Test of Discrete Event Systems - 28.11.2013

Exercise 1

A mobile robot randomly moves along a circular path divided into three sectors. At every sampling instant the robot decides with probability p to move clockwise, and with probability $1-p$ to move counterclockwise. During a sampling interval the robot accomplishes the length of a sector.

1. Define a discrete time Markov chain for the random walk of the robot.
2. Study the stationary probability that the robot is localized in each sector for $p \in [0,1]$.

Let $p = 1/3$.

3. Choose an initial sector, and compute the average number of sampling intervals needed by the robot to return to it.
4. Choose an initial sector, and compute the probability that the robot returns to it in at most five sampling intervals.

Exercise 2

A virus can exist in N different strains, numbered from 1 to N. At each generation the virus mutates with probability $\alpha \in (0,1)$ to another strain which is chosen at random with equal probability.

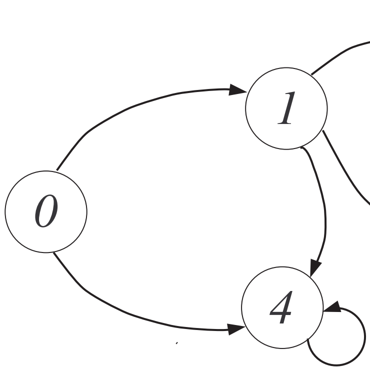
1. Compute the average number of generations to find the virus again in the same strain.

Let $N = 4$ and $\alpha = 2/3$.

2. Compute the probability that the strain in the sixth generation of the virus is the same as that in the first.
3. Assuming that the virus is initially either in strain 1 or 4, compute the probability that the virus never exists in strains 2 and 3 during the first six generations.

Exercise 3

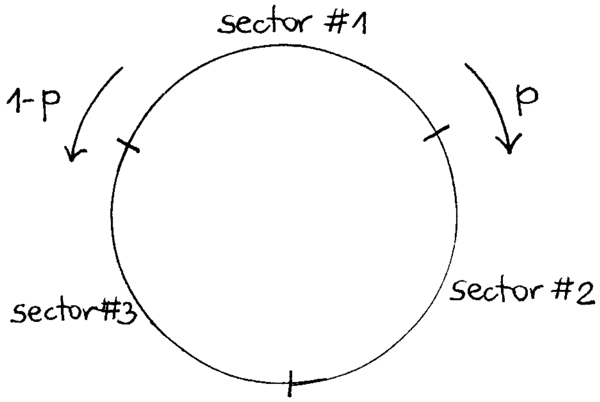
Consider the discrete-time homogeneous Markov chain whose state transition diagram is represented in the figure, and with transition probabilities $p_{0,1} = 1/3$, $p_{1,2} = 1/8$, $p_{1,3} = 1/4$ and $p_{2,3} = 4/5$.



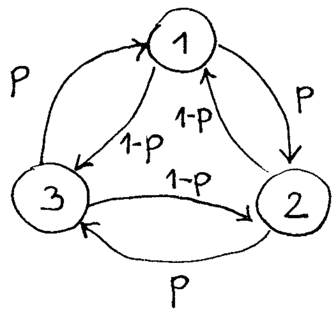
1. Compute the average recurrence time for each recurrent state.
2. Compute the stationary probabilities of all states, assuming the probability vector of the initial state $\pi(0) = [\ 1\ 0\ 0\ 0\ 0\ ]$.

Exercise 1

1. model

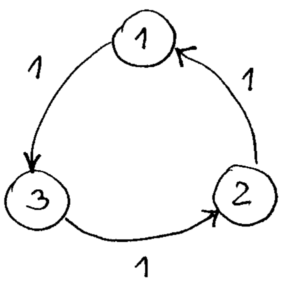


state X = sector occupied by the robot $\in \{1,2,3\}$



$$P = \begin{bmatrix} 0 & p & 1-p \\ 1-p & 0 & p \\ p & 1-p & 0 \end{bmatrix}$$

2. If $p=0$, the state transition diagram is as follows:



It turns out that each state of the Markov chain is periodic with period $d_i = 3$, $i=1,2,3$.
Therefore, the stationary state probabilities do not exist.

If $p=1$, the case is similar to $p=0$.

