Test of Discrete Event Systems - 12.11.2014

Exercise 1

In a post office there are two desks. Desk 1 serves with highest priority customers needing to perform payments (type A), whereas desk 2 serves with highest priority customers requiring mail services (type B). However, if desk 1 is busy and there are no type B waiting customers, desk 2 can serve type A customers; vice versa, if desk 2 is busy and there are no type A waiting customers, desk 1 can serve type B customers. Both interarrival and service times follow exponential distributions. For the arrivals of type A customers, the average rate is 14 customers/hour, whereas for the arrivals of type B customers, the average rate is 8 customers/hour. Average service times do not depend on the desk, and are equal to 3 minutes for type A customers and 5 minutes for type B customers.

1. Model the post office through a stochastic timed automaton $(\mathcal{E}, \mathcal{X}, \Gamma, f, x_0, F)$, assuming that the system is initially empty.

2. Compute the average holding time in a state where there are two type A customers in the system, and none of type B.

3. Assume that there are two type A customers in the system, and none of type B. Compute the probability that the next arriving customer is of type B and finds desk 2 available.

4. Assume that there are one type A customer and one type B customer in the system. Compute the probability that there are exactly 6 arrivals of customers in 4 minutes, and no one of them is admitted to the service in the same time interval.

Exercise 2

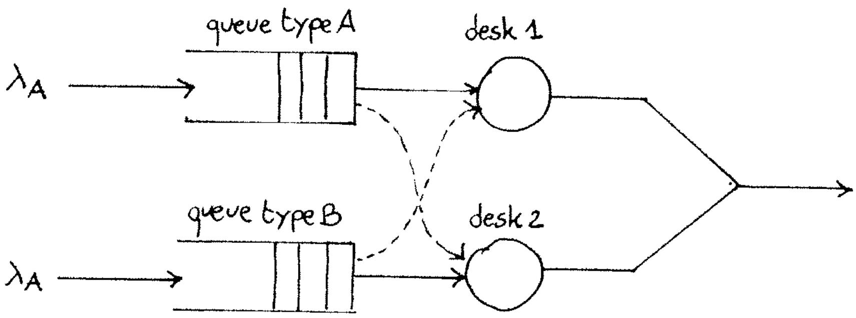
The call-center of an insurance company is equipped with 10 phone lines, but only three operators. This means that calling customers exceeding the number of three, are put on hold. Phone calls arrive at the call-center as generated by a Poisson process with average rate equal to 2 calls/minute. Time needed to answer a query is independent of the operator, and follows an exponential distribution with average 5 minutes. The call-center is open 24 hours a day.

1. Model the call-center through a stochastic timed automaton $(\mathcal{E}, \mathcal{X}, \Gamma, f, x_0, F)$, assuming that the three operators are initially idle.

2. Assume that the three operators are idle. Compute the probability that the fourth calling customer is put on hold and, in such a case, the average waiting time to be served.

Exercise 1

1. The system can be represented as follows:



Definition of state:

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \begin{array}{l} \to \text{desk 1: idle (0), serving A (A), serving B (B)} \\ \to \text{desk 2: idle (0), serving A (A), serving B (B)} \\ \to \text{length of queue type A} \in \{0,1,2,\dots\} \\ \to \text{length of queue type B} \in \{0,1,2,\dots\} \end{array}$$

Events: $\mathcal{E} = \{a_A, a_B, d_{A1}, d_{A2}, d_{B1}, d_{B2}\}$

a_A: arrival type A

a_B: arrival type B

d_{A1}: termination of a type A service in desk 1

...

Since in this case the cardinality of the state space is infinite, it is more convenient to define the state automaton in a formal way.

- Function Γ:

$$\Gamma\left(\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right) = \{a_A, a_B\}$$

$$\Gamma\left(\begin{bmatrix} A \\ 0 \\ 0 \\ 0 \end{bmatrix}\right) = \{a_A, a_B, d_{A1}\}, \quad \Gamma\left(\begin{bmatrix} B \\ 0 \\ 0 \\ 0 \end{bmatrix}\right) = \{a_A, a_B, d_{B1}\},$$

$$\Gamma\left(\begin{bmatrix} 0 \\ A \\ 0 \\ 0 \end{bmatrix}\right) = \{a_A, a_B, d_{A2}\}, \quad \Gamma\left(\begin{bmatrix} 0 \\ B \\ 0 \\ 0 \end{bmatrix}\right) = \{a_A, a_B, d_{B2}\}$$

$$\Gamma\left(\begin{bmatrix} A \\ A \\ x_3 \\ x_4 \end{bmatrix}\right) = \{a_A, a_B, d_{A1}, d_{A2}\}, \quad \Gamma\left(\begin{bmatrix} A \\ B \\ x_3 \\ x_4 \end{bmatrix}\right) = \{a_A, a_B, d_{A1}, d_{B2}\},$$

$$\Gamma\left(\begin{bmatrix} B \\ A \\ x_3 \\ x_4 \end{bmatrix}\right) = \{a_A, a_B, d_{B1}, d_{A2}\}, \quad \Gamma\left(\begin{bmatrix} B \\ B \\ x_3 \\ x_4 \end{bmatrix}\right) = \{a_A, a_B, d_{B1}, d_{B2}\}, \quad x_3 \geq 0,\ x_4 \geq 0$$

