Test of Discrete Event Systems - 09.01.2014

Exercise 1

Define the age of a device as the number of whole days the device has operated. A device of age $j \geq 0$ at the beginning of a day, fails during the day with probability $q_j = jq$, where $q = 0.25$, and in such a case it is replaced by a new identical device, which starts operating at the beginning of the next day.

1. Model the system described above through a discrete time homogenous Markov chain.
2. Compute the probability that a new device operates for the maximum number of days.
3. If the device at day 1 is new, compute the probability that the device at day 10 is new.
4. Compute the average duration (in days) of a device.

Exercise 2

A multi-processor computer is equipped with three identical CPUs. N different processes run on the computer. Each process can be either in the waiting state or in the execution state. The duration of the execution of a process on a CPU is an exponential random variable with expected value 0.77 ms. At the end of the execution, the process is put in the waiting state. A process in waiting state requests again the use of a CPU after a random time which follows an exponential distribution with expected value 1.12 ms. At the moment of the request, if at least one CPU is available, the process is executed. Otherwise, it remains in the waiting state and will request again the use of a CPU as described above.

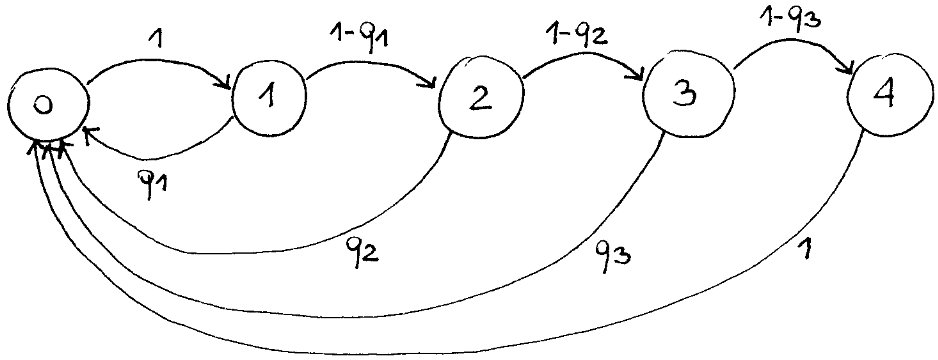
1. Compute the minimum number N such that the steady-state probability that all the three CPUs are busy, is at least 0.75.

With the value of N computed above:

2. Verify the condition $\lambda_{eff} = \mu_{eff}$ at steady-state for the system composed by the three CPUs.
3. Compute the utilization of a generic CPU at steady state, assuming that, when more than one CPU are available at the moment of a request of execution, the choice is equally probable.
4. Assuming that all the three CPUs are initially idle, compute the probability that at time $t = 0.6$ ms only one CPU is busy.

EXERCISE 1

1. state x = age of the device $\in \{0,1,2,3,4\}$



where $q_0 = 0$, $q_1 = \frac{1}{4}$, $q_2 = \frac{1}{2}$, $q_3 = \frac{3}{4}$ and $q_4 = 1$.

$$\Rightarrow \underset{\substack{\uparrow \\ \text{transition} \\ \text{probability} \\ \text{matrix}}}{P} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ q_1 & 0 & 1-q_1 & 0 & 0 \\ q_2 & 0 & 0 & 1-q_2 & 0 \\ q_3 & 0 & 0 & 0 & 1-q_3 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ \frac{1}{4} & 0 & \frac{3}{4} & 0 & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 \\ \frac{3}{4} & 0 & 0 & 0 & \frac{1}{4} \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

2. $P(X(t+4)=4 \mid X(t)=0) = p_{0,4}(4) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \end{bmatrix} P^4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \frac{3}{32} \simeq 0.0938$

3. $P(X(10)=0 \mid X(1)=0) = p_{0,0}(9) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \end{bmatrix} P^9 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \frac{39}{128} \simeq 0.3047$

