Test of Discrete Event Systems - 05.11.2014

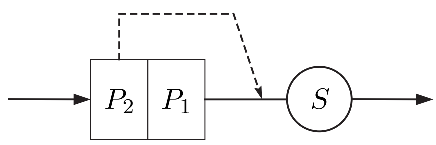
Exercise 1

A small hair salon has two chairs and two hairdressers. Customers arrive according to a Poisson process with rate 3 arrivals/hour. A customer is male with probability $p = 1/3$. The duration of a hair-cut is independent of the hairdresser, but depends on the sex of the customer. It is exponentially distributed with expected value 20 minutes for men, and 45 minutes for women. Since the hair salon does not have a waiting room, customers arriving when both chairs are busy decide to abandon the hair salon. The hair salon is empty at the opening.

1. Model the hair salon through a stochastic timed automaton $(\mathcal{E}, \mathcal{X}, \Gamma, p, x_0, F)$.
2. Compute the probability that the third customer arriving after the opening abandons the hair salon.
3. Assume that one hairdresser is serving a man, and the other is serving a woman. Compute the probability that both customers leave the hair salon before another customer arrives.
4. Assume that one hairdresser is serving a man, and the other is serving a woman. Compute the probability that, in the next hour, both customers leave the hair salon and no other customer arrives.
5. Compute the probability that at least three customers arrive in the next hour.

Exercise 2

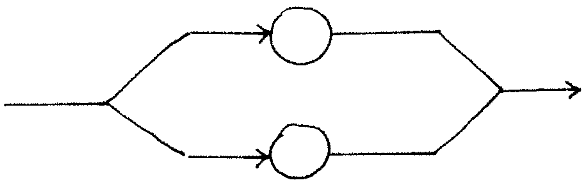
Consider the queueing system depicted in the figure, and formed by a server S preceded by a queue with two places P_1 and P_2. When the queue is full and the server S completes the service, with probability $p = 1/4$ the customer waiting in P_2 is admitted to the service before the customer waiting in P_1 ("overtaking"). A customer arriving when the queue in full, is not admitted to the system. Assume that the customers arrive according to a Poisson process with rate $\lambda = 0.1$ arrivals/min, and the service times in S are exponentially distributed with rate $\mu = 0.2$ services/min, respectively. The system is initially empty.



1. Model the queueing system through a stochastic timed automaton $(\mathcal{E}, \mathcal{X}, \Gamma, p, x_0, F)$.
2. Assume that the queue is full and the server S is busy. Compute the probability that the customer waiting in P_1 is overtaken exactly twice before being admitted to the service.
3. Assume that the queue is full and the server S is serving a customer who did not overtake. Compute the probability that, after the next event, the server S is serving a customer who did not overtake.
4. Determine the probability distribution function of the holding time in a state where the server S is busy and only one customer is waiting.

Exercise 1

1. The system can be represented as a queueing system with two servers and no queue:



REMARK: the two servers are equal, therefore there is no need to distinguish them in the definition of state.

Definition of state: $x = \begin{bmatrix} x_M \\ x_F \end{bmatrix}$ $x_M \to$ number of male customers $\in \{0,1,2\}$, $x_F \to$ number of female customers $\in \{0,1,2\}$

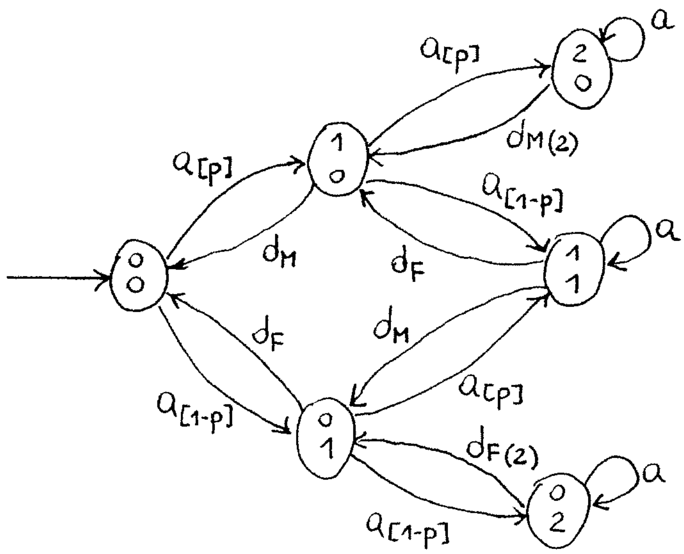
State space: $\mathcal{X} = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \end{bmatrix} \right\}$ => 6 states