If $p \in (0,1)$, the Markov chain is irreducible, aperiodic and finite. Therefore, the stationary state probabilities exist.
The unique steady state probability vector is given by

$$\pi = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

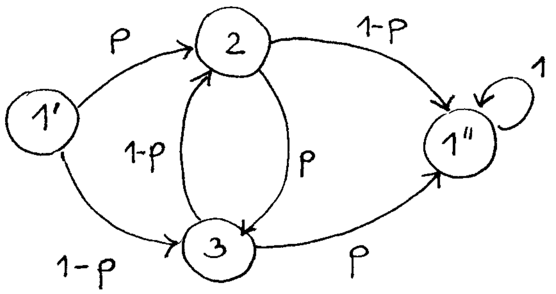
which is solution of the system of equations $\begin{cases} \pi = \pi P \\ \sum_{i=1}^{3} \pi_i = 1 \end{cases}$

3. Choose sector #1 as the initial sector (the solution does not depend on the particular initial sector). The answer corresponds to:

$$M_1 = E[T_{1,1}] = \frac{1}{\pi_1} = 3$$

for $p=\frac{1}{3}$ the Markov chain is irreducible, aperiodic and finite

4. Choose sector #1, and modify the model of item 1 as follows:



$\Rightarrow$ State 1 has been duplicated. The copy is made an absorbing state.

$$\tilde{P} = \begin{bmatrix} 0 & p & 1-p & 0 \\ 0 & 0 & p & 1-p \\ 0 & 1-p & 0 & p \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{with } p=\frac{1}{3}$$

$$\tilde{\pi}(t) = \left[P(\tilde{X}(t)=1') \quad P(\tilde{X}(t)=2) \quad P(\tilde{X}(t)=3) \quad P(\tilde{X}(t)=1'') \right]$$

$$\tilde{\pi}(0) = [\, 1 \quad 0 \quad 0 \quad 0 \,]$$

The answer corresponds to:

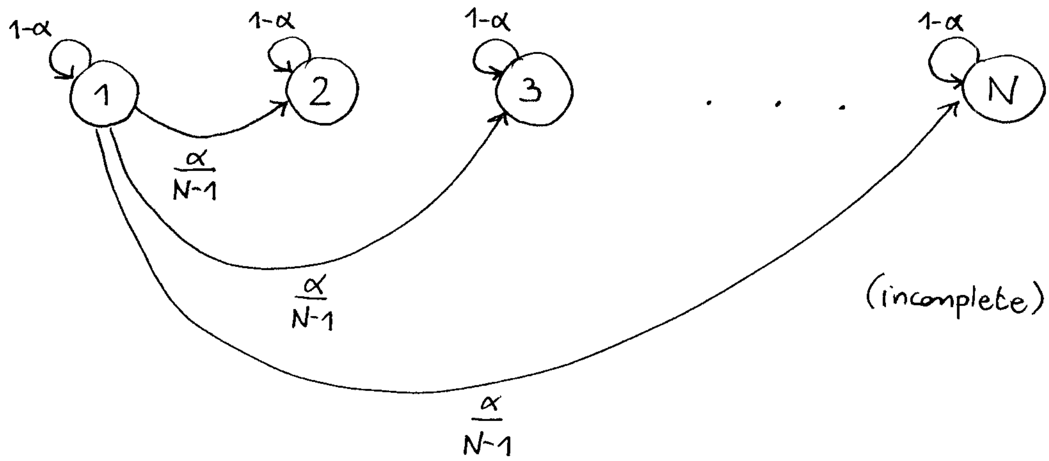
$$P(\tilde{X}(5) = 1'') = \frac{77}{81} \simeq 0.9506$$

$$\tilde{\pi}(5) = \tilde{\pi}(0)\tilde{P}^5 = \left[\, 0 \quad \frac{4}{243} \quad \frac{8}{243} \quad \frac{77}{81} \,\right]$$

Exercise 2

0. model (not explicitly required, but needed to answer the other questions).

state X = current strain of the virus $\in \{1, \dots, N\}$



(incomplete)

$$P = \begin{bmatrix} 1-\alpha & \frac{\alpha}{N-1} & \frac{\alpha}{N-1} & \cdots & \frac{\alpha}{N-1} \\ \frac{\alpha}{N-1} & 1-\alpha & \frac{\alpha}{N-1} & \cdots & \frac{\alpha}{N-1} \\ \vdots & & \ddots & & \vdots \\ \frac{\alpha}{N-1} & \frac{\alpha}{N-1} & \frac{\alpha}{N-1} & \cdots & 1-\alpha \end{bmatrix}$$

1. The answer corresponds to:

$$M_i = E[T_{i,i}] = \frac{1}{\pi_i} = N$$

irreducible, aperiodic and finite Markov chain

since the unique steady state probability vector is given by:

$$\pi = \left[\frac{1}{N} \; \frac{1}{N} \; \cdots \; \frac{1}{N} \right]$$

being solution of the system of equations $\begin{cases} \pi = \pi P \\ \sum_{i=1}^{N} \pi_i = 1 \end{cases}$

