Test of Discrete Event Systems - 03.12.2014

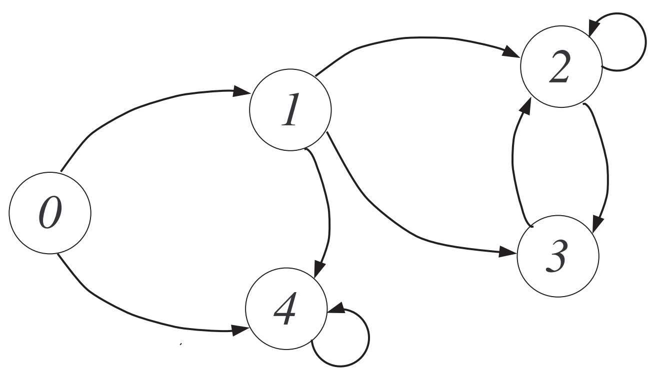
Exercise 1

A study of the strengths of the football teams of the main American universities shows that, if a university has a strong team one year, it is equally likely to have a strong team or an average team next year; if it has an average team, it is average next year with 50% probability, and if it changes, it is just as likely to become strong as weak; if it is weak, it has 65% probability of remaining so, otherwise it becomes average.

1. What is the probability that a team is strong on the long run?
2. Assume that a team is weak. How many years are needed on average for it to become strong?
3. Assume that a team is weak. Compute the probability that, in the next ten years, it is strong at least two years.

Exercise 2

Consider the discrete-time homogeneous Markov chain whose state transition diagram is represented in the figure, and with transition probabilities $p_{0,1} = 1/3$, $p_{1,2} = 1/8$, $p_{1,3} = 1/4$ and $p_{2,3} = 4/5$.

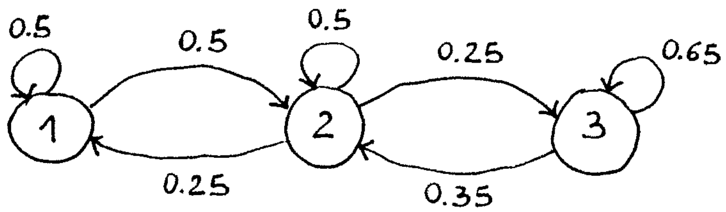


1. Compute the average recurrence time for each recurrent state.
2. Compute the stationary probabilities of all states, assuming the probability vector of the initial state $\pi(0) = [\ 1\ 0\ 0\ 0\ 0\ ]$.

EXERCISE #1

1. **model #1**

$$\text{state } x = \begin{cases} 1 : \text{strong} \\ 2 : \text{average} \\ 3 : \text{weak} \end{cases}$$



$$P = \begin{bmatrix} 0.5 & 0.5 & 0 \\ 0.25 & 0.5 & 0.25 \\ 0 & 0.35 & 0.65 \end{bmatrix}$$

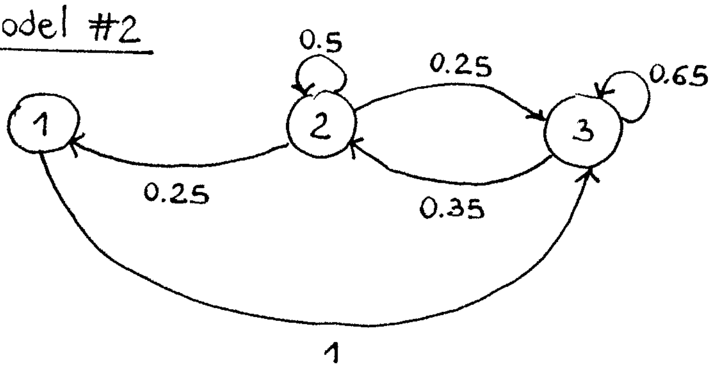
The Markov chain is irreducible, aperiodic and finite. This implies that stationary state probabilities can be computed by solving the set of linear equations:

$$\begin{cases} \pi = \pi P \\ \pi_1 + \pi_2 + \pi_3 = 1 \end{cases} \Rightarrow \pi \simeq [\underset{\pi_1}{0.2258} \quad \underset{\pi_2}{0.4516} \quad \underset{\pi_3}{0.3226}]$$

The answer to question #1 is $\pi_1 \simeq 0.2258$.