Definition of events: $\mathcal{E} = \{ a, d_M, d_F \}$

a → arrival of a customer

d_M → termination of the service of a male customer

d_F → termination of the service of a female customer



$F_a(t) = 1 - e^{-\lambda t}, \ t \geq 0$

where $\lambda = 3$ arrivals/hour

$F_{d_M}(t) = 1 - e^{-\mu_M t}, \ t \geq 0$

where $\frac{1}{\mu_M} = 20$ minutes $= \frac{1}{3}$ hours

=> $\mu_M = 3$ services/hour

$F_{d_F}(t) = 1 - e^{-\mu_F t}, \ t \geq 0$

where $\frac{1}{\mu_F} = 45$ minutes $= \frac{3}{4}$ hours

=> $\mu_F = \frac{4}{3}$ services/hour

2. When the third customer arrives, the state must be one of $\binom{2}{0}$, $\binom{1}{1}$ and $\binom{0}{2}$
=> Starting from the initial state $\binom{0}{0}$, the first three events must be arrivals.

We identify four favorable cases, corresponding to the following paths on the state transition diagram:

1. $\binom{0}{0} \xrightarrow{a} \binom{1}{0} \xrightarrow{a} \binom{2}{0} \xrightarrow{a}$

2. $\binom{0}{0} \xrightarrow{a} \binom{1}{0} \xrightarrow{a} \binom{1}{1} \xrightarrow{a}$

3. $\binom{0}{0} \xrightarrow{a} \binom{0}{1} \xrightarrow{a} \binom{1}{1} \xrightarrow{a}$

4. $\binom{0}{0} \xrightarrow{a} \binom{0}{1} \xrightarrow{a} \binom{0}{2} \xrightarrow{a}$

The probability we are looking for, is the sum of the probabilities of these four cases.

To compute the probability of case 1, we proceed as follows.

- In state $\binom{0}{0}$, the only active event is a, Therefore, the probability that the next state is $\binom{1}{0}$, corresponds to the probability that the arrival is a man. This probability is p.
- In state $\binom{1}{0}$, the next event is a with probability $\dfrac{\lambda}{\lambda+\mu_M}$.

  rate of event a

  The arrival is a man, and therefore the next state is $\binom{2}{0}$, with probability p.

  sum of the rates of active events in state $\binom{1}{0}$
- In state $\binom{2}{0}$, the next event is a with probability $\dfrac{\lambda}{\lambda+2\mu_M}$

  two male customers are being served, therefore we have scheduled two distinct events d_M

By multiplying all these probabilities, we obtain the probability of case 1:

$$P(1) = p \cdot \frac{\lambda}{\lambda+\mu_M} \cdot p \cdot \frac{\lambda}{\lambda+2\mu_M} = \frac{(\lambda p)^2}{(\lambda+\mu_M)(\lambda+2\mu_M)}$$

In the same fashion, we can compute the probabilities of all other cases:

$$P(2) = p \cdot \frac{\lambda}{\lambda+\mu_M} \cdot (1-p) \cdot \frac{\lambda}{\lambda+\mu_M+\mu_F}$$

$$P(3) = (1-p) \cdot \frac{\lambda}{\lambda+\mu_F} \cdot p \cdot \frac{\lambda}{\lambda+\mu_M+\mu_F}$$

$$P(4) = (1-p) \cdot \frac{\lambda}{\lambda+\mu_F} \cdot (1-p) \cdot \frac{\lambda}{\lambda+2\mu_F}$$

Finally:

$$P(\ldots) = P(1) + P(2) + P(3) + P(4) =$$

$$= \frac{(\lambda p)^2}{(\lambda+\mu_M)(\lambda+2\mu_M)} + \frac{\lambda^2 p(1-p)}{(\lambda+\mu_M)(\lambda+\mu_M+\mu_F)} +$$

$$+ \frac{\lambda^2 p(1-p)}{(\lambda+\mu_F)(\lambda+\mu_M+\mu_F)} + \frac{[\lambda(1-p)]^2}{(\lambda+\mu_F)(\lambda+2\mu_F)} \simeq 0.2898$$

3. The current state is $X_k = \binom{1}{1}$. Event a must occur after both event d_M and event d_F. We identify two favorable cases, corresponding to the following paths on the state transition diagram:

