Test of Discrete Event Systems - 26.11.2013

Exercise 1

An electronic device driving the opening of a safe generates one of three numbers, 0, 1, or 2, according to the following rules:

i) if 0 was generated last, then the next number is 0 again with probability 1/2 or 1 with probability 1/2;

ii) if 1 was generated last, then the next number is 1 again with probability 2/5 or 2 with probability 3/5;

iii) if 2 was generated last, then the next number is either 0 with probability 7/10 or 1 with probability 3/10.

Moreover, the first generated number is 0 with probability 3/10, 1 with probability 3/10, and 2 with probability 2/5. The safe is opened the first time the sequence 120 takes place.

1. Define a discrete time Markov chain for the above described mechanism of opening the safe.
2. Compute the probability that the safe is opened with the fourth generated number.
3. Compute the average length of the sequence generated to open the safe.

Exercise 2

A communication node receives messages formed by sequences of bytes. The length L of a generic message (expressed in Kbytes) is a geometric random variable with parameter $q = 2/3$, namely $P(L = n) = (1 - q)^{n-1}q$, $n = 1, 2, \ldots$. The node has a dedicated buffer which may contain at most two messages waiting to be transmitted (independently of their length), and transmits over a dedicated line with constant channel capacity equal to 1000 Kbytes/s. During the time needed to transmit a single Kbyte, at most one new message may arrive with probability $p = 3/5$. If the buffer is full, the arriving message is rejected. It is assumed that the node is initially empty.

1. Define a discrete time Markov chain for the communication node.
2. Compute the probability that, after 5 ms from the initial time instant, there are exactly two messages in the node (one waiting and one being transmitted).
3. Compute the average number of messages in the node at steady state.

Exercise 1

1. model #1

We define the following values for the state:

1 : last number is ∅, not preceded by the subsequence 12

2 : last number is 1

3 : last number is 2, not preceded by 1

4 : last two numbers are 1 and 2 (subsequence 12)

5 : last three numbers are 1, 2 and ∅ (subsequence 120) => the safe is opened

The problen description provides the following conditional probabilities:

$p(0|0) = \frac{1}{2}$, $p(1|0) = \frac{1}{2}$, $p(1|1) = \frac{2}{5}$, $p(2|1) = \frac{3}{5}$,

next number / last number

$p(0|2) = \frac{7}{10}$, $p(1|2) = \frac{3}{10}$

Therefore, we have:

$p_{1,1} = p(0|0) = \frac{1}{2}$, $p_{1,2} = p(1|0) = \frac{1}{2}$

$p_{2,2} = p(1|1) = \frac{2}{5}$, $p_{2,4} = p(2|1) = \frac{3}{5}$

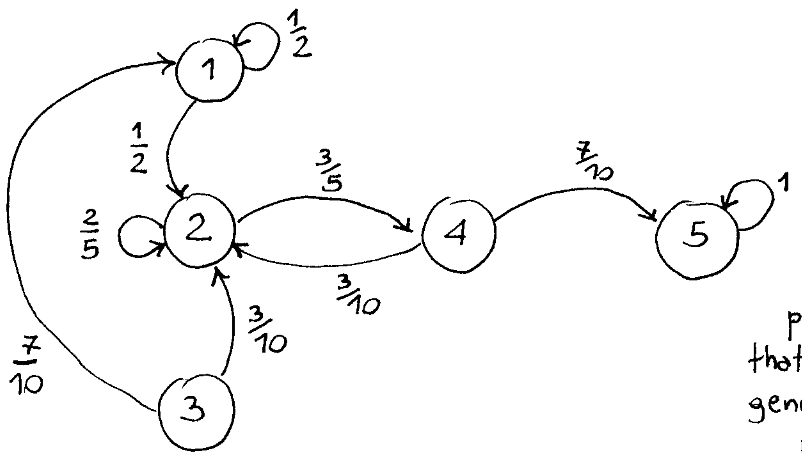
$p_{3,1} = p(0|2) = \frac{7}{10}$, $p_{3,2} = p(1|2) = \frac{3}{10}$

$p_{4,2} = p(1|2) = \frac{3}{10}$, $p_{4,5} = p(0|2) = \frac{7}{10}$

$p_{5,5} = 1$ (the safe is opened)

