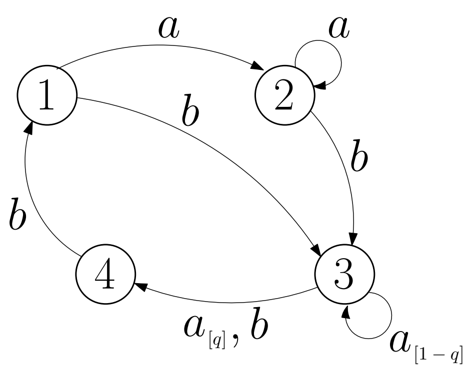
Test of Discrete Event Systems - 31.10.2019

Exercise 1

Consider the stochastic timed automaton in the figure. The initial state is drawn from a discrete uniform distribution over the set $\{1, 3, 4\}$. Lifetimes of event a are uniformly distributed over the interval $[6, 10]$ min, while lifetimes of event b follow an exponential distribution with expected value 5.5 min. The value of probability q is 0.4.



1. Compute $P(X_2 = 3)$.

2. Compute the cumulative distribution function of the state holding time of state 1.

Exercise 2

Consider a queueing system with one server and total storage capacity equal to three. Interarrival times follow a uniform distribution over the interval $[5, 20]$ minutes, while service times are all equal to $T = 12$ minutes. Assume that the queueing system is initially empty.

1. Compute the probability that only one customer is in the system when the third arrival occurs.

2. Compute the cumulative distribution function of the waiting time of the second customer.

EXERCISE 1

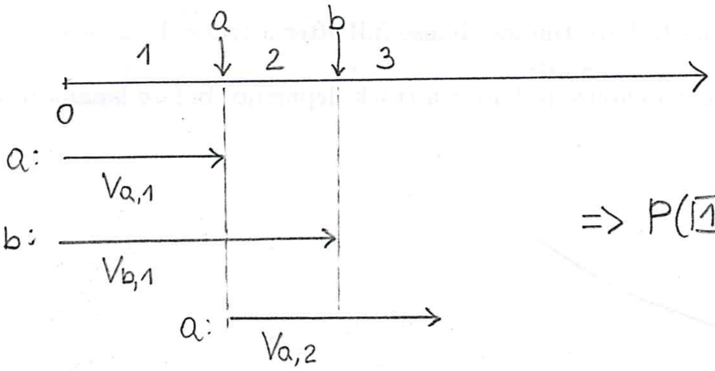
$(\mathcal{E}, \mathcal{X}, \Gamma, p, p_0, F)$

with:

- $p_0(1) = p_0(3) = p_0(4) = \frac{1}{3}$, $p_0(2) = 0$
- $F = \{F_a, F_b\}$
  $V_a \sim U(6,10)$ $\quad V_b \sim Exp(5.5)$
- $p(4|3,a) = q = 0.4$, $p(3|3,a) = 1-q = 0.6$

1. To compute $P(X_2=3)$, there are 4 possible cases, taking into account that $p_0(2)=0$.

$\boxed{1}$ $\quad 1 \xrightarrow{a} 2 \xrightarrow{b} 3$

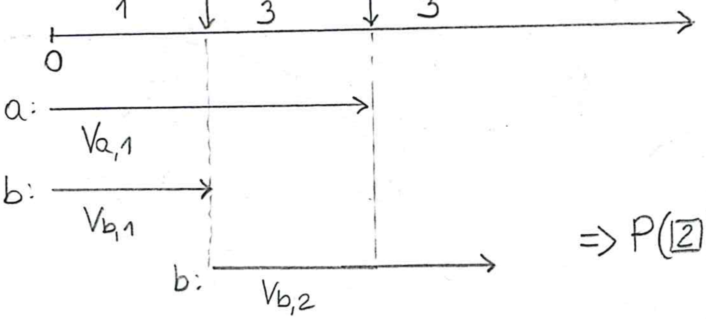


$\Rightarrow P(\boxed{1}) = p_0(1)\, P(V_{a,1} < V_{b,1} < V_{a,1} + V_{a,2})$

$\simeq 0.0606$

estimated with Matlab

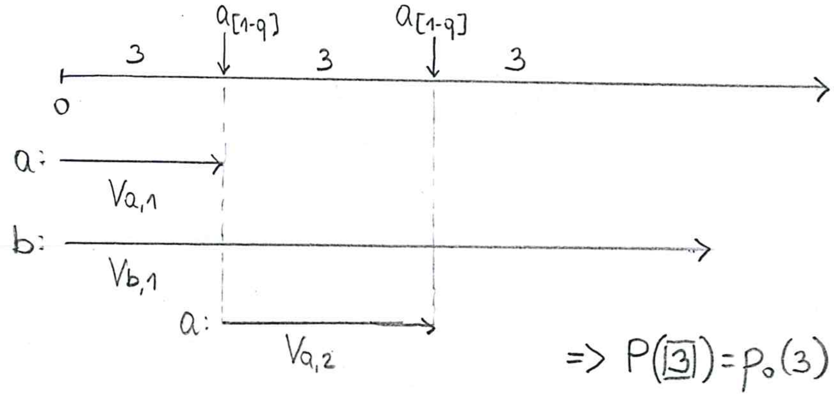
$\boxed{2}$ $\quad 1 \xrightarrow{b} 3 \xrightarrow{a_{[1-q]}} 3$