1 $\binom{1}{1} \xrightarrow{d_M} \binom{0}{1} \xrightarrow{d_F}$

2 $\binom{1}{1} \xrightarrow{d_F} \binom{1}{0} \xrightarrow{d_M}$

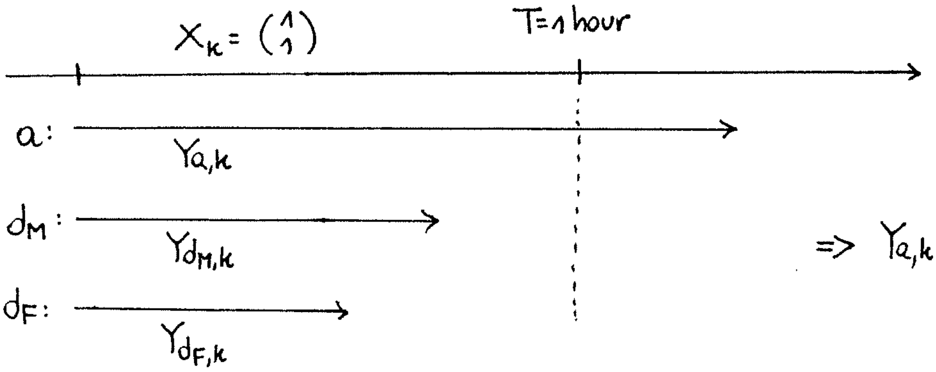
The probability we are looking for, is the sum of the probabilities of these two cases. We have:

$$P(1) = \frac{\mu_M}{\lambda+\mu_M+\mu_F} \frac{\mu_F}{\lambda+\mu_F} \quad ; \quad P(2) = \frac{\mu_F}{\lambda+\mu_M+\mu_F} \frac{\mu_M}{\lambda+\mu_M}$$

Therefore:

$$P(\ldots) = P(1) + P(2) = \frac{\mu_M \mu_F}{(\lambda+\mu_M+\mu_F)(\lambda+\mu_F)} + \frac{\mu_M \mu_F}{(\lambda+\mu_M+\mu_F)(\lambda+\mu_M)} \simeq 0.2168$$

4. The current state is $X_k = \binom{1}{1}$. We require that the residual lifetimes of both events d_M and d_F are smaller than one hour, and the residual lifetime of event a is larger than one hour.



$$\Rightarrow Y_{a,k} > T,\ Y_{d_M,k} \le T,\ Y_{d_F,k} \le T$$

$$\Rightarrow P(\ldots) = P(Y_{a,k} > T,\ Y_{d_M,k} \le T,\ Y_{d_F,k} \le T) = P(Y_{a,k} > T)\,P(Y_{d_M,k} \le T)\,P(Y_{d_F,k} \le T)$$

independent random variables

$$= e^{-\lambda T} \cdot (1-e^{-\mu_M T})(1-e^{-\mu_F T}) \simeq 0.0348$$

5. Arrivals are generated by a Poisson process. We apply the Poisson distribution with $T = 1$ hour.

$$P(N_a(T) \ge 3) = 1 - P(N_a(T)=0) - P(N_a(T)=1) - P(N_a(T)=2)$$

$$= 1 - e^{-\lambda T} - (\lambda T)e^{-\lambda T} - \frac{(\lambda T)^2}{2} e^{-\lambda T} = 1 - \left[1 + (\lambda T) + \frac{(\lambda T)^2}{2}\right] e^{-\lambda T} \simeq 0.5768$$

EXERCISE 2

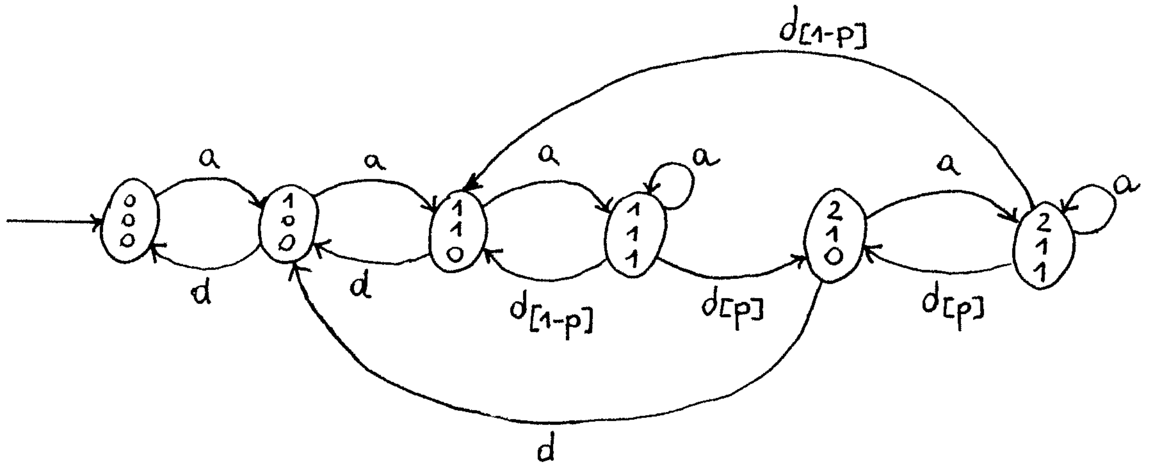
1. In view of the next questions, we consider a definition of state which takes into account the "type" of the customer being served.