2. For $N=4$ and $\alpha=\frac{2}{3}$, we have:

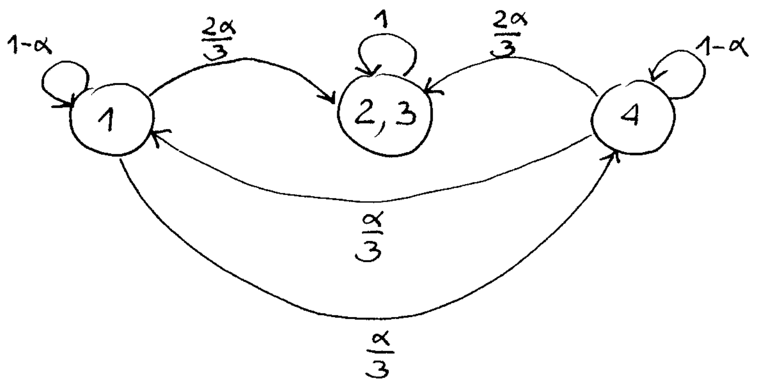
$$P=\begin{bmatrix} \frac{1}{3} & \frac{2}{9} & \frac{2}{9} & \frac{2}{9} \\ \frac{2}{9} & \frac{1}{3} & \frac{2}{9} & \frac{2}{9} \\ \frac{2}{9} & \frac{2}{9} & \frac{1}{3} & \frac{2}{9} \\ \frac{2}{9} & \frac{2}{9} & \frac{2}{9} & \frac{1}{3} \end{bmatrix}$$

The answer is

$$p_{i,i}(5)=\frac{4921}{19683}\simeq 0.2500 \quad \text{independent of } i=1,..,4$$

element (i,i) of $P^5=\begin{bmatrix} \frac{4921}{19683} & & * \\ & \ddots & \\ * & & \frac{4921}{19683} \end{bmatrix}$

3. We modify the model of item 0 for $N=4$ as follows:



$\Rightarrow$ states 2 and 3 are merged into one state, which is made absorbing.

$$\tilde{P}=\begin{bmatrix} 1-\alpha & \frac{2\alpha}{3} & \frac{\alpha}{3} \\ 0 & 1 & 0 \\ \frac{\alpha}{3} & \frac{2\alpha}{3} & 1-\alpha \end{bmatrix}$$

$$\tilde{\pi}(t)=\left[P(\tilde{X}(t)=1) \;\; P(\tilde{X}(t)=2,3) \;\; P(\tilde{X}(t)=4)\right]$$

$$\tilde{\pi}(0)=\left[\gamma \;\; 0 \;\; 1-\gamma\right], \quad \gamma\in[0,1]$$

The answer corresponds to:

$$P(\tilde{X}(5)\neq 2,3)=1-P(\tilde{X}(5)=2,3)=1-\frac{55924}{59049}=\frac{278}{5253}\simeq 0.0529 \quad \text{independent of } \gamma$$

$$\tilde{\pi}(5)=\tilde{\pi}(0)\tilde{P}^5=\left[\frac{\gamma+1562}{59049} \;\; \frac{55924}{59049} \;\; \frac{1563-\gamma}{59049}\right]$$

Exercise 3

1. The recurrent states are 2, 3 and 4. States 0 and 1 are transient.

   In order to compute $E[T_{2,2}]$ and $E[T_{3,3}]$, we observe that states 2 and 3 form a closed subset which is irreducible, aperiodic and finite, with transition probability matrix:

$$\tilde{P} = \begin{bmatrix} p_{2,2} & p_{2,3} \\ p_{3,2} & p_{3,3} \end{bmatrix} = \begin{bmatrix} \frac{1}{5} & \frac{4}{5} \\ 1 & 0 \end{bmatrix}$$

Solving:

$$\begin{cases} \tilde{\pi} = \tilde{\pi}\tilde{P} \\ \tilde{\pi}_2 + \tilde{\pi}_3 = 1 \end{cases}$$

we obtain $\tilde{\pi} = \begin{bmatrix} \frac{5}{9} & \frac{4}{9} \end{bmatrix}$. Therefore:

$$E[T_{2,2}] = \frac{1}{\left(\frac{5}{9}\right)} = \frac{9}{5} = 1.80$$

$$E[T_{3,3}] = \frac{1}{\left(\frac{4}{9}\right)} = \frac{9}{4} = 2.25$$

State 4 is absorbing, therefore:

$E[T_{4,4}] = 1.$

2. The transition probability matrix of the Markov chain is

$$P = \begin{bmatrix} 0 & \frac{1}{3} & 0 & 0 & \frac{2}{3} \\ 0 & 0 & \frac{1}{8} & \frac{1}{4} & \frac{5}{8} \\ 0 & 0 & \frac{1}{5} & \frac{4}{5} & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

We are asked to compute

$$\pi = \lim_{t\to\infty} \pi(0) P^t, \quad \text{with } \pi(0) = [1\ 0\ 0\ 0\ 0].$$

Notice that the Markov chain is <u>non-irreducible</u>!

A possible way to circumvent the computation of the limit, is as follows:

$$\left.\begin{array}{l}\pi_0 = 0\\ \pi_1 = 0\end{array}\right\} \text{states 0 and 1 are transient}$$

$$\pi_2 = \underbrace{P\left(\text{the chain enters the closed subset } \{2,3\}\right)}_{\substack{P(0\to 1)\left[P(1\to 2)+P(1\to 3)\right] \\ = \frac{1}{3}\left(\frac{1}{8}+\frac{1}{4}\right) = \frac{1}{8}}} \cdot \tilde{\pi}_2 = \frac{1}{8}\cdot\frac{5}{9} = \frac{5}{72}$$

computed in item 1: $\tilde{\pi}_2 = \frac{5}{9}$

$$\pi_3 = P\left(\text{the chain enters the closed subset } \{2,3\}\right)\cdot\tilde{\pi}_3 = \frac{1}{8}\cdot\frac{4}{9} = \frac{1}{18}$$

computed in item 1: $\tilde{\pi}_3 = \frac{4}{9}$

$$\pi_4 = P\left(\text{the chain enters the closed subset } \{4\}\right) =$$

$$= P(0\to 1)P(1\to 4) + P(0\to 4) = \frac{1}{3}\cdot\frac{5}{8} + \frac{2}{3} = \frac{7}{8}.$$

Therefore,

$$\pi = \left[\begin{array}{ccccc} 0 & 0 & \frac{5}{72} & \frac{1}{18} & \frac{7}{8}\end{array}\right].$$

<u>Remark</u>: Since the Markov chain is non-irreducible, it was not possible to compute π by solving $\begin{cases}\pi = \pi P\\ \sum_{i=0}^{4}\pi_i = 1\end{cases}$. Indeed,

$$\begin{cases} 0 = \pi_0 \\ \frac{1}{3}\pi_0 = \pi_1 \\ \frac{1}{8}\pi_1 + \frac{1}{5}\pi_2 + \pi_3 = \pi_2 \\ \frac{1}{4}\pi_1 + \frac{4}{5}\pi_2 = \pi_3 \\ \frac{2}{3}\pi_0 + \frac{5}{8}\pi_1 + \pi_4 = \pi_4 \\ \pi_0+\pi_1+\pi_2+\pi_3+\pi_4 = 1 \end{cases} \Rightarrow \begin{cases} \pi_0 = 0 \\ \pi_1 = 0 \\ \frac{4}{5}\pi_2 = \pi_3 \\ \cancel{\frac{4}{5}\pi_2 = \pi_3} \quad \text{redundant} \\ \pi_4 = \pi_4 \quad \Rightarrow \pi_4 \text{ can be chosen arbitrarily: } \pi_4 = \gamma,\ \gamma\in[0,1] \\ \pi_2+\pi_3+\pi_4 = 1 \end{cases}$$

$$\Rightarrow \pi_2 + \frac{4}{5}\pi_2 + \gamma = 1 \Rightarrow \pi_2 = \frac{5}{9}(1-\gamma), \quad \pi_3 = \frac{4}{9}(1-\gamma)$$

It follows that the system of equations:

$$\begin{cases} \pi = \pi P \\ \sum_{i=0}^{4} \pi_i = 1 \end{cases}$$

has infinite solutions parameterized by $\gamma \in [0,1]$.

Notice that γ can be interpreted as the probability that the chain enters the closed subset $\{4\}$, and therefore $1-\gamma$ is the probability that the chain enters the closed subset $\{2,3\}$. Some examples:

- if the initial state is 0, then $\gamma = \frac{7}{8}$, and therefore $1-\gamma = \frac{1}{8}$;
- if the initial state is either 2 or 3, then $\gamma = 0$, and therefore $1-\gamma = 1$,

ecc.