**Test of Discrete Event Systems - 10.11.2014**

**Exercise 1**

The buffer of a production plant may contain up to three semifinished parts. Parts can be of two types (type 1 and type 2), with parts of type 2 being of double weight with respect to those of type 1. Parts arrive at the buffer according to Poisson processes with rates $\lambda_1 = 4$ arrivals/hour and $\lambda_2 = 3$ arrivals/hour, respectively. Parts arriving when the buffer is full are rejected.

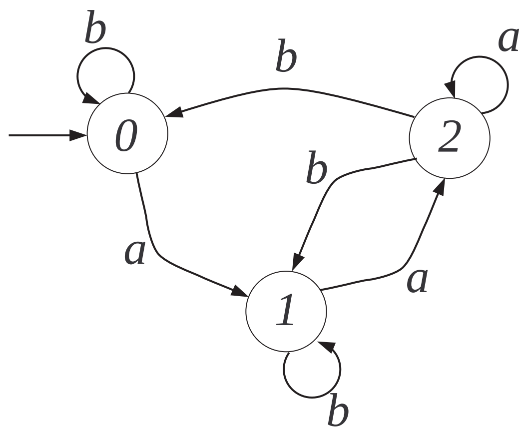
A lift truck loads the parts from the buffer and carries them to the assembly station. The maximum weight the lift truck may carry, is equivalent to three parts of type 1. When the cart arrives at the buffer, it loads the available parts so as to maximize the carried weight. If the buffer is empty, the lift truck leaves without waiting for the arrival of the next part. The lift truck returns to the buffer after random times following an exponential distribution with expected value 30 minutes.

1. Model the system through a stochastic state automaton $(\mathcal{E}, \mathcal{X}, \Gamma, f, x_0, F)$, assuming that the buffer is initially empty.
2. Compute the probability that at least three arrivals of parts occur before the arrival of the lift truck.
3. Assume that only one part of type 2 is present in the buffer, and none of type 1. Compute the probability that, the next time the lift truck arrives, it may load its maximum weight.
4. Compute the probability that the buffer is empty for at least one hour, and in the meantime the lift truck returns to the buffer exactly twice.
5. Show the procedure to compute the probability that the system is empty after the 5th event.
6. Show the procedure to compute the probability that the 5th event is the arrival of the lift truck.

**Exercise 2**

Consider the stochastic timed automaton $(\mathcal{E}, \mathcal{X}, \Gamma, p, x_0, F)$ in the figure, with $p(0|2, b) = 1/4$. Event $a$ is characterized by constant lifetimes equal to $t_a = 1.2$, whereas

$$F_b(t) = P(V_b \leq t) = \begin{cases} 0 & \text{if } t < 1.0 \\ 8(t-1)^2 & \text{if } 1.0 \leq t < 1.25 \\ 1 - 2(2t-3)^2 & \text{if } 1.25 \leq t < 1.5 \\ 1 & \text{otherwise.} \end{cases}$$




1. Compute the probability $P(X_2 = 1 | X_0 = 2)$.

Assume $X_0 = 0$.

2. For all $t \geq 0$, compute the probability to visit all the three states in the time interval $[0, t]$.
3. Compute the probability distribution function of the first interevent time, i.e. the function $P(Y_0^* \leq t | X_0 = 0)$ for all $t \geq 0$.

# Exercise 1

1. definition of state: $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $x_i \in \{0,1,2,3\}$

number of parts of type $i$ in the buffer, $i=1,2$

constraint: $x_1 + x_2 \leq 3$ (maximum buffer capacity)

$\Rightarrow \mathcal{X} = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$ $\Rightarrow$ 10 states

