}1\\1\end{pmatrix}$

Hence,

$$P(\ldots) = P(1) + P(2) = \frac{\mu_M \mu_F}{(\mu_M+\mu_F)(\lambda+\mu_F)} + \frac{\mu_M \mu_F}{(\mu_M+\mu_F)(\lambda+\mu_M)} \simeq 0.3669$$

8. The current state is $X_k = \binom{1}{1}$. We require that the residual lifetimes of both events $d_M$ and $d_F$ are smaller than one hour, and the residual lifetime of event $a$ is larger than one hour.




$\Rightarrow Y_{a,k} > T,\ Y_{d_M,k} \leq T,\ Y_{d_F,k} \leq T$

$$\Rightarrow P(\ldots) = P(Y_{a,k} > T,\ Y_{d_M,k} \leq T,\ Y_{d_F,k} \leq T) \underset{\substack{\text{independent} \\ \text{random variables}}}{=} P(Y_{a,k} > T)P(Y_{d_M,k} \leq T)P(Y_{d_F,k} \leq T)$$

$$= e^{-\lambda T} \cdot (1 - e^{-\mu_M T})(1 - e^{-\mu_F T}) \simeq 0.0348$$

9. Arrivals are generated by a Poisson process. We apply the Poisson distribution with $T = 1$ hour.

$$P(N_a(T) \geq 3) = 1 - P(N_a(T) = 0) - P(N_a(T) = 1) - P(N_a(T) = 2)$$

$$= 1 - e^{-\lambda T} - (\lambda T) e^{-\lambda T} - \frac{(\lambda T)^2}{2} e^{-\lambda T} = 1 - \left[1 + (\lambda T) + \frac{(\lambda T)^2}{2}\right] e^{-\lambda T} \simeq 0.5768$$

10.a. The current state is $X_k = \binom{1}{0}$.

$$E\left[V\binom{1}{0}\right] = \frac{1}{(\lambda + \mu_M)}$$

sum of the rates of the events which take the system away from state $\binom{1}{0}$.

10.b The current state is $X_k = \binom{1}{1}$.

$$E\left[V\binom{1}{1}\right] = \frac{1}{(\mu_M + \mu_F)}$$

The rate $\lambda$ does not appear because event $a$ leaves state $\binom{1}{1}$ unchanged.