All other transition probabilities are ∅.

$$P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{2}{5} & 0 & \frac{3}{5} & 0 \\ \frac{7}{10} & \frac{3}{10} & 0 & 0 & 0 \\ 0 & \frac{3}{10} & 0 & 0 & \frac{7}{10} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$



The initial state probability vector is:

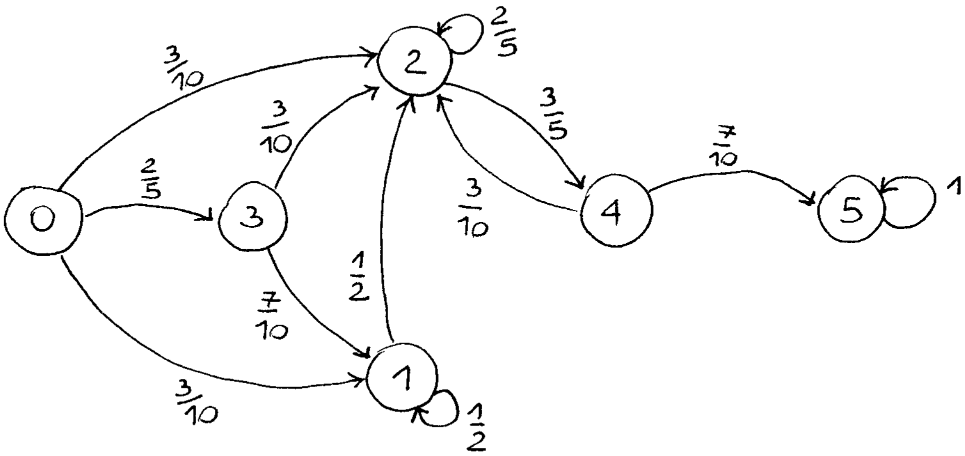
$$\Pi_0 = \begin{bmatrix} \frac{3}{10} & \frac{3}{10} & \frac{2}{5} & 0 & 0 \end{bmatrix}$$

probability that the first generated number is ∅

etc.

model #2

Another model can be derived from model #1 by adding an initial state 0 corresponding to the fact that no. number has been generated yet:



For this model, the matrix P is as follows:

$$P = \begin{bmatrix} 0 & \frac{3}{10} & \frac{3}{10} & \frac{2}{5} & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{2}{5} & 0 & \frac{3}{5} & 0 \\ 0 & \frac{7}{10} & \frac{3}{10} & 0 & 0 & 0 \\ 0 & 0 & \frac{3}{10} & 0 & 0 & \frac{7}{10} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

and the initial state probability vector is $\pi_0 = [1\ 0\ 0\ 0\ 0\ 0]$.

2. Using model #1, the answer is

$$P(X(3)=5, X(2)\neq 5) = P(X(3)=5, X(2)=4) = P(X(3)=5 \mid X(2)=4)\, P(X(2)=4)$$

$$= p_{4,5} \cdot \pi_4(2)$$

Note that, for model #1,
time t = (number of generated numbers − 1)

where $p_{4,5} = \frac{7}{10}$ and $\pi_4(2)$ can be computed through

$\pi(2) = \pi_0 P^2$, where $\pi(2) = [\pi_1(2)\ \pi_2(2)\ \pi_3(2)\ \pi_4(2)\ \pi_5(2)]$

It turns out that

$$\pi(2) = \left[\frac{43}{200} \quad \frac{17}{40} \quad 0 \quad \frac{117}{500} \quad \frac{63}{500} \right]$$

$\pi_4(2)$

and therefore

$$P(X(3)=5 \mid X(2) \neq 5) = \frac{7}{10} \cdot \frac{117}{500} = \frac{819}{5000} \simeq 0.1638$$

Using model #2, the answer is:

$$P(X(4)=5, X(3) \neq 5) = P(X(4)=5, X(3)=4) = P(X(4)=5 \mid X(3)=4)\, P(X(3)=4)$$