$\Rightarrow P(\boxed{2}) = p_0(1)(1-q)\, P(V_{b,1} < V_{a,1} < V_{b,1} + V_{b,2})$

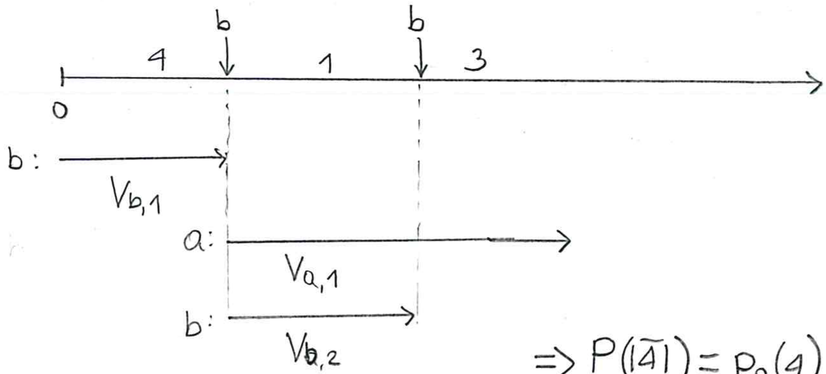
$\simeq 0.0673$

$\boxed{3}$ $\quad 3 \xrightarrow{a[1-q]} 3 \xrightarrow{a[1-q]} 3$



$$\Rightarrow P(\boxed{3}) = p_o(3)(1-q)^2 P(V_{a,1}+V_{a,2}<V_{b,1})$$
$$\simeq 0.0068$$

$\boxed{4}$ $\quad 4 \xrightarrow{b} 1 \xrightarrow{b} 3$



$$\Rightarrow P(\boxed{4}) = p_o(4) P(V_{b,2}<V_{a,1})$$
$$\simeq 0.2538$$

$$\Rightarrow P(X_2=3) = \sum_{i=1}^{4} P(\boxed{i}) \simeq 0.3885$$

2. Let $V(1)$ be the state holding time of state 1. It turns out that

$$V(1) = \min\{V_a, V_b\}$$

$$\Rightarrow P(V(1)\leq t) = 1-P(V(1)>t) = 1-P(\min\{V_a,V_b\}>t) = 1-P(V_a>t, V_b>t)$$
$$= 1-P(V_a>t)P(V_b>t) = 1-\left[1-P(V_a\leq t)\right]\left[1-P(V_b\leq t)\right]$$

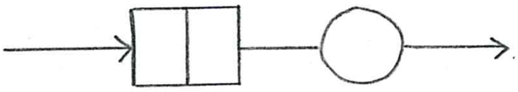
Recall that

$$P(V_a\leq t) = \begin{cases} 0 & \text{if } t<6 \\ \frac{t-6}{4} & \text{if } 6\leq t\leq 10 \\ 1 & \text{if } t>10 \end{cases} \qquad P(V_b\leq t) = 1-e^{-\frac{t}{5.5}}, \quad t\geq 0$$

Hence,

$$P(V(1)\leq t)=\begin{cases} 0 & \text{if } t<0 \\ 1-e^{-\frac{t}{5.5}} & \text{if } 0\leq t<6 \\ 1-\frac{(10-t)}{4}e^{-\frac{t}{5.5}} & \text{if } 6\leq t\leq 10 \\ 1 & \text{if } t>0 \end{cases}$$

