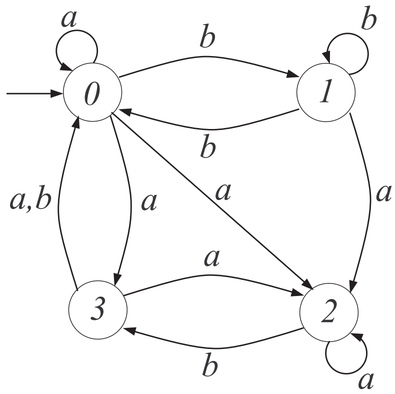
Test of Discrete Event Systems - 12.12.2013

Exercise 1

Consider the stochastic timed automaton $(\mathcal{E}, \mathcal{X}, \Gamma, p, x_0, F)$ in the figure, with $\mathcal{E} = \{a, b\}$, $p(0|0, a) = 1/2$, $p(2|0, a) = 1/3$, $p(1|1, b) = 1/4$, F_a and F_b exponential distributions with rates $\lambda_a = 1$ and $\lambda_b = 2$, respectively, and $P(X_{k+1} = 0|X_k = 3) = 8/9$.



1. Define a continuous-time homogeneous Markov chain with the same stochastic behavior as the given stochastic timed automaton

Exercise 2

An enzyme can be in one of three states: 0, when it is inhibited (it cannot act upon the substrate); 1, when it is not inhibited but does not act upon the substrate; 2, when it acts upon the substrate. Denote by $V(0)$, $V(1)$ and $V(2)$ the time spent by the enzyme in each of the three states. After time $V(0)$ spent in state 0, the enzyme enters state 1. After time $V(1)$ spent in state 1, the enzyme starts acting upon the substrate. After time $V(2)$ spent in state 2, the enzyme can be with probability p in state 0 (substrate inhibition), otherwise in state 1.

1. Discuss the conditions under which the enzyme dynamics can be modeled by a continuous-time homogeneous Markov chain, and define the corresponding Markov chain.

Exercise 3

A low-cost hotel has a small fitness centre with only two identical equipments. An arriving guest uses one of the equipments, if available, otherwise he/she comes back to his/her room. The fitness centre opens at 10 AM and closes at 10 PM. The guest arrival process can be modeled by a Poisson process with average frequency 4 guests/hour, whereas the use of an equipment has an exponentially distributed duration with expected value 30 minutes for each guest.

1. Compute the probability that at 5 PM both equipments are available.

Exercise 1

1. $p(0|0,a) = \frac{1}{2}$, $p(2|0,a) = \frac{1}{3}$ $\Rightarrow$ $p(3|0,a) = \frac{1}{6}$

$p(1|1,b) = \frac{1}{4}$ $\Rightarrow$ $p(0|1,b) = \frac{3}{4}$

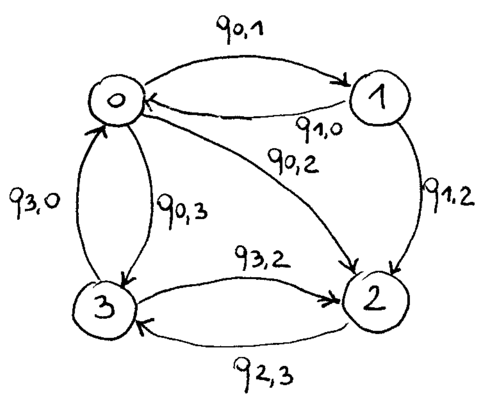
$$P(X_{k+1}=0|X_k=3) = \frac{\lambda_a}{\lambda_a+\lambda_b} p(0|3,a) + \frac{\lambda_b}{\lambda_a+\lambda_b} = \frac{1}{3} p(0|3,a) + \frac{2}{3}$$

$$P(X_{k+1}=0|X_k=3) = \frac{8}{9}$$

$$\frac{1}{3} p(0|3,a) + \frac{2}{3} = \frac{8}{9} \Rightarrow p(0|3,a) = \frac{2}{3}$$