4. If we define D the duration (in days) of a device, we have

$D = T_{0,0} - 1$

where $T_{0,0}$ is the recurrence time of state 0. Therefore,

$E[D] = E[T_{0,0}] - 1$

The discrete-time homogeneous Markov chain is:

- irreducible
- aperiodic
- finite

Hence,

$$E[T_{0,0}] = \frac{1}{\pi_0}$$

and the vector of stationary probabilities

$$\pi = [\pi_0\ \pi_1\ \pi_2\ \pi_3\ \pi_4]$$

can be computed by solving

$$\begin{cases} \pi = \pi P \\ \sum_{i=0}^{4} \pi_i = 1 \end{cases}$$

It follows that

$$\pi = \left[\frac{32}{103}\ \ \frac{32}{103}\ \ \frac{24}{103}\ \ \frac{12}{103}\ \ \frac{3}{103} \right]$$

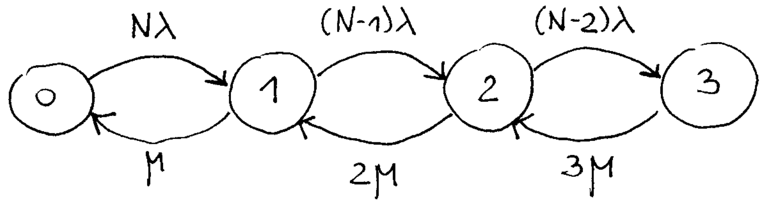
and finally

$$E[D] = \frac{103}{32} - 1 = \frac{71}{32} \simeq 2.2188 \text{ days}$$

EXERCISE 2

model of the system: continuous-time homogeneous Markov chain

state x = number of processes being executed $\in \{0,1,2,3\}$



where $\lambda = \frac{1}{1.12} \simeq 0.8929$ requests/ms and $\mu = \frac{1}{0.77} \simeq 1.2987$ services/ms.

$$\Rightarrow \quad Q = \begin{bmatrix} -N\lambda & N\lambda & 0 & 0 \\ \mu & -[(N-1)\lambda+\mu] & (N-1)\lambda & 0 \\ 0 & 2\mu & -[(N-2)\lambda+2\mu] & (N-2)\lambda \\ 0 & 0 & 3\mu & -3\mu \end{bmatrix}$$

transition rate matrix

1. We have to solve the optimization problem

$$\begin{cases} \min N \\ \text{s.t.} \\ \quad \pi_3 \geq 0.75 \end{cases}$$

where π_3 is the stationary probability of state 3 (where all the three CPUs are busy).

The continuous-time homogeneous Markov chain is:

- irreducible
- finite

Therefore, the vector of stationary probabilities:

$$\pi = [\pi_0 \; \pi_1 \; \pi_2 \; \pi_3]$$

can be computed by solving

$$\begin{cases} \pi Q = 0 \\ \sum_{i=0}^{3} \pi_i = 1 \end{cases}$$

the optimal solution of the optimization problem

Using Matlab, one could find N^* by solving the above system of linear equalities for different values of N. The result is $N^* = 18$.

Otherwise, analytically:

$$\begin{cases} -N\lambda \pi_0 + \mu \pi_1 = 0 \\ N\lambda\pi_0 - [(N-1)\lambda + \mu]\pi_1 + 2\mu\pi_2 = 0 \\ (N-2)\lambda\pi_2 - 3\mu\pi_3 = 0 \\ \pi_0 + \pi_1 + \pi_2 + \pi_3 = 1 \end{cases} \Rightarrow \begin{cases} -N\lambda\pi_0 + \mu\pi_1 = 0 \\ -(N-1)\lambda\pi_1 + 2\mu\pi_2 = 0 \\ -(N-2)\lambda\pi_2 + 3\mu\pi_3 = 0 \\ \pi_0 + \pi_1 + \pi_2 + \pi_3 = 1 \end{cases}$$

Let $\rho = \frac{\lambda}{\mu}$. We have:

$$\left.\begin{aligned} \pi_1 &= N\rho\pi_0 \\ \pi_2 &= \frac{N-1}{2}\rho\pi_1 = \frac{N(N-1)}{2}\rho^2\pi_0 \\ \pi_3 &= \frac{N-2}{3}\rho\pi_2 = \frac{N(N-1)(N-2)}{6}\rho^3\pi_0 \end{aligned}\right\} \left[1 + N\rho + \frac{N(N-1)}{2}\rho^2 + \frac{N(N-1)(N-2)}{6}\rho^3\right]\pi_0 = 1$$

$$\Rightarrow \quad \pi_3 = \frac{\frac{N(N-1)(N-2)}{6}\rho^3}{1 + N\rho + \frac{N(N-1)}{2}\rho^2 + \frac{N(N-1)(N-2)}{6}\rho^3} \geq 0.75$$

$$\Rightarrow \quad 0.0135\,N^3 - 0.2179\,N^2 - 0.3113\,N - 0.75 \geq 0$$

The roots of the polynomial are 17.6277, $-0.7435 \pm 1.6121j$.
The minimum integer greater than 17.6277 is $N^* = 18$.

2.
$$\left.\begin{aligned} \lambda_{eff} &= N\lambda\pi_0 + (N-1)\lambda\pi_1 + (N-2)\lambda\pi_2 \simeq 3.5257 \\ \mu_{eff} &= \mu\pi_1 + 2\mu\pi_2 + 3\mu\pi_3 \simeq 3.5257 \end{aligned}\right\} \Rightarrow \lambda_{eff} = \mu_{eff}\ !$$

3.
$$U = \frac{1}{3}\pi_1 + \frac{2}{3}\pi_2 + \pi_3 \simeq 0.9049$$

because, if only one CPU is busy, there is one possibility over three that the considered CPU is busy

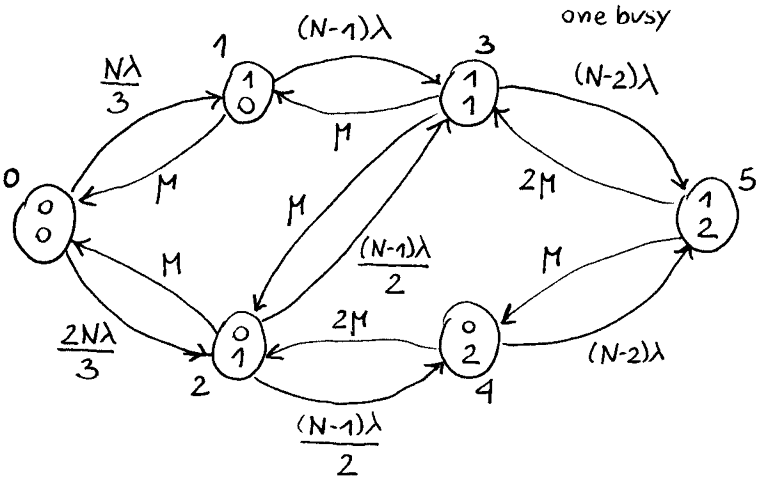
because, if two CPUs are busy, there are two possibilities over three that the considered CPU is busy

Alternative solution

Modify the model as follows.

$$\text{state} \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \begin{aligned} &\to \text{considered CPU: } x_1 \in \{0, 1\} \quad (0 = \text{idle},\ 1 = \text{busy}) \\ &\to \text{other two CPUs: } x_2 \in \{0, 1, 2\} \end{aligned}$$

0: both idle; 1: one idle one busy; 2: both busy



Using this model, the utilization of the considered CPU is:

$$U = \pi_1 + \pi_3 + \pi_5 \simeq 0.9049$$

4. $P(X(0.6)=1 \mid X(0)=0) = p_{0,1}(0.6) = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} e^{Q \cdot 0.6} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \simeq 0.0381$ 5