events: $\mathcal{E} = \{a_1, a_2, d\}$

$a_1$: arrival of a part of type 1

$a_2$: arrival of a part of type 2

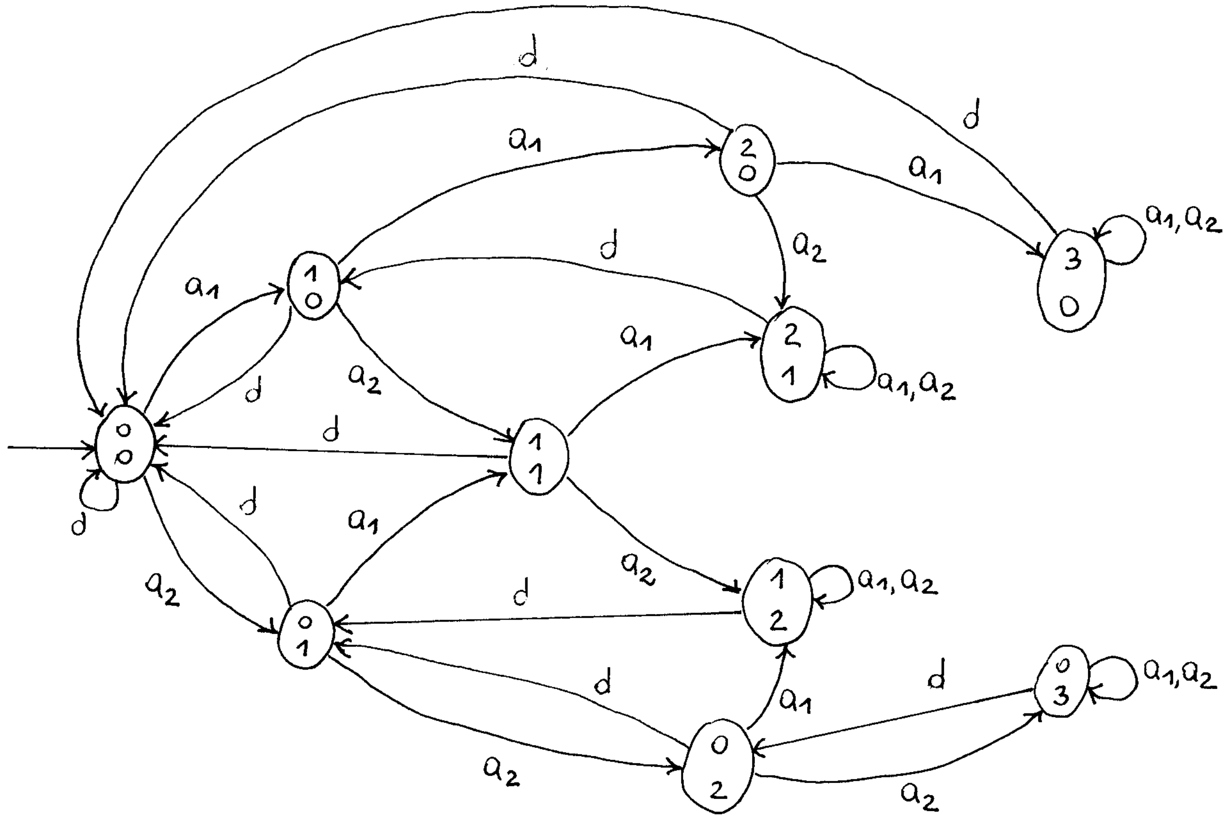
$d$: arrival of the lift truck

lift truck

maximum loadable weight: 3

weight type 1: 1

weight type 2: 2

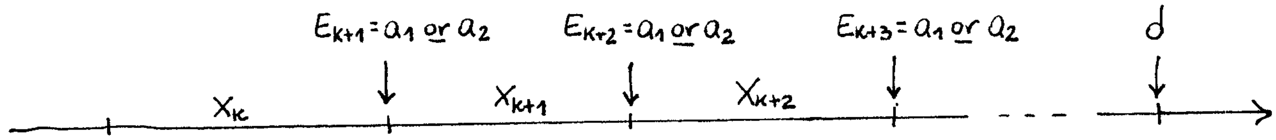



$F_{a_1}(t) = 1 - e^{-\lambda_1 t}$, $t \geq 0$, where $\lambda_1 = 4$ arrivals/hour

$F_{a_2}(t) = 1 - e^{-\lambda_2 t}$, $t \geq 0$, where $\lambda_2 = 3$ arrivals/hour

$F_d(t) = 1 - e^{-\mu t}$, $t \geq 0$, where $\frac{1}{\mu} = 30$ minutes $= 0.5$ hours $\Rightarrow \mu = 2$ arrivals/hour