2. To answer question #2, we modify model #1 as follows:

model #2



$$\tilde{P} = \begin{bmatrix} 0 & 0 & 1 \\ 0.25 & 0.5 & 0.25 \\ 0 & 0.35 & 0.65 \end{bmatrix}$$

Using model #2, we can write

$$T_{1,1} = 1 + T_{3,1}$$

$T_{3,1}$ → time to reach state 1 from state 3

$T_{1,1}$ → recurrence time of state 1

Taking expectations of both sides, we have:

$$\underbrace{E[T_{3,1}]}_{\text{answer to question \#2}} = \underbrace{E[T_{1,1}]} - 1$$

Since the Markov chain is irreducible, aperiodic and finite, $E[T_{1,1}] = \dfrac{1}{\tilde{\pi}_1}$, where $\tilde{\pi}_1$ is the stationary probability of state 1.

$$\begin{cases} \tilde{\pi} = \tilde{\pi}\tilde{P} \\ \tilde{\pi}_1 + \tilde{\pi}_2 + \tilde{\pi}_3 = 1 \end{cases} \Rightarrow \tilde{\pi} \simeq \begin{array}{c} \begin{bmatrix} 0.0933 & 0.3733 & 0.5333 \end{bmatrix} \\ \quad \tilde{\pi}_1 \qquad\quad \tilde{\pi}_2 \qquad\quad \tilde{\pi}_3 \end{array}$$

$$\Rightarrow E[T_{3,1}] = \frac{1}{\tilde{\pi}_1} - 1 \simeq 9.7143$$

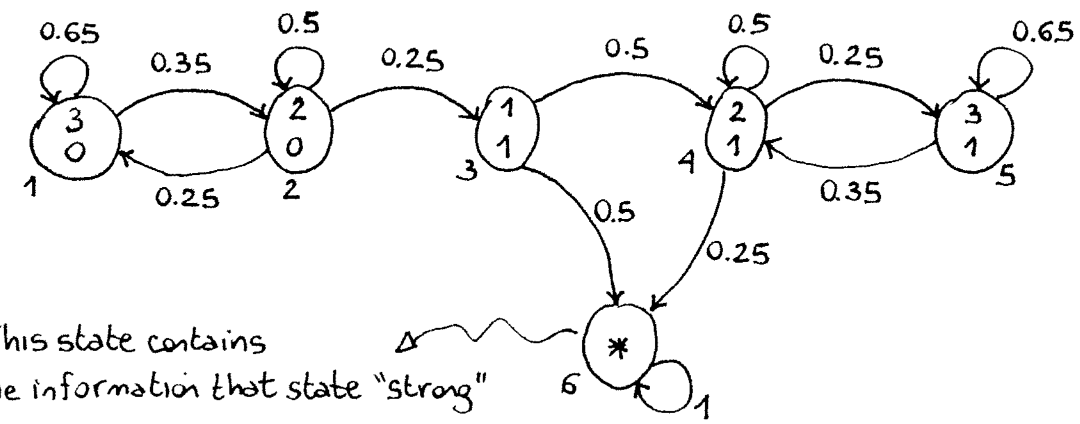
3. <u>model #3</u>

Now the state must take into account the number of steps spent in state "strong"

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

where:

- $x_1 = \begin{cases} 1: \text{strong} \\ 2: \text{average} \\ 3: \text{weak} \end{cases}$ - current strength of the football team
- $x_2 \in \{0, 1, 2, ..\}$ - number of steps spent in state "strong"



This state contains the information that state "strong" has been visited at least two time steps.

Enumerate the states of model #3 from 1 to 6 as reported in the figure. Then:

$$\hat{P} = \begin{bmatrix} 0.65 & 0.35 & 0 & 0 & 0 & 0 \\ 0.25 & 0.5 & 0.25 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.5 & 0 & 0.5 \\ 0 & 0 & 0 & 0.5 & 0.25 & 0.25 \\ 0 & 0 & 0 & 0.35 & 0.65 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

The answer to question #3 is:

$$P(\hat{X}(10)=6) = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{\hat{\pi}(0)} \hat{P}^{10} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \simeq 0.4593$$

we start from the state where the team is weak and was never strong in the past.

Note that $\hat{\pi}(10) = \hat{\pi}(0)\hat{P}^{10}$ is a row vector and we need the last component $\hat{\pi}_6(10)$. This is obtained by multiplying $\hat{\pi}(10)$ on the right by a suitable vector.

Exercise 2

1. The recurrent states are 2, 3 and 4. States 0 and 1 are transient.

In order to compute $E[T_{2,2}]$ and $E[T_{3,3}]$, we observe that states 2 and 3 form a closed subset which is irreducible, aperiodic and finite, with transition probability matrix:

$$\tilde{P} = \begin{bmatrix} p_{2,2} & p_{2,3} \\ p_{3,2} & p_{3,3} \end{bmatrix} = \begin{bmatrix} \frac{1}{5} & \frac{4}{5} \\ 1 & 0 \end{bmatrix}$$

