Test of Discrete Event Systems - 29.10.2018

Exercise 1

A zero-order holder transforms an asynchronous sequence of bits 0 and 1 in a continuous-time binary signal $y(t)$. The zero-order holder implements the logical XOR of the last two bits received. At initialization, assume to have received an indefinite sequence of bits 0 terminated by a bit 1. Moreover, assume that bits 0 and 1 arrive with independent interarrival times following uniform distributions over the intervals [2.5,5] ms and [3,6] ms, respectively.

1. Compute the probability that the output $y(t)$ is 0 after the arrival of the second bit.
2. Compute the probability that the output $y(t)$ is 1 at time $t = 4.5$ ms.

Model

$(\mathcal{E}, \mathcal{X}, \Gamma, f, x_0, F, \mathcal{Y}, g)$

state $x = x_2 x_1$

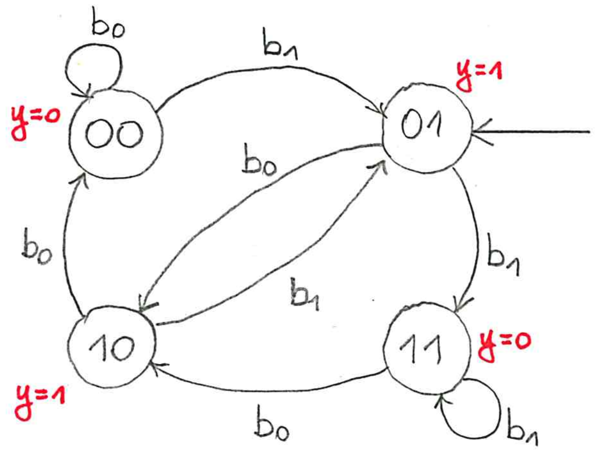
x_2: 2nd last bit

x_1: last bit

events $\mathcal{E} = \{b_0, b_1\}$

b_0: arrival of bit $\emptyset$

b_1: arrival of bit 1



$V_{b_0} \sim U(2.5, 5)$

generic lifetime of event b_0

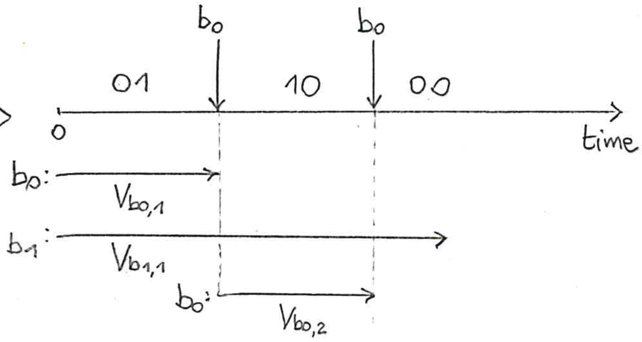
$V_{b_1} \sim U(3, 6)$

generic lifetime of event b_1

1. Two possible paths:

$\boxed{1}$ $\quad 01 \xrightarrow{b_0} 10 \xrightarrow{b_0} 00$

$\Rightarrow$



$\Rightarrow P(\boxed{1}) = P(V_{b_0,1} + V_{b_0,2} \leq V_{b_1,1})$

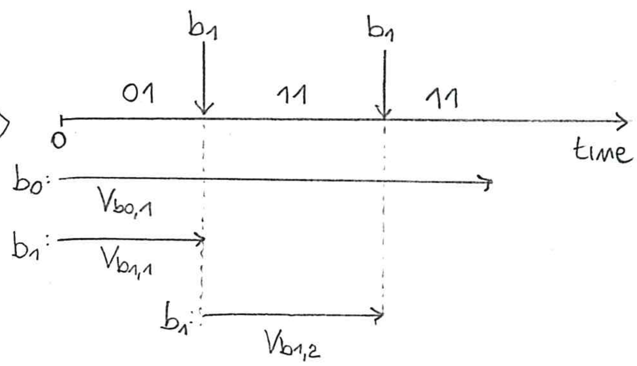
$\simeq 0.0089$

Notice that, being the lifetimes nonnegative numbers, this condition implies also