2. The situation is as follows:




$$\Rightarrow P(\ldots) = \ldots = \underbrace{P(E_{k+1} = a_1 \text{ or } a_2 \mid X_k)}_{\frac{\lambda_1+\lambda_2}{\lambda_1+\lambda_2+\mu}} \underbrace{P(E_{k+2} = a_1 \text{ or } a_2 \mid X_{k+1})}_{\frac{\lambda_1+\lambda_2}{\lambda_1+\lambda_2+\mu}} \underbrace{P(E_{k+3} = a_1 \text{ or } a_2 \mid X_{k+2})}_{\frac{\lambda_1+\lambda_2}{\lambda_1+\lambda_2+\mu}} =$$

$$= \left( \frac{\lambda_1+\lambda_2}{\lambda_1+\lambda_2+\mu} \right)^3 = \left( \frac{7}{9} \right)^3 = \frac{343}{729} \simeq 0.4705$$

3. The current state is $X_k = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

$$P(\ldots) = P(E_{k+1} = a_1 \mid X_k) + P(E_{k+1} = a_2, E_{k+2} = a_1 \mid X_k) =$$

$$= \frac{\lambda_1}{\lambda_1+\lambda_2+\mu} + \frac{\lambda_2}{\lambda_1+\lambda_2+\mu} \cdot \frac{\lambda_1}{\lambda_1+\lambda_2+\mu} = \frac{16}{27} \simeq 0.5926$$

4. $P(\ldots) = P(N_{a_1}(T) = 0, N_{a_2}(T) = 0, N_d(T) = 2) =$ events are independent

$$= P(N_{a_1}(T) = 0) P(N_{a_2}(T) = 0) P(N_d(T) = 2) =$$

$$= e^{-\lambda_1 T} \cdot e^{-\lambda_2 T} \cdot \frac{(\mu T)^2}{2} e^{-\mu T} \simeq 2.47 \cdot 10^{-4}$$

where $T = 1$ hour.

5. Enumerate the states in $\mathcal{X}$ from 1 to 10:

$$\mathcal{X} = \left\{ \underset{\substack{\downarrow \\ 1}}{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}, \underset{\substack{\downarrow \\ 2}}{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}, \underset{\substack{\downarrow \\ 3}}{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}, \underset{\substack{\downarrow \\ 4}}{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}, \underset{\substack{\downarrow \\ 5}}{\begin{bmatrix} 2 \\ 0 \end{bmatrix}}, \underset{\substack{\downarrow \\ 6}}{\begin{bmatrix} 0 \\ 2 \end{bmatrix}}, \underset{\substack{\downarrow \\ 7}}{\begin{bmatrix} 2 \\ 1 \end{bmatrix}}, \underset{\substack{\downarrow \\ 8}}{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}, \underset{\substack{\downarrow \\ 9}}{\begin{bmatrix} 3 \\ 0 \end{bmatrix}}, \underset{\substack{\downarrow \\ 10}}{\begin{bmatrix} 0 \\ 3 \end{bmatrix}} \right\}$$

and define the row vector:

$$\Pi_X(k) = \left[ P(X_k = 1) \; P(X_k = 2) \; \ldots \; P(X_k = 10) \right]$$

It is known that:

$$\Pi_X(k+1) = \Pi_X(k) P_X \quad \Rightarrow \quad \Pi_X(k) = \Pi_X(0) P_X^k$$

where $\Pi_X(0)$ is the vector containing probabilities of the initial state:

$$\Pi_X(0) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

the initial state $X_0 = 1$ is certain

and

$$P_X = \begin{bmatrix} P(X_{k+1}=1 | X_k=1) & \cdots & P(X_{k+1}=10 | X_k=1) \\ \vdots & \ddots & \vdots \\ P(X_{k+1}=1 | X_k=10) & \cdots & P(X_{k+1}=10 | X_k=10) \end{bmatrix} =$$

$$= \begin{bmatrix} \frac{\mu}{\lambda_1+\lambda_2+\mu} & \frac{\lambda_1}{\lambda_1+\lambda_2+\mu} & \frac{\lambda_2}{\lambda_1+\lambda_2+\mu} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{\mu}{\lambda_1+\lambda_2+\mu} & 0 & 0 & \frac{\lambda_2}{\lambda_1+\lambda_2+\mu} & \frac{\lambda_1}{\lambda_1+\lambda_2+\mu} & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \frac{\mu}{\lambda_1+\lambda_2+\mu} & 0 & 0 & 0 & \frac{\lambda_1+\lambda_2}{\lambda_1+\lambda_2+\mu} \end{bmatrix}$$

$\Rightarrow$ matrix $10 \times 10$

Since we need $P(X_5 = 1)$, we have to compute

$$\Pi_X(5) = \Pi_X(0) P_X^5$$

and then take the first element of $\Pi_X(5)$.