Solving:

$$\begin{cases} \tilde{\pi} = \tilde{\pi}\tilde{P} \\ \tilde{\pi}_2 + \tilde{\pi}_3 = 1 \end{cases}$$

we obtain $\tilde{\pi} = \begin{bmatrix} \frac{5}{9} & \frac{4}{9} \end{bmatrix}$. Therefore:

$$E[T_{2,2}] = \frac{1}{\left(\frac{5}{9}\right)} = \frac{9}{5} = 1.80$$

$$E[T_{3,3}] = \frac{1}{\left(\frac{4}{9}\right)} = \frac{9}{4} = 2.25$$

State 4 is absorbing, therefore:

$E[T_{4,4}] = 1.$

2. The transition probability matrix of the Markov chain is

$$P = \begin{bmatrix} 0 & \frac{1}{3} & 0 & 0 & \frac{2}{3} \\ 0 & 0 & \frac{1}{8} & \frac{1}{4} & \frac{5}{8} \\ 0 & 0 & \frac{1}{5} & \frac{4}{5} & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

We are asked to compute

$$\pi = \lim_{t\to\infty} \pi(0)P^t, \quad \text{with } \pi(0) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Notice that the Markov chain is <u>non-irreducible</u>!

A possible way to circumvent the computation of the limit, is as follows:

$$\left.\begin{array}{l}\pi_0 = 0\\ \pi_1 = 0\end{array}\right\}\text{ states 0 and 1 are transient}$$

$$\pi_2 = \underbrace{P(\text{the chain enters the closed subset } \{2,3\})}_{\substack{P(0\to 1)\left[P(1\to 2)+P(1\to 3)\right] \\ = \frac{1}{3}\left(\frac{1}{8}+\frac{1}{4}\right) = \frac{1}{8}}} \cdot \underbrace{\tilde{\pi}_2}_{\substack{\text{computed in item 1:}\\ \tilde{\pi}_2 = \frac{5}{9}}} = \frac{1}{8}\cdot\frac{5}{9} = \frac{5}{72}$$

$$\pi_3 = P(\text{the chain enters the closed subset } \{2,3\}) \cdot \underbrace{\tilde{\pi}_3}_{\substack{\text{computed in item 1:}\\ \tilde{\pi}_3 = \frac{4}{9}}} = \frac{1}{8}\cdot\frac{4}{9} = \frac{1}{18}$$

$$\pi_4 = P(\text{the chain enters the closed subset } \{4\}) =$$

$$= P(0\to 1)P(1\to 4) + P(0\to 4) = \frac{1}{3}\cdot\frac{5}{8} + \frac{2}{3} = \frac{7}{8}.$$

Therefore,

$$\pi = \begin{bmatrix} 0 & 0 & \frac{5}{72} & \frac{1}{18} & \frac{7}{8} \end{bmatrix}.$$

Remark: Since the Markov chain is non-irreducible, it was not possible to compute π by solving $\begin{cases} \pi = \pi P \\ \sum_{i=0}^{4} \pi_i = 1 \end{cases}$. Indeed,

$$\begin{cases} 0 = \pi_0 \\ \frac{1}{3}\pi_0 = \pi_1 \\ \frac{1}{8}\pi_1 + \frac{1}{5}\pi_2 + \pi_3 = \pi_2 \\ \frac{1}{4}\pi_1 + \frac{4}{5}\pi_2 = \pi_3 \\ \frac{2}{3}\pi_0 + \frac{5}{8}\pi_1 + \pi_4 = \pi_4 \\ \pi_0 + \pi_1 + \pi_2 + \pi_3 + \pi_4 = 1 \end{cases} \Rightarrow \begin{cases} \pi_0 = 0 \\ \pi_1 = 0 \\ \frac{4}{5}\pi_2 = \pi_3 \\ \cancel{\frac{4}{5}\pi_2 = \pi_3} \quad \text{redundant} \\ \pi_4 = \pi_4 \quad \Rightarrow \pi_4 \text{ can be chosen arbitrarily: } \pi_4 = \gamma,\ \gamma \in [0,1] \\ \pi_2 + \pi_3 + \pi_4 = 1 \end{cases}$$

$$\Rightarrow \pi_2 + \frac{4}{5}\pi_2 + \gamma = 1 \Rightarrow \pi_2 = \frac{5}{9}(1-\gamma), \quad \pi_3 = \frac{4}{9}(1-\gamma)$$

It follows that the system of equations:

$$\begin{cases} \pi = \pi P \\ \sum_{i=0}^{4} \pi_i = 1 \end{cases}$$

has infinite solutions parameterized by $\gamma \in [0,1]$.

Notice that γ can be interpreted as the probability that the chain enters the closed subset $\{4\}$, and therefore $1-\gamma$ is the probability that the chain enters the closed subset $\{2,3\}$. Some examples:

- if the initial state is 0, then $\gamma = \frac{7}{8}$, and therefore $1-\gamma = \frac{1}{8}$;
- if the initial state is either 2 or 3, then $\gamma = 0$, and therefore $1-\gamma = 1$,

ecc.