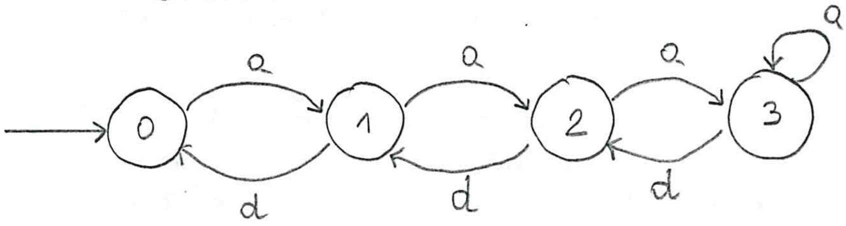
EXERCISE 2



state $X = \#$ customers in the system $\in \{0,1,2,3\}$

events $\mathcal{E} = \{a, d\}$

- a: arrival of a customer
- d: termination of a service

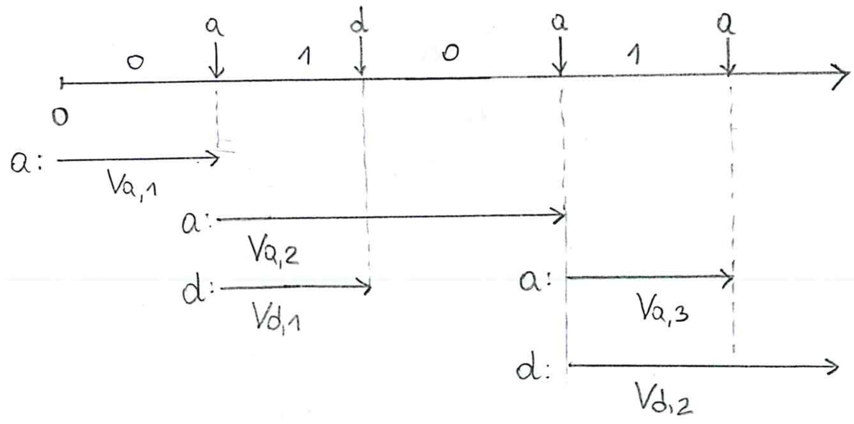


$V_a \sim U(5,20)$

$V_d = T = 12$

1. There are only two possible cases: with one d before the third a:

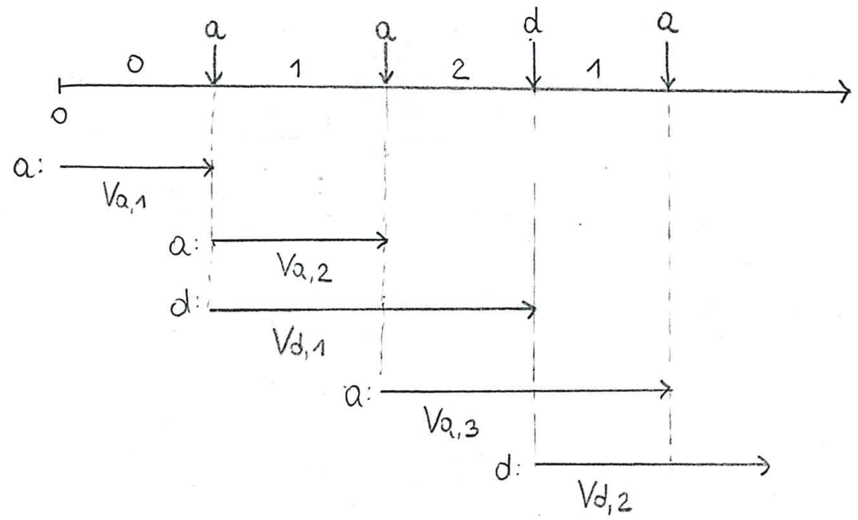
$\boxed{1}$ $\quad 0 \xrightarrow{a} 1 \xrightarrow{d} 0 \xrightarrow{a} 1 \xrightarrow{a}$



$$\Rightarrow P(\boxed{1}) = P(V_{d,1} < V_{a,2},\ V_{a,3} < V_{d,2}) \simeq 0.2489$$

estimated with Matlab

$\boxed{2}$ $\quad 0 \xrightarrow{a} 1 \xrightarrow{a} 2 \xrightarrow{d} 1 \xrightarrow{a}$



$$\Rightarrow P(\boxed{2}) = P(V_{a,2} < V_{d,1} < V_{a,2} + V_{a,3} < V_{d,1} + V_{d,2}) \simeq 0.3178$$

$$\Rightarrow P(\ldots) = P(\boxed{1}) + P(\boxed{2}) \simeq 0.5667$$

2. The waiting time of the second customer can be expressed as:

$$W_2 = \max\{0, V_{d,1} - V_{a,2}\}$$

$$\Rightarrow P(W_2 \leq t) = P(\max\{0, V_{d,1} - V_{a,2}\} \leq t) =$$

$$= P(0 \leq t,\ V_{d,1} - V_{a,2} < 0) + P(V_{d,1} - V_{a,2} \leq t,\ V_{d,1} - V_{a,2} \geq 0)$$

$$= P(V_{a,2} > 12) + P(12 - t \leq V_{a,2} \leq 12)$$

for $t \geq 0$

$$\frac{20-12}{20-5} = \frac{8}{15}$$

$$\begin{cases} P(12-t \leq V_{a,2} \leq 12) = \dfrac{12-(12-t)}{20-5} = \dfrac{t}{15} & \text{if } t < 7 \\ P(5 \leq V_{a,2} \leq 12) = \dfrac{12-5}{20-5} = \dfrac{7}{15} & \text{if } t \geq 7 \end{cases}$$

$$\Rightarrow P(W_2 \leq t) = \begin{cases} 0 & \text{if } t < 0 \\ \dfrac{t+8}{15} & \text{if } 0 \leq t < 7 \\ 1 & \text{if } t \geq 7 \end{cases}$$