6. Enumerate the events in $\mathcal{E}$ from 1 to 3:

$$\mathcal{E} = \{ a_1, a_2, d \}$$

$$\begin{array}{ccc} a_1 & a_2 & d \\ \downarrow & \downarrow & \downarrow \\ 1 & 2 & 3 \end{array}$$

and define the row vector:

$$\Pi_E(k) = \left[\, P(E_k=1) \;\; P(E_k=2) \;\; P(E_k=3) \,\right].$$

It is known that

$$\Pi_E(k+1) = \Pi_X(k) P_E \quad \Rightarrow \quad \Pi_E(k+1) = \Pi_X(0) P_X^k P_E$$

where

$$P_E = \begin{bmatrix} P(E_{k+1}=1 \mid X_k=1) & P(E_{k+1}=2 \mid X_k=1) & P(E_{k+1}=3 \mid X_k=1) \\ \vdots & \vdots & \vdots \\ P(E_{k+1}=1 \mid X_k=10) & P(E_{k+1}=2 \mid X_k=10) & P(E_{k+1}=3 \mid X_k=10) \end{bmatrix} =$$

$$= \begin{bmatrix} \frac{\lambda_1}{\lambda_1+\lambda_2+\mu} & \frac{\lambda_2}{\lambda_1+\lambda_2+\mu} & \frac{\mu}{\lambda_1+\lambda_2+\mu} \\ \vdots & \vdots & \vdots \\ \frac{\lambda_1}{\lambda_1+\lambda_2+\mu} & \frac{\lambda_2}{\lambda_1+\lambda_2+\mu} & \frac{\mu}{\lambda_1+\lambda_2+\mu} \end{bmatrix}$$

matrix $10 \times 3$

Since we need $P(E_5=3)$, we have to compute

$$\Pi_E(5) = \Pi_X(0) P_X^4 P_E$$

and then take the last element of $\Pi_E(5)$.

# Exercise 2

1. We have to compute $P(X_2=1|X_0=2)$. There are three possible paths:

$$2 \xrightarrow{b} 0 \xrightarrow{a} 1 \qquad (\text{case } \#1)$$

$$2 \xrightarrow{a} 2 \xrightarrow{b} 1 \qquad (\text{case } \#2)$$

$$2 \xrightarrow{b} 1 \xrightarrow{b} 1 \qquad (\text{case } \#3)$$

Therefore:

$$P(X_2=1|X_0=2) = \underbrace{P(X_2=1|X_0=2,E_1=b,X_1=0,E_2=a)}_{1}\underbrace{P(E_2=a|X_0=2,E_1=b,X_1=0)\underbrace{P(X_1=0|X_0=2,E_1=b)}_{\frac{1}{4}}P(E_1=b|X_0=2)}_{P(V_{b,1}<V_{a,1},\ V_{a,1}<V_{b,1}+V_{b,2})} +$$

$$+\underbrace{P(X_2=1|X_0=2,E_1=a,X_1=2,E_2=b)}_{\frac{3}{4}}\underbrace{P(E_2=b|X_0=2,E_1=a,X_1=2)\underbrace{P(X_1=2|X_0=2,E_1=a)}_{1}P(E_1=a|X_0=2)}_{P(V_{a,1}<V_{b,1},\ V_{b,1}<V_{a,1}+V_{a,2})} +$$

$$+\underbrace{P(X_2=1|X_0=2,E_1=b,X_1=1,E_2=b)}_{1}\underbrace{P(E_2=b|X_0=2,E_1=b,X_1=1)\underbrace{P(X_1=1|X_0=2,E_1=b)}_{\frac{3}{4}}P(E_1=b|X_0=2)}_{P(V_{b,1}<V_{a,1},\ V_{b,1}+V_{b,2}<V_{a,1})}$$

$$= \frac{1}{4}P(V_{b,1}<V_{a,1}, V_{a,1}<V_{b,1}+V_{b,2}) + \frac{3}{4}P(V_{a,1}<V_{b,1}, V_{b,1}<V_{a,1}+V_{a,2}) + \frac{3}{4}P(V_{b,1}<V_{a,1}, V_{b,1}+V_{b,2}<V_{a,1})$$

$$P(V_{b,1}<V_{a,1}, V_{a,1}<V_{b,1}+V_{b,2}) \underset{V_{a,1}=1.2}{=} P(V_{b,1}<1.2, V_{b,1}+V_{b,2}>1.2) = P(V_{b,1}<1.2) = F_b(1.2) = \frac{8}{25}$$

note that $V_{b,1}\geq 1$ and $V_{b,2}\geq 1$, therefore $V_{b,1}+V_{b,2}>1.2$ is always true

$$P(V_{a,1}<V_{b,1},\ V_{b,1}<V_{a,1}+V_{a,2}) \underset{V_{a,1}=V_{a,2}=1.2}{=} P(1.2<V_{b,1}<2.4) = F_b(2.4)-F_b(1.2) =$$

$$= 1-\frac{8}{25} = \frac{17}{25}$$

$$P(V_{b,1}<V_{a,1},\ V_{b,1}+V_{b,2}<V_{a,1}) \underset{V_{a,1}=1.2}{=} P(V_{b,1}<1.2,\ V_{b,1}+V_{b,2}<1.2) = 0$$

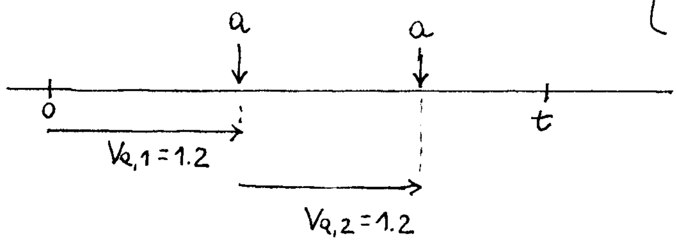
note that $V_{b,1}\geq 1$ and $V_{b,2}\geq 1$, therefore $V_{b,1}+V_{b,2}<1.2$ is always false

In conclusion:

$$P(X_2=1|X_0=2) = \frac{1}{4}\cdot\frac{8}{25}+\frac{3}{4}\cdot\frac{17}{25}+\frac{3}{4}\cdot 0 = \frac{59}{100} = 0.59$$

2. It must happen that at least two events $a$ occur in the time interval $[0,t]$.

$$\Rightarrow P(V_{a,1}+V_{a,2}\leq t) = P(2.4\leq t) = \begin{cases} 0 & \text{if } t<2.4 \\ 1 & \text{otherwise} \end{cases}$$




3. $P(Y_0^*\leq t|X_0=0) = P(\min\{V_{a,1},V_{b,1}\}\leq t) = 1-P(\min\{V_{a,1},V_{b,1}\}>t) =$

$$= 1-\underbrace{P(V_{a,1}>t,\ V_{b,1}>t)}_{\text{independent}} = 1-P(V_{a,1}>t)P(V_{b,1}>t) \underset{V_{a,1}=1.2}{=} 1-P(t<1.2)P(V_{b,1}>t)$$

If $t<1.2$, $P(t<1.2)=1$. Therefore

$$P(Y_0^*\leq t|X_0=0) = 1-P(V_{b,1}>t) = (V_{b,1}\leq t) = \begin{cases} 0 & \text{if } t<1.0 \\ 8(t-1)^2 & \text{if } 1.0\leq t<1.2 \end{cases}$$

If $t\geq 1.2$, $P(t<1.2)=0$. Therefore

$$P(Y_0^*\leq t|X_0=0) = 1$$

In conclusion:

$$P(Y_0^*\leq t|X_0=0) = \begin{cases} 0 & \text{if } t<1.0 \\ 8(t-1)^2 & \text{if } 1.0\leq t<1.2 \\ 1 & \text{otherwise} \end{cases}$$