$V_{b_0,1} \leq V_{b_1,1}$

$\boxed{2}$ $\quad 01 \xrightarrow{b_1} 11 \xrightarrow{b_1} 11$

$\Rightarrow$



$\Rightarrow P(\boxed{2}) = P(V_{b_1,1} + V_{b_1,2} \leq V_{b_0,1})$

$= 0$

why? ;-)

$\Rightarrow P(\ldots) = P(\boxed{1}) + P(\boxed{2}) \simeq 0.0089$

2. First of all, we notice that, given the stochastic clock structure F, at most one occurrence of event b_0 and one occurrence of event b_1 can be observed before time $t=4.5$. This restricts the attention to the following cases:

$\boxed{1}$ • no event before $t=4.5$ $\Rightarrow$ ok, the state at $t=4.5$ is 01, and the output is 1.

$\boxed{2}$ • one occurrence of b_0 and no occurrence of b_1 before $t=4.5$

$\Rightarrow$ ok, the state at $t=4.5$ is 10, and the output is 1.

• one occurrence of b_1 and no occurrence of b_0 before $t=4.5$

$\Rightarrow$ no, the state at $t=4.5$ is 11, and the output is $\emptyset$.

$\boxed{3}$ • both b_0 and b_1 occur before $t=4.5$, b_0 occurs first

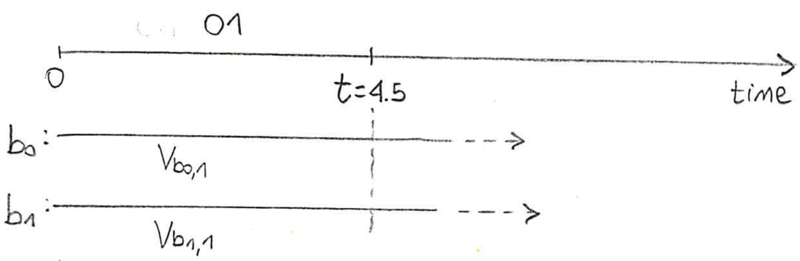
$\Rightarrow$ ok, the state at $t=4.5$ is 01, and the output is 1.

$\boxed{4}$ • both b_0 and b_1 occur before $t=4.5$, b_1 occurs first

$\Rightarrow$ ok, the state at $t=4.5$ is 10, and the output is 1.

We have thus identified four possible cases:

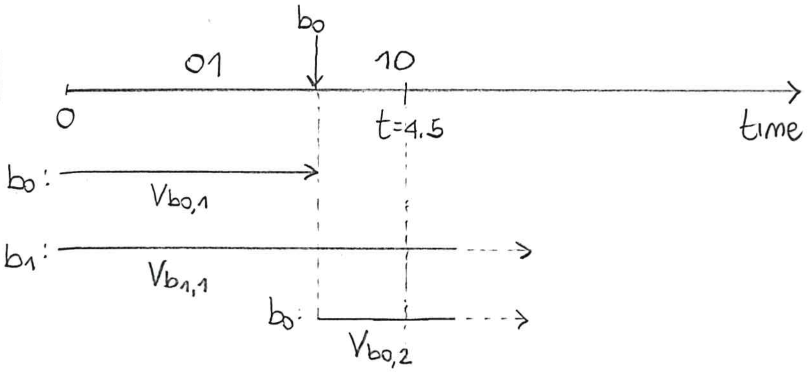
$\boxed{1}$



$$\Rightarrow P(\boxed{1}) = P(V_{b_0,1} > t, V_{b_1,1} > t) \underset{\text{independent}}{=} P(V_{b_0,1} > t)\, P(V_{b_1,1} > t)$$