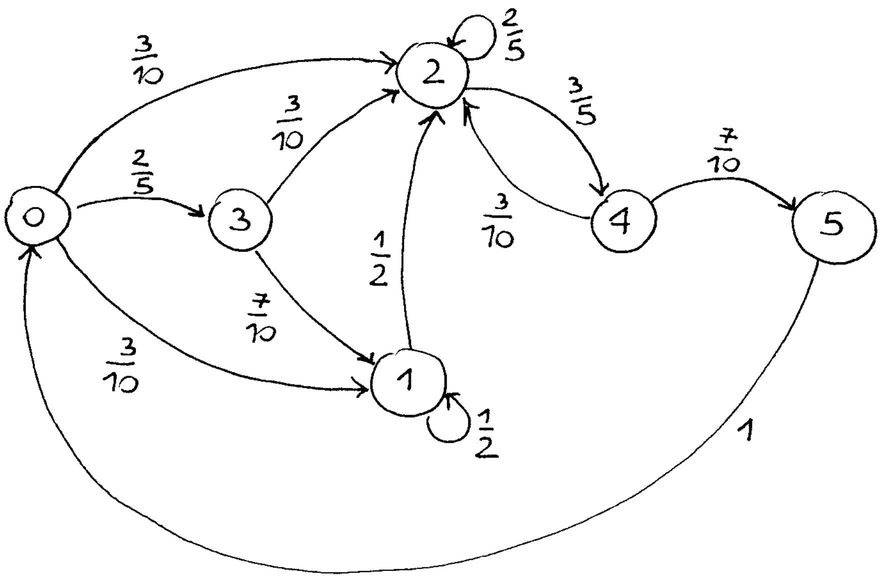
$$= p_{4,5} \cdot \pi_4(3)$$

$$= \frac{7}{10} \cdot \frac{117}{500} = \frac{819}{5000}$$

Note that, for model #2,
time t = number of generated numbers

Of course, the results obtained using the two models are equal.

3. We modify model #2 as follows:



In this way, the average length of the sequence generated to open the safe (denote it $E[N]$) is equal to the average recurrence time of state 0 ($M_0 = E[T_{0,0}]$) minus 1:

$$E[N] = E[T_{0,0}] - 1$$

Since the modified Markov chain is irreducible, aperiodic and finite, we know that we can compute M_0 as:

$$M_0 = \frac{1}{\pi_0}$$

where π_0 is the first element of the stationary state probability vector π obtained by solving:

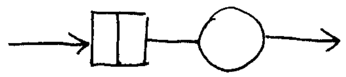
$$\begin{cases} \pi = \pi \tilde{P} \\ \sum_{i=0}^{5} \pi_i = 1 \end{cases} \qquad \text{where} \quad \tilde{P} = \begin{bmatrix} 0 & \frac{3}{10} & \frac{3}{10} & \frac{2}{5} & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{2}{5} & 0 & \frac{3}{5} & 0 \\ 0 & \frac{7}{10} & \frac{3}{10} & 0 & 0 & 0 \\ 0 & 0 & \frac{3}{10} & 0 & 0 & \frac{7}{10} \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \leftarrow !$$

$$\Rightarrow \pi = \left[\underbrace{\frac{525}{3869}}_{\pi_0} \; \frac{609}{3869} \; \frac{1250}{3869} \; \frac{210}{3869} \; \frac{271}{1398} \; \frac{525}{3869} \right]$$

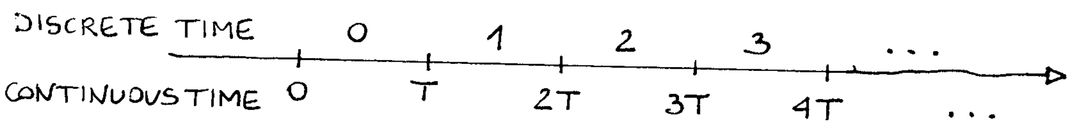
$$\Rightarrow E[N] = \frac{3869}{525} - 1 = \frac{3344}{525} \simeq 6.3695$$

Exercise 2

1. This is a queueing system with one server (the transmission line) and a buffer of capacity equal to 2:



The transmission rate is 1000 Kbytes/s. This implies that the time needed to transmit 1 Kbyte is $T = 1$ ms. We discretize the continuous time into intervals of length T:

