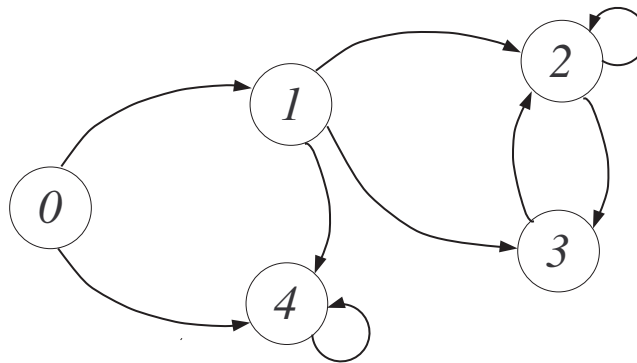


Test of Discrete Event Systems - 18.01.2017

NOTE: Other exercises with solutions are available online at the URL http://www3.diism.unisi.it/~paoletti/teaching/sed/1617/#tth_sEc5.2

Exercise 1

Consider the discrete-time homogeneous Markov chain whose state transition diagram is represented in the figure, and with transition probabilities $p_{0,1} = 1/3$, $p_{1,2} = 1/8$, $p_{1,3} = 1/4$ and $p_{2,3} = 4/5$.



1. Compute the average recurrence time for each recurrent state.
2. Compute the stationary probabilities of all states, assuming the probability vector of the initial state $\pi(0) = [1 \ 0 \ 0 \ 0 \ 0]$.

Exercise 2

A simplified telephone call process in discrete time over a single telephone line works as follows.

- At most one telephone call can go through the line in a single time slot.
- The probability of an incoming call during a time slot is α . Assume that an incoming call may arrive only at the end of a time slot. If the line is busy, the call is lost; otherwise, the call is processed.
- The probability that a call in process terminates in any one time slot is β .

Assume that $\alpha = 1/3$ and $\beta = 1/2$.

1. Model the telephone call process through a discrete-time homogeneous Markov chain.
2. Compute the utilization of the telephone line at steady state.
3. Assume that the telephone line is idle. Compute the probability that the telephone line is busy at least 30% of the time over the next 10 time slots.

Exercise 1

1

1. The recurrent states are 2, 3 and 4. States 0 and 1 are transient.

In order to compute $E[T_{2,2}]$ and $E[T_{3,3}]$, we observe that states 2 and 3 form a closed subset which is irreducible, aperiodic and finite, with transition probability matrix:

$$\tilde{P} = \begin{bmatrix} p_{2,2} & p_{2,3} \\ p_{3,2} & p_{3,3} \end{bmatrix} = \begin{bmatrix} \frac{1}{5} & \frac{4}{5} \\ 1 & 0 \end{bmatrix}$$

Solving:

$$\begin{cases} \tilde{\pi} = \tilde{\pi} \tilde{P} \\ \tilde{\pi}_2 + \tilde{\pi}_3 = 1 \end{cases}$$

we obtain $\tilde{\pi} = \left[\frac{5}{9} \quad \frac{4}{9} \right]$. Therefore:

$$E[T_{2,2}] = \frac{1}{\left(\frac{5}{9}\right)} = \frac{9}{5} = 1.80$$

$$E[T_{3,3}] = \frac{1}{\left(\frac{4}{9}\right)} = \frac{9}{4} = 2.25$$

State 4 is absorbing, therefore:

$$E[T_{4,4}] = 1.$$

2. The transition probability matrix of the Markov chain is

$$P = \begin{bmatrix} 0 & \frac{1}{3} & 0 & 0 & \frac{2}{3} \\ 0 & 0 & \frac{1}{8} & \frac{1}{4} & \frac{5}{8} \\ 0 & 0 & \frac{1}{5} & \frac{4}{5} & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

We are asked to compute

$$\pi = \lim_{t \rightarrow \infty} \pi(0) P^t, \quad \text{with } \pi(0) = [1 \ 0 \ 0 \ 0 \ 0].$$

Notice that the Markov chain is non-irreducible!

2

A possible way to circumvent the computation of the limit, is as follows:

$$\begin{cases} \pi_0 = 0 \\ \pi_1 = 0 \end{cases} \left\{ \begin{array}{l} \text{states 0 and 1 are transient} \end{array} \right.$$

$$\begin{aligned} \pi_2 &= P(\text{the chain enters the closed subset } \{2,3\}) \cdot \tilde{\pi}_2 = \frac{1}{8} \cdot \frac{5}{9} = \frac{5}{72} \\ &\quad \underbrace{P(0 \rightarrow 1) [P(1 \rightarrow 2) + P(1 \rightarrow 3)]}_{\text{computed in item 1: } \tilde{\pi}_2 = \frac{5}{9}} \\ &= \frac{1}{3} \left(\frac{1}{8} + \frac{1}{4} \right) = \frac{1}{8} \end{aligned}$$

$$\begin{aligned} \pi_3 &= P(\text{the chain enters the closed subset } \{2,3\}) \cdot \tilde{\pi}_3 = \frac{1}{8} \cdot \frac{4}{9} = \frac{1}{18} \\ &\quad \downarrow \text{computed in item 1: } \tilde{\pi}_3 = \frac{4}{9} \end{aligned}$$

$$\begin{aligned} \pi_4 &= P(\text{the chain enters the closed subset } \{4\}) = \\ &= P(0 \rightarrow 1)P(1 \rightarrow 4) + P(0 \rightarrow 4) = \frac{1}{3} \cdot \frac{5}{8} + \frac{2}{3} = \frac{7}{8} \end{aligned}$$

Therefore,

$$\pi = \left[0 \quad 0 \quad \frac{5}{72} \quad \frac{1}{18} \quad \frac{7}{8} \right].$$

Remark: Since the Markov chain is non-irreducible, it was not possible to compute π

by solving $\begin{cases} \pi = \pi P \\ \sum_{i=0}^4 \pi_i = 1 \end{cases}$. Indeed,

$$\begin{cases} 0 = \pi_0 \\ \frac{1}{3}\pi_0 = \pi_1 \\ \frac{1}{8}\pi_1 + \frac{1}{5}\pi_2 + \pi_3 = \pi_2 \\ \frac{1}{4}\pi_1 + \frac{4}{5}\pi_2 = \pi_3 \\ \frac{2}{3}\pi_0 + \frac{5}{8}\pi_1 + \pi_4 = \pi_4 \\ \pi_0 + \pi_1 + \pi_2 + \pi_3 + \pi_4 = 1 \end{cases} \Rightarrow \begin{cases} \pi_0 = 0 \\ \pi_1 = 0 \\ \frac{4}{5}\pi_2 = \pi_3 \\ \frac{4}{5}\pi_2 = \pi_3 \text{ redundant} \\ \pi_4 = \pi_4 \Rightarrow \pi_4 \text{ can be chosen arbitrarily: } \pi_4 = \gamma, \gamma \in [0,1] \\ \pi_2 + \pi_3 + \pi_4 = 1 \end{cases}$$

$$\Rightarrow \pi_2 + \frac{4}{5}\pi_2 + \gamma = 1 \Rightarrow \pi_2 = \frac{5}{9}(1-\gamma), \quad \pi_3 = \frac{4}{9}(1-\gamma)$$

③

It follows that the system of equations:

$$\begin{cases} \pi = \pi P \\ \sum_{i=0}^4 \pi_i = 1 \end{cases}$$

has infinite solutions parameterized by $\gamma \in [0, 1]$.

Notice that γ can be interpreted as the probability that the chain enters the closed subset $\{4\}$, and therefore $1-\gamma$ is the probability that the chain enters the closed subset $\{2, 3\}$. Some examples:

- if the initial state is 0, then $\gamma = \frac{7}{8}$, and therefore $1-\gamma = \frac{1}{8}$;
- if the initial state is either 2 or 3, then $\gamma = 0$, and therefore $1-\gamma = 1$,
ecc.

Exercise 2

4

1. state

$$x = \begin{cases} 0: & \text{line idle} \\ 1: & \text{line busy} \end{cases}$$

transition probabilities

$$\begin{aligned} p_{0,0} &= P(X(t+1)=0 | X(t)=0) \\ &= P(\text{no incoming call} \mid \text{line idle}) = 1-\alpha \end{aligned}$$

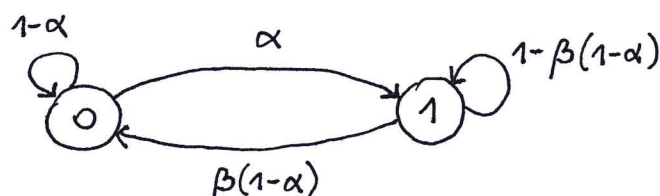
$$\begin{aligned} p_{0,1} &= P(X(t+1)=1 | X(t)=0) \\ &= P(\text{one incoming call} \mid \text{line idle}) = \alpha \end{aligned}$$

$$\begin{aligned} p_{1,0} &= P(X(t+1)=0 | X(t)=1) \\ &= P(\text{termination of a call} \underline{\text{AND}} \text{ no incoming call} \mid \text{line busy}) = \beta \cdot (1-\alpha) \end{aligned}$$