$$= \frac{5-t}{5-2.5} \cdot \frac{6-t}{6-3} = 0.1$$

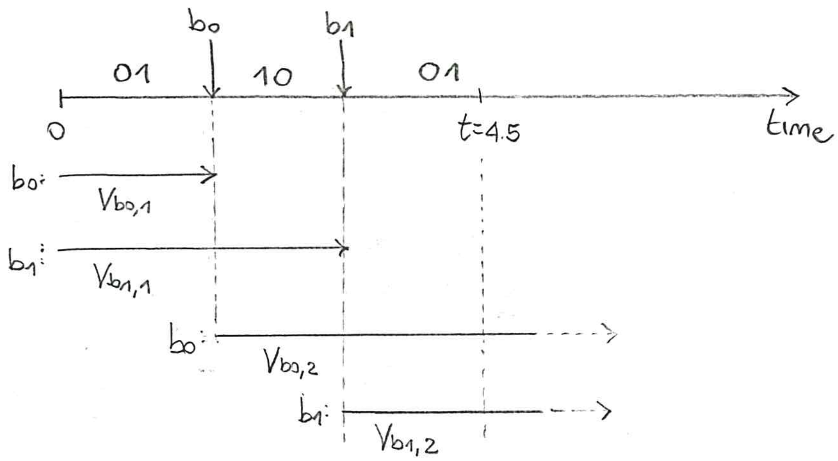
$\boxed{2}$



$$\Rightarrow P(\boxed{2}) = P(\underbrace{V_{b_0,1} \leq t,\ V_{b_1,1} > t}_{\text{implies } V_{b_0,1} < V_{b_1,1}},\ \underbrace{V_{b_0,1} + V_{b_0,2} > t}_{\text{always true}}) \underset{\text{independent}}{=} P(V_{b_0,1} \leq t)P(V_{b_1,1} > t)$$

$$= \frac{t-2.5}{5-2.5} \cdot \frac{6-t}{6-3} = 0.4$$

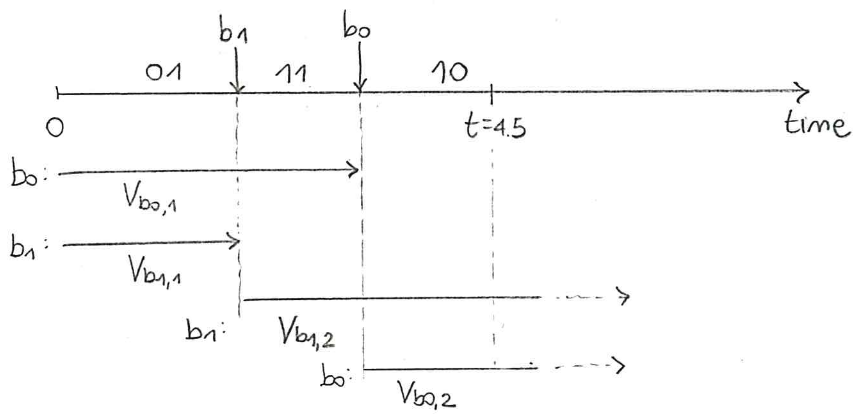
$\boxed{3}$



$$\Rightarrow P(\boxed{3}) = P(\underbrace{V_{b_0,1} \leq V_{b_1,1},\ V_{b_1,1} \leq t}_{\text{implies } V_{b_0,1} \leq t},\ \underbrace{V_{b_0,1} + V_{b_0,2} > t,\ V_{b_1,1} + V_{b_1,2} > t}_{\text{always true}})$$