- function f:

$$f\left(\begin{bmatrix}0\\0\\0\\0\end{bmatrix}, a_A\right)=\begin{bmatrix}A\\0\\0\\0\end{bmatrix}, \quad f\left(\begin{bmatrix}0\\0\\0\\0\end{bmatrix}, a_B\right)=\begin{bmatrix}B\\0\\0\\0\end{bmatrix}$$

$$f\left(\begin{bmatrix}A\\0\\0\\0\end{bmatrix}, a_A\right)=\begin{bmatrix}A\\A\\0\\0\end{bmatrix}, \quad f\left(\begin{bmatrix}A\\0\\0\\0\end{bmatrix}, a_B\right)=\begin{bmatrix}A\\B\\0\\0\end{bmatrix}, \quad f\left(\begin{bmatrix}A\\0\\0\\0\end{bmatrix}, d_{A1}\right)=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

... (omitted)

$$f\left(\begin{bmatrix}A\\A\\x_3\\x_4\end{bmatrix}, a_A\right)=\begin{bmatrix}A\\A\\x_3+1\\x_4\end{bmatrix}, \quad f\left(\begin{bmatrix}A\\A\\x_3\\x_4\end{bmatrix}, a_B\right)=\begin{bmatrix}A\\A\\x_3\\x_4+1\end{bmatrix}$$

$$f\left(\begin{bmatrix}A\\A\\x_3\\x_4\end{bmatrix}, d_{A1}\right)=\begin{cases}\begin{bmatrix}A\\A\\x_3-1\\x_4\end{bmatrix} & \text{if } x_3\geq 1\\ \begin{bmatrix}B\\A\\0\\x_4-1\end{bmatrix} & \text{if } x_3=0,\ x_4\geq 1\\ \begin{bmatrix}0\\A\\0\\0\end{bmatrix} & \text{if } x_3=0,\ x_4=0\end{cases}$$

$$f\left(\begin{bmatrix}A\\A\\x_3\\x_4\end{bmatrix}, d_{A2}\right)=\begin{cases}\begin{bmatrix}A\\B\\x_3\\x_4-1\end{bmatrix} & \text{if } x_4\geq 1\\ \begin{bmatrix}A\\A\\x_3-1\\0\end{bmatrix} & \text{if } x_3\geq 1,\ x_4=0\\ \begin{bmatrix}A\\0\\0\\0\end{bmatrix} & \text{if } x_3=0,\ x_4=0\end{cases}$$

... (omitted)

- initial state x_0:

$$x_0=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

- stochastic clock structure F:

$$F_{a_A}(t) = 1 - e^{-\lambda_A t}, \quad t \geq 0, \qquad \text{where } \lambda_A = 14 \text{ arrivals/hour}$$

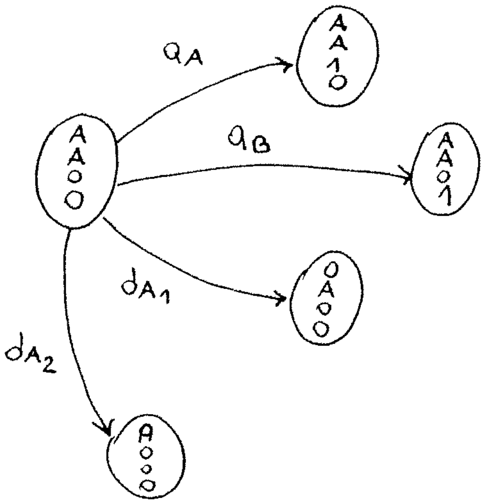
$$F_{a_B}(t) = 1 - e^{-\lambda_B t}, \quad t \geq 0, \qquad \text{where } \lambda_B = 8 \text{ arrivals/hour}$$

$$F_{d_{A_1}}(t) = F_{d_{A_2}}(t) = 1 - e^{-\mu_A t}, \quad t \geq 0, \quad \text{where } \mu_A = 20 \text{ services/hour}$$

$$F_{d_{B_1}}(t) = F_{d_{B_2}}(t) = 1 - e^{-\mu_B t}, \quad t \geq 0, \quad \text{where } \mu_B = 12 \text{ services/hour}$$

2. The current state is $X_k = \begin{bmatrix} A \\ A \\ 0 \\ 0 \end{bmatrix}$.

"Local" modeling:



The holding time $V(X_k)$ in state X_k follows an exponential distribution with rate $\lambda_A + \lambda_B + 2\mu_A$.

$$\Rightarrow E[V(X_k)] = \frac{1}{\lambda_A + \lambda_B + 2\mu_A} = \frac{1}{62} \text{ hours} \simeq 58 \text{ seconds}$$

3. The current state is $X_k = \begin{bmatrix} A \\ A \\ 0 \\ 0 \end{bmatrix}$. There must be an event d_{A_2} before any arrival; then the next arrival must be of type B:

$$\Rightarrow P(\ldots) = \frac{\mu_A}{\lambda_A + \lambda_B + \mu_A} \cdot \frac{\lambda_B}{\lambda_A + \lambda_B} = \frac{40}{231} \simeq 0.1732$$

Note that we ignore event d_{A_1} in state $X_k = \begin{bmatrix} A \\ A \\ 0 \\ 0 \end{bmatrix}$...