$$x = \begin{bmatrix} x_S \\ x_{P_1} \\ x_{P_2} \end{bmatrix} \begin{array}{l} \rightarrow \text{server: idle (0), busy - customer did not overtake (1), busy - customer did overtake (2)} \\ \rightarrow \text{place P1: free (0), busy (1)} \\ \rightarrow \text{place P2: free (0), busy (1)} \end{array}$$

Event set: $\mathcal{E} = \{a, d\}$ — a: arrival of a customer, d: termination of a service

$p = \frac{1}{4}$ (probability of overtaking)

State transition diagram:



Stochastic clock structure $F = \{F_a, F_d\}$:

$F_a(t) = 1 - e^{-\lambda t}$, $t \geq 0$ where $\lambda = 0.1$ arrivals/min

$F_d(t) = 1 - e^{-\mu t}$, $t \geq 0$ where $\mu = 0.2$ services/min

2. The current state is either $\begin{pmatrix}1\\1\\1\end{pmatrix}$ or $\begin{pmatrix}2\\1\\1\end{pmatrix}$.

We proceed in the following way.

- When event d occurs, first overtaking must take place. Overtaking occurs with probability p. The next state is $\begin{pmatrix}2\\1\\0\end{pmatrix}$.

  REMARK: Arrivals are ignored in the current state because they are rejected, and do not change the composition of the queue.

- In state $\begin{pmatrix}2\\1\\0\end{pmatrix}$, the next event must be an arrival, because then we want a second overtaking. Event a occurs with probability $\frac{\lambda}{\lambda+\mu}$. The next state is $\begin{pmatrix}2\\1\\1\end{pmatrix}$.

- In state $\begin{pmatrix}2\\1\\1\end{pmatrix}$, when event d occurs, second overtaking must take place. Overtaking occurs with probability p. The next state is $\begin{pmatrix}2\\1\\0\end{pmatrix}$.
  REMARK: As before, arrivals are ignored in $\begin{pmatrix}2\\1\\1\end{pmatrix}$.

- In state $\begin{pmatrix}2\\1\\0\end{pmatrix}$, we have now two possible favorable cases:

CASE #1

- The next event is d. This occurs with probability $\frac{\mu}{\lambda+\mu}$.
  The customer in P1 is finally admitted to the service.

CASE #2

- The next event is a. This occurs with probability $\frac{\lambda}{\lambda+\mu}$.
  The next state is $\begin{pmatrix}2\\1\\1\end{pmatrix}$.

- In state $\begin{pmatrix}2\\1\\1\end{pmatrix}$, when event d occurs, overtaking must not take place.
  This happens with probability $(1-p)$.
  The customer in P1 is finally admitted to the service.
  REMARK: As before, arrivals are ignored in $\begin{pmatrix}2\\1\\1\end{pmatrix}$.

To conclude:

$$P(\ldots) = p\cdot\frac{\lambda}{\lambda+\mu}\cdot p\cdot\left[\frac{\mu}{\lambda+\mu}+\frac{\lambda}{\lambda+\mu}(1-p)\right] \simeq 0.0191$$

3. The current state is $X_k = \begin{pmatrix}1\\1\\1\end{pmatrix}$.
   There are two

(1) The next event is a, so that the customer in the server does not change.
This occurs with probability $\frac{\lambda}{\lambda+\mu}$.

(2) The next event is d, and there is not overtaking.

with probability $\frac{\mu}{\lambda+\mu}$ ... with probability $1-p$

$$\Rightarrow P(\dots) = P(1) + P(2) = \frac{\lambda}{\lambda+\mu} + \frac{\mu}{\lambda+\mu}(1-p) \simeq 0.8333$$

4. The current state is either $X_k = \begin{pmatrix}1\\1\\0\end{pmatrix}$ or $X_k = \begin{pmatrix}2\\1\\0\end{pmatrix}$. The answer is the same for both states, so let us consider $X_k = \begin{pmatrix}1\\1\\0\end{pmatrix}$.

The state holding time is the time spent by the system in a given state.

If we denote by $V\begin{pmatrix}1\\1\\0\end{pmatrix}$ the state holding time of $X_k = \begin{pmatrix}1\\1\\0\end{pmatrix}$, we have that $V\begin{pmatrix}1\\1\\0\end{pmatrix}$ is the superposition of the residual lifetimes of all events that take the system away from $\begin{pmatrix}1\\1\\0\end{pmatrix}$. In this case, $V\begin{pmatrix}1\\1\\0\end{pmatrix} = \min\{Y_{a,k}, Y_{d,k}\}$.

exponential with rate λ ... exponential with rate μ

This implies that $V\begin{pmatrix}1\\1\\0\end{pmatrix}$ is exponentially distributed with rate $\lambda+\mu$.

$$\Rightarrow P\left(V\begin{pmatrix}1\\1\\0\end{pmatrix} \leq t\right) = 1 - e^{-(\lambda+\mu)t} = 1 - e^{-0.3t}, \quad t \geq 0.$$