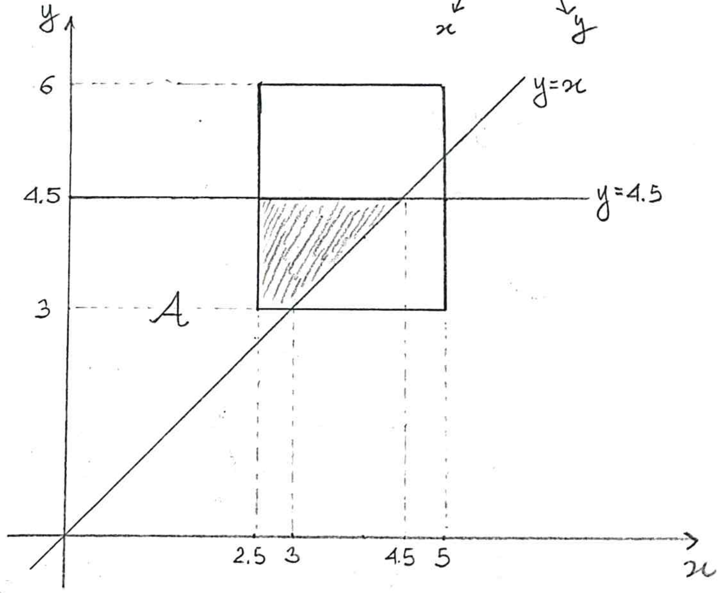
$$= P(V_{b_0,1} \leq V_{b_1,1},\ V_{b_1,1} \leq t) = 0.25$$ [estimated using Matlab or computed analytically(*)]

$\boxed{4}$



$$\Rightarrow P(\boxed{4}) = \underbrace{\ldots}_{\text{similar to case } \boxed{3}} = P(V_{b_1,1} < V_{b_0,1},\ V_{b_0,1} \leq t) = 0.15$$ [estimated using Matlab or computed analytically]

(*) $P(V_{b_0,1} \leq V_{b_1,1}\ ,\ V_{b_1,1} \leq t) = P((\underbrace{V_{b_0,1}}_{x}, \underbrace{V_{b_1,1}}_{y}) \in A)$



$$= \frac{\text{AREA(trapezoid)}}{\text{AREA(rectangle)}} = \frac{\frac{[(4.5-2.5)+(3-2.5)]\cdot(4.5-3)}{2}}{(5-2.5)(6-3)} = 0.25$$

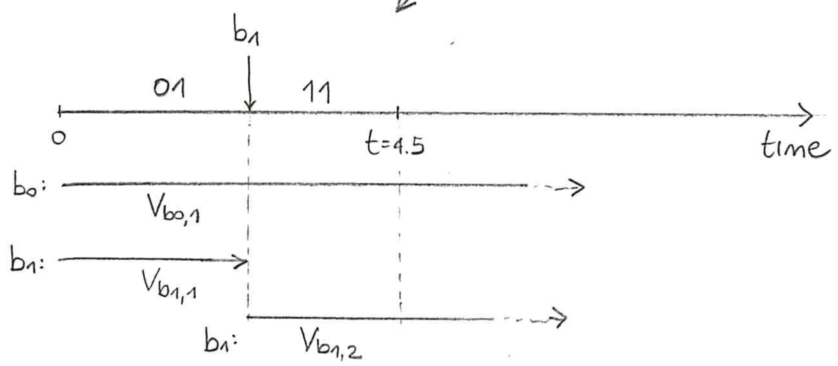
↑ why? ;-)

⇒ Finally:

$$P(\ldots) = P(\boxed{1}) + P(\boxed{2}) + P(\boxed{3}) + P(\boxed{4}) = 0.1 + 0.4 + 0.25 + 0.15 = 0.9$$

Notice that, in this case (see the discussion at the beginning of this point):

$$P(\ldots) = 1 - P(\text{one occurrence of } b_1 \text{ and no occurrence of } b_0 \text{ before } t = 4.5)$$



$$\Rightarrow P(V_{b_1,1} \leq t, V_{b_0,1} > t, \underbrace{V_{b_1,1} + V_{b_1,2} > t}_{\text{always true}}) = P(V_{b_1,1} \leq t, V_{b_0,1} > t) \underset{\text{independent}}{=} P(V_{b_1,1} \leq t)\, P(V_{b_0,1} > t)$$

$$= \frac{t-3}{6-3} \cdot \frac{5-t}{5-2.5} = 0.1$$