$$\begin{aligned} p_{1,1} &= P(X(t+1)=1 | X(t)=1) \\ &= P((\text{no termination of a call}) \underline{\text{OR}} (\text{termination of a call} \underline{\text{AND}} \\ &\quad \text{one incoming call}) \mid \text{line busy}) \\ &= (1-\beta) + \beta \cdot \alpha = 1-\beta(1-\alpha) \end{aligned}$$

$$\Rightarrow P = \begin{bmatrix} p_{0,0} & p_{0,1} \\ p_{1,0} & p_{1,1} \end{bmatrix} = \begin{bmatrix} 1-\alpha & \alpha \\ \beta(1-\alpha) & 1-\beta(1-\alpha) \end{bmatrix} = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

The initial state probability vector is unspecified.



2. The utilization of the telephone line at steady state is the stationary probability of state 1.

The Markov chain is irreducible, aperiodic and finite.

$$\Rightarrow \begin{cases} \pi = \pi P \\ \pi_0 + \pi_1 = 1 \end{cases} \quad \text{where } \pi = [\pi_0 \ \pi_1]$$

$$\begin{cases} \pi_0 = \frac{2}{3}\pi_0 + \frac{1}{3}\pi_1 \\ \pi_1 = \frac{1}{3}\pi_0 + \frac{2}{3}\pi_1 \text{ redundant} \\ \pi_0 + \pi_1 = 1 \end{cases} \quad \begin{cases} \pi_1 = \pi_0 \\ 2\pi_0 = 1 \end{cases} \Rightarrow \begin{cases} \pi_0 = \frac{1}{2} \\ \pi_1 = \frac{1}{2} \end{cases}$$

$$\Rightarrow U_{\text{line}} = \pi_1 = 0.5 = 50\%$$

↑
utilization
of the line

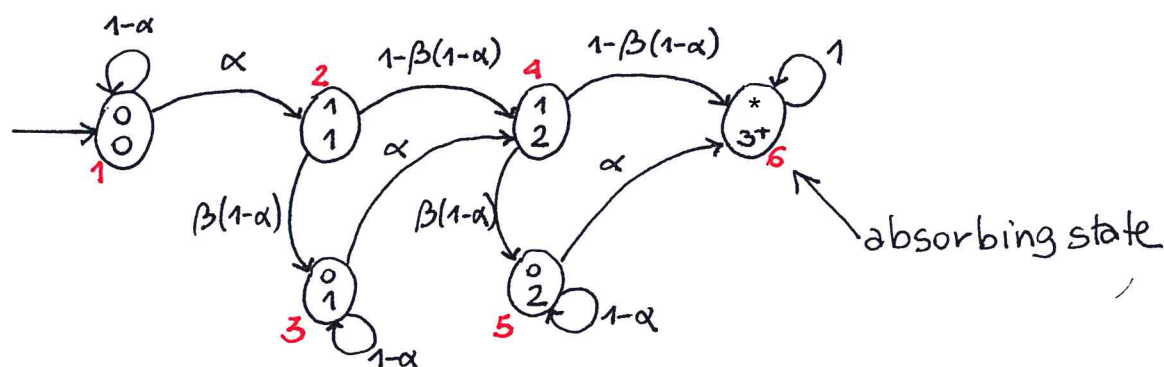
3. The telephone line is initially idle.

busy at least 30% of the time over the next 10 time slots

\equiv busy at least 3 time slots over the next 10 time slots

We modify the previous model as follows:

state $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \rightarrow \begin{matrix} \text{line: } 0 \text{ (idle), } 1 \text{ (busy)} \\ \text{counter of time slots spent in state 'busy' } \in \{0, 1, 2, 3^+\} \end{matrix}$
↓
3 or more



$$\tilde{\pi}(0) = [1 \ 0 \ 0 \ 0 \ 0 \ 0]$$

6

$$\tilde{P} = \begin{bmatrix} 1-\alpha & \alpha & 0 & 0 & 0 & 0 \\ 0 & 0 & \beta(1-\alpha) & 1-\beta(1-\alpha) & 0 & 0 \\ 0 & 0 & 1-\alpha & \alpha & 0 & 0 \\ 0 & 0 & 0 & 0 & \beta(1-\alpha) & 1-\beta(1-\alpha) \\ 0 & 0 & 0 & 0 & 1-\alpha & \alpha \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Using this model, the answer to the question is:

$$P(\tilde{X}_{(10)} = 6) = \tilde{\pi}(0) \tilde{P}^{10} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \simeq 0.8439$$

because
state 6 is
absorbing.