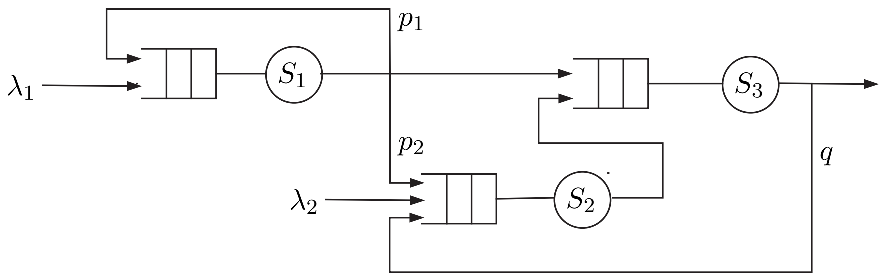
Test of Discrete Event Systems - 19.12.2013

Exercise 1

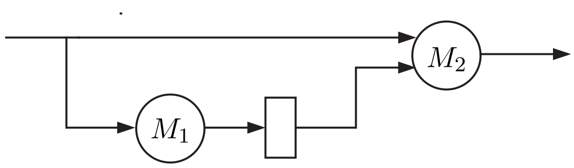
Consider the queueing network in the figure, where each node is represented by a M/M/1 queueing system, $\lambda_1 = 45$ arrivals/hour, $\lambda_2 = 3$ arrivals/hour, $p_1 = 0.2$, $p_2 = 0.4$ and $q = 0.75$. The service rate μ_1 of server S_1 is 81 services/hour.



1. Design the service rates μ_2 and μ_3 of servers S_2 and S_3, respectively, such that at steady state the average queue length is the same in each node of the network.

Exercise 2

Consider the queueing network in the figure.



Arriving parts may require preprocessing in M_1 with probability $p = 1/3$, otherwise they go directly to M_2. When a part arrives and the corresponding machine is not available, the part is rejected. There is a unitary buffer between M_1 and M_2. When M_1 terminates preprocessing of a part and M_2 is busy, the part is moved to the buffer, if it is empty. Otherwise, the part is kept by M_1, that therefore remains unavailable for a new job until M_2 terminates its job. Parts arrive according to a Poisson process with expected interarrival time equal to 5 min, whereas service times in M_1 and M_2 follow exponential distributions with rates $\mu_1 = 0.5$ services/min and $\mu_2 = 0.8$ services/min, respectively.

1. Compute the expected number of parts in the system at steady state.

2. Compute the expected time spent by a part in M_1 at steady state.

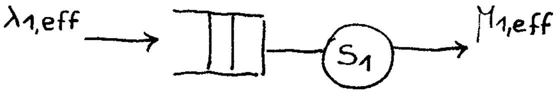
3. Verify the condition $\lambda_{eff} = \mu_{eff}$ for the whole system at steady state.

4. Compute the utilization of M_1 and M_2 at steady state.

5. Compute the blocking probability of the system at steady state for those parts requiring preprocessing in M_1.

Exercise 1

1. Consider the first node of the network at steady state:



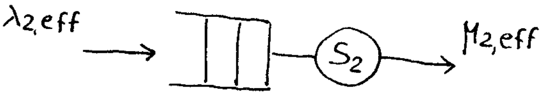
$$\begin{cases} \lambda_{1,eff} = \lambda_1 + p_1 \mu_{1,eff} \\ \lambda_{1,eff} = \mu_{1,eff} \end{cases} \Rightarrow \lambda_{1,eff} = \lambda_1 + p_1 \lambda_{1,eff} \Rightarrow \lambda_{1,eff} = \frac{\lambda_1}{1-p_1} = \frac{225}{4}$$

$$\rho_1 = \frac{\lambda_{1,eff}}{\mu_1} = \frac{25}{36} < 1 \quad \underline{ok}$$

$$E[X_1] = \frac{\rho_1}{1-\rho_1} = \frac{25}{11} \simeq 2.2727 \text{ customers}$$

$\downarrow$
length of the queue in the first node at steady state

Consider the second and third node of the network at steady state:



$$\lambda_{2,eff} = p_2 \mu_{1,eff} + \lambda_2 + q \mu_{3,eff}$$

$$\lambda_{2,eff} = \mu_{2,eff}$$

$$\lambda_{3,eff} = (1-p_1-p_2)\mu_{1,eff} + \mu_{2,eff}$$

$$\lambda_{3,eff} = \mu_{3,eff}$$

$$\begin{cases} -p_2 \lambda_{1,eff} + \lambda_{2,eff} - q \lambda_{3,eff} = \lambda_2 \\ (1-p_1-p_2)\lambda_{1,eff} + \lambda_{2,eff} - \lambda_{3,eff} = 0 \end{cases} \qquad \begin{cases} \lambda_{2,eff} - \frac{3}{4}\lambda_{3,eff} = \frac{51}{2} \\ \lambda_{2,eff} - \lambda_{3,eff} = -\frac{45}{2} \end{cases}$$

$$\begin{cases} \frac{1}{4}\lambda_{3,eff} = 48 \\ \lambda_{2,eff} = \lambda_{3,eff} - \frac{45}{2} \end{cases} \qquad \begin{cases} \lambda_{2,eff} = \frac{339}{2} \\ \lambda_{3,eff} = 192 \end{cases}$$

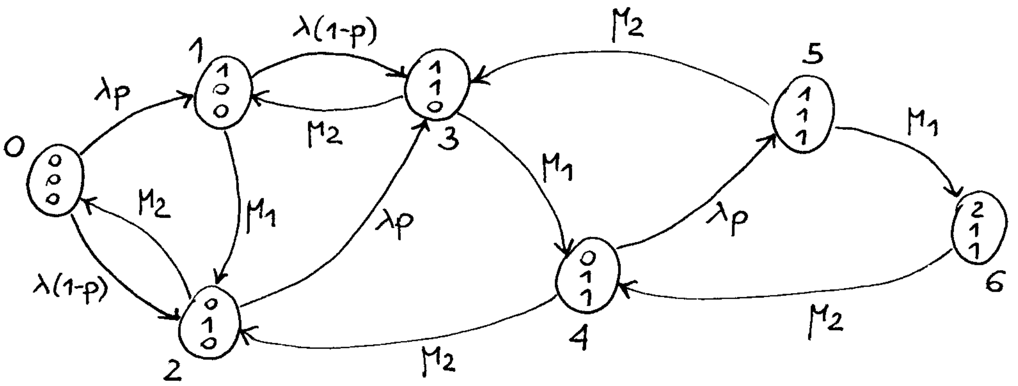
$$\rho_2 = \frac{\lambda_{2,eff}}{\mu_2} = \frac{\left(\frac{339}{2}\right)}{\mu_2} = \overset{=\rho_1}{\frac{25}{36}} \Rightarrow \mu_2 = \frac{6102}{25} \simeq 244.08$$

$$\rho_3 = \frac{\lambda_{3,eff}}{\mu_3} = \frac{192}{\mu_3} = \overset{=\rho_1}{\frac{25}{36}} \Rightarrow \mu_3 = \frac{6912}{25} \simeq 276.48$$

Exercise 2

For the stochastic timed automaton model of the system, see Exercise 4 of November 5, 2013.

Equivalent continuous-time homogeneous Markov chain (possible because the stochastic clock structure is exponential):



$$Q=\begin{bmatrix} -\lambda & \lambda p & \lambda(1-p) & 0 & 0 & 0 & 0 \\ 0 & -[\lambda(1-p)+\mu_1] & \mu_1 & \lambda(1-p) & 0 & 0 & 0 \\ \mu_2 & 0 & -(\lambda p+\mu_2) & \lambda p & 0 & 0 & 0 \\ 0 & \mu_2 & 0 & -(\mu_1+\mu_2) & \mu_1 & 0 & 0 \\ 0 & 0 & \mu_2 & 0 & -(\lambda p+\mu_2) & \lambda p & 0 \\ 0 & 0 & 0 & \mu_2 & 0 & -(\mu_1+\mu_2) & \mu_1 \\ 0 & 0 & 0 & 0 & \mu_2 & 0 & -\mu_2 \end{bmatrix}$$

The Markov chain is irreducible and finite; the stationary probabilities can be computed by solving:

$$\begin{cases} \pi Q = 0 \\ \sum \pi_i = 1 \end{cases}$$

Using Matlab:

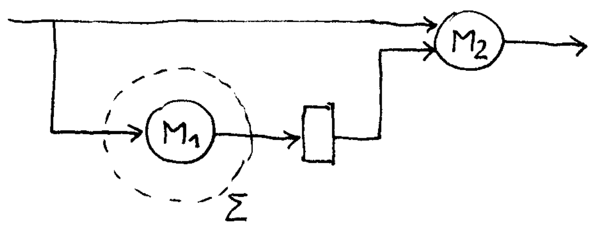
$$\begin{array}{cccccccc} \pi \simeq & [\,0.6964 & 0.0977 & 0.1741 & 0.0193 & 0.0115 & 0.0006 & 0.0004\,] \\ & \pi_0 & \pi_1 & \pi_2 & \pi_3 & \pi_4 & \pi_5 & \pi_6 \end{array}$$

1. $E[X] = 0 \cdot \pi_0 + 1 \cdot (\pi_1 + \pi_2) + 2 \cdot (\pi_3 + \pi_4) + 3 \cdot (\pi_5 + \pi_6) \simeq 0.3363$

$E[X]$: number of parts in the system at steady state

$X \in \{0, 1, 2, 3\}$

2. Consider a closed curve surrounding M_1 only:



$\Rightarrow$

$\lambda_\Sigma = \lambda p (\pi_0 + \pi_2 + \pi_4) \simeq 0.0588$

$E[X_\Sigma] = 0 \cdot (\pi_0 + \pi_2 + \pi_4) + 1 (\pi_1 + \pi_3 + \pi_5 + \pi_6)$

$\simeq 0.1180$

$E[X_\Sigma]$: number of parts in Σ at steady state

$X_\Sigma \in \{0, 1\}$

and apply the Little's law to Σ:

$$E[S_\Sigma] = \frac{E[X_\Sigma]}{\lambda_\Sigma} \simeq 2.0063$$

$E[S_\Sigma]$: time spent by a part in M_1 at steady state

Notice that $E[S_\Sigma] > \frac{1}{\mu_1} = 2.0$

Indeed, the time spent by a part in M_1 may include also the waiting time that the buffer is empty.

3. $\lambda_{eff} = \lambda p (\pi_0 + \pi_2 + \pi_4) + \lambda (1-p) (\pi_0 + \pi_1) \simeq 0.1647$

$\mu_{eff} = \mu_2 (\pi_2 + \pi_3 + \pi_4 + \pi_5 + \pi_6) \simeq 0.1647$

4. $U_1 = \pi_1 + \pi_3 + \pi_5 \simeq 0.1176$

U_1: utilization of M_1 at steady state

$U_2 = \pi_2 + \pi_3 + \pi_4 + \pi_5 + \pi_6 \simeq 0.2059$

U_2: utilization of M_2 at steady state

5. $P_{B,preproc} = \pi_1 + \pi_3 + \pi_5 + \pi_6 \simeq 0.1180$

$P_{B,preproc}$: blocking probability at steady state for those parts requiring preprocessing in M_1