4. The current state is either $X_k = \begin{bmatrix} A \\ B \\ 0 \\ 0 \end{bmatrix}$ or $X_k = \begin{bmatrix} B \\ A \\ 0 \\ 0 \end{bmatrix}$.

$$\Rightarrow P(\ldots) = P\left(N_a(T)=6,\ Y_{d_{A,k}} > T,\ Y_{d_{B,k}} > T\right) =$$

events are independent

residual lifetime of event d_B

residual lifetime of event d_A

$$= P(N_a(T)=6)\,P(Y_{d_{A,k}} > T)\,P(Y_{d_{B,k}} > T) =$$

$$= \frac{(\lambda T)^6}{6!} e^{-\lambda T} \cdot e^{-\mu_A T} \cdot e^{-\mu_B T}$$

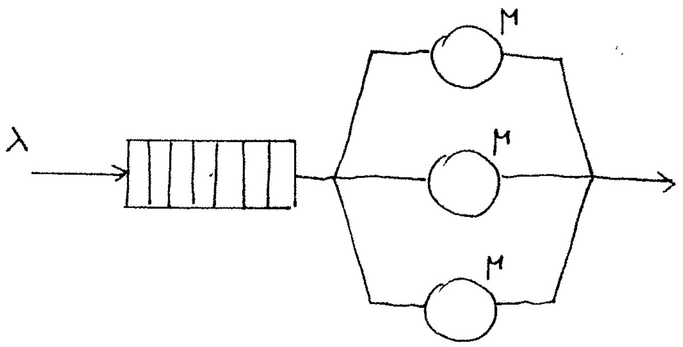
The arrival process is a Poisson process with rate $\lambda = \lambda_A + \lambda_B$ (superposition of two Poisson processes with rates λ_A and λ_B)

$$\simeq 3.78 \cdot 10^{-4}$$

$T = 4$ minutes $= \frac{1}{15}$ hours

Exercise 2

1. The system can be represented as follows:



$\lambda = 2$ arrivals/min

$\frac{1}{\mu} = 5$ minutes $\Rightarrow \mu = 0.2$ services/min

Definition of state:

$x = \#$ calls in the system $\in \{0, 1, 2, \ldots, 10\}$

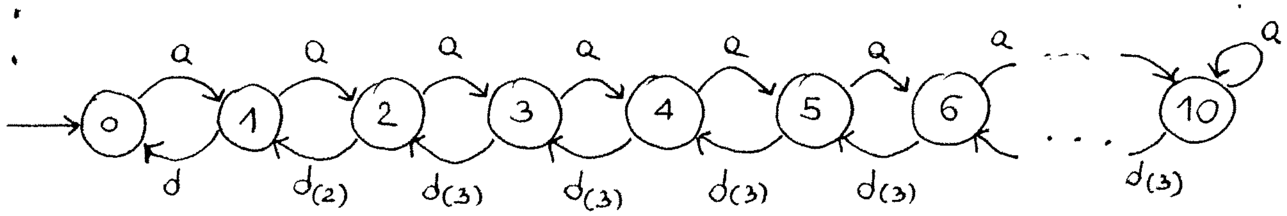
Note that there is no need to distinguish which server is busy and which not, since all servers are equal.

Events:

$\mathcal{E} = \{a, d\}$

a: arrival of a new call

d: termination of a service



$F_a(t) = 1 - e^{-\lambda t}, \; t \geq 0$

$F_d(t) = 1 - e^{-\mu t}, \; t \geq 0$

2. The current state is $X_k = 0$. There is only one possible case:

$$0 \xrightarrow{a} 1 \xrightarrow{a} 2 \xrightarrow{a} 3 \xrightarrow{a} 4$$

$$\Rightarrow P(\ldots) = 1 \cdot \frac{\lambda}{\lambda+\mu} \cdot \frac{\lambda}{\lambda+2\mu} \cdot \frac{\lambda}{\lambda+3\mu} = \frac{250}{429} \simeq 0.5828$$

When the fourth call arrives, all servers are busy. Therefore, the expected waiting time of the fourth customer is:

$$E[W_4] = \frac{1}{3\mu} = \frac{5}{3} \simeq 1 \text{ min } 40 \text{ sec}$$