$p(0|3,a) = \frac{2}{3}$ $\Rightarrow$ $p(2|3,a) = \frac{1}{3}$

Equivalent continuous-time homogeneous Markov chain:



$q_{0,1} = \lambda_b = 2$

$q_{0,2} = \lambda_a p(2|0,a) = \frac{1}{3}$

$q_{0,3} = \lambda_a p(3|0,a) = \frac{1}{6}$

$q_{1,0} = \lambda_b p(0|1,b) = \frac{3}{2}$

$q_{1,2} = \lambda_a = 1$

$q_{2,3} = \lambda_b = 2$

$q_{3,0} = \lambda_a p(0|3,a) + \lambda_b = \frac{8}{3}$

$q_{3,2} = \lambda_a p(2|3,a) = \frac{1}{3}$

$\Rightarrow (\mathcal{X}, Q, \pi_0)$, with:

- $$Q = \begin{bmatrix} -\frac{5}{2} & 2 & \frac{1}{3} & \frac{1}{6} \\ \frac{3}{2} & -\frac{5}{2} & 1 & 0 \\ 0 & 0 & -2 & 2 \\ \frac{8}{3} & 0 & \frac{1}{3} & -3 \end{bmatrix}$$

- $\pi_0 = [\,1 \quad 0 \quad 0 \quad 0\,]$ because the initial state is 0 with probability 1.

Exercise 2

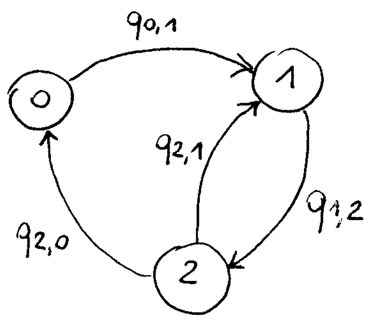
1. • The state holding times $V(0)$, $V(1)$ and $V(2)$ must be exponentially distributed.

   • The transition probabilities $p_{i,j}$ from state i to state j must depend only on i and j.

   ↳ this condition is satisfied since, according to the problem description:

$$p_{0,1} = 1$$
$$p_{1,2} = 1$$
$$p_{2,0} = p$$
$$p_{2,1} = 1-p$$

Assume that $V(0)$, $V(1)$ and $V(2)$ are exponentially distributed with rates λ_0, λ_1 and λ_2, respectively. Then, the enzyme dynamics can be modeled by the continuous-time homogeneous Markov chain:



$$Q = \begin{bmatrix} q_{0,0} & q_{0,1} & 0 \\ 0 & q_{1,1} & q_{1,2} \\ q_{2,0} & q_{2,1} & q_{2,2} \end{bmatrix}$$

with:

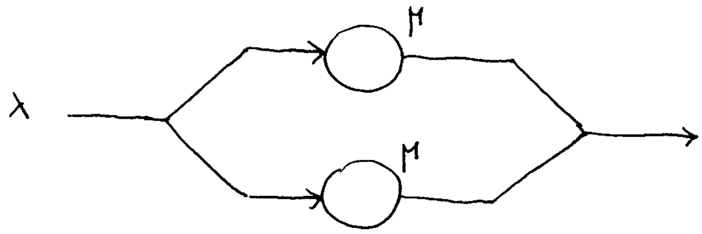
- $q_{0,0} = -\lambda_0 \Rightarrow q_{0,1} = \lambda_0$
- $q_{1,1} = -\lambda_1 \Rightarrow q_{1,2} = \lambda_1$
- $q_{2,2} = -\lambda_2$

$$p = p_{2,0} = \frac{q_{2,0}}{-q_{2,2}} = \frac{q_{2,0}}{\lambda_2} \Rightarrow q_{2,0} = \lambda_2 p$$

$$1-p = p_{2,1} = \frac{q_{2,1}}{-q_{2,2}} = \frac{q_{2,1}}{\lambda_2} \Rightarrow q_{2,1} = \lambda_2(1-p)$$

Exercise 3

The system can be modeled as a queueing system of the type:



⇓

equipments ≡ servers

no queue

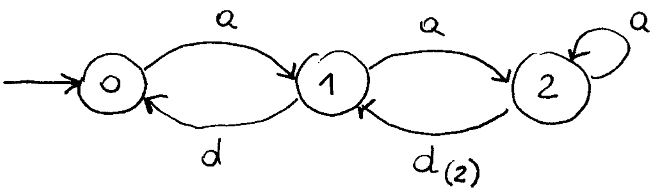
where $\lambda = 4$ guests/hour and $\frac{1}{\mu} = 30$ minutes $= 0.5$ hours $\Rightarrow \mu = 2$ guests/hour.

Stochastic timed automaton with exponential clock structure:

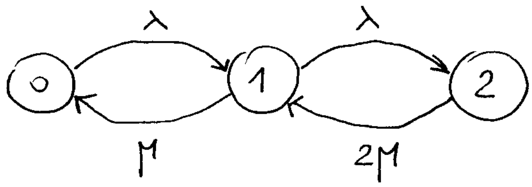
state x = number of guests in the fitness centre $\in \{0,1,2\}$

events $\mathcal{E} = \{a, d\}$

- a: arrival of a new guest
- d: termination of the use of an equipment



Equivalent continuous-time homogeneous Markov chain:



$$Q = \begin{bmatrix} -\lambda & \lambda & 0 \\ \mu & -(\lambda+\mu) & \lambda \\ 0 & 2\mu & -2\mu \end{bmatrix} = \begin{bmatrix} -4 & 4 & 0 \\ 2 & -6 & 4 \\ 0 & 4 & -4 \end{bmatrix}, \quad \pi_0 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

We are asked to compute $P(X(7)=0)$.

(7: time interval from 10 AM to 5 PM)

Recall that

$$\pi(7) = \left[\; P(X(7)=0) \quad P(X(7)=1) \quad P(X(7)=2) \;\right]$$

and

$$\pi(7) = \pi_0 \, e^{Qt}.$$

Using Matlab [>> pi7 = pi0 * expm(Q*7)]:

$$\pi(7) \simeq \left[\; \frac{1}{5} \quad \frac{2}{5} \quad \frac{2}{5} \;\right]$$

$$\Rightarrow P(X(7)=0) \simeq \frac{1}{5}.$$

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Procedure using Laplace transform:

$$\mathcal{L}\left[e^{Qt}\right] = sI - Q = \begin{bmatrix} s+4 & -4 & 0 \\ -2 & s+6 & -4 \\ 0 & -4 & s+4 \end{bmatrix} \qquad \det(sI-Q) = s(s+4)(s+10)$$

$$\Rightarrow (sI-Q)^{-1} = \frac{1}{\det(sI-Q)} \cdot \mathrm{Adj}(sI-Q) = \frac{1}{s(s+4)(s+10)} \begin{bmatrix} s^2+10s+8 & * & * \\ * & * & * \\ * & * & * \end{bmatrix}$$

↘ * denotes values that are not needed here

$$\Rightarrow P(X(t)=0) = \mathcal{L}^{-1}\left[\frac{s^2+10s+8}{s(s+4)(s+10)}\right] = \mathcal{L}^{-1}\left[\frac{1}{5}\cdot\frac{1}{s} + \frac{2}{3}\cdot\frac{1}{s+4} + \frac{2}{15}\cdot\frac{1}{s+10}\right]$$

$$= \frac{1}{5} + \frac{2}{3}e^{-4t} + \frac{2}{15}e^{-10t}, \quad t \geq 0$$

$$\Rightarrow P(X(7)=0) = \frac{1}{5} + \frac{2}{3}e^{-28} + \frac{2}{15}e^{-70} \simeq \frac{1}{5}$$

Notice that $e^{-4t} \simeq 0$ and $e^{-10t} \simeq 0$ for $t=7$

⇒ the system is in practice at steady state, and the approximate solution could be computed by solving

$$\begin{cases} \pi Q = 0 \\ \sum \pi_i = 1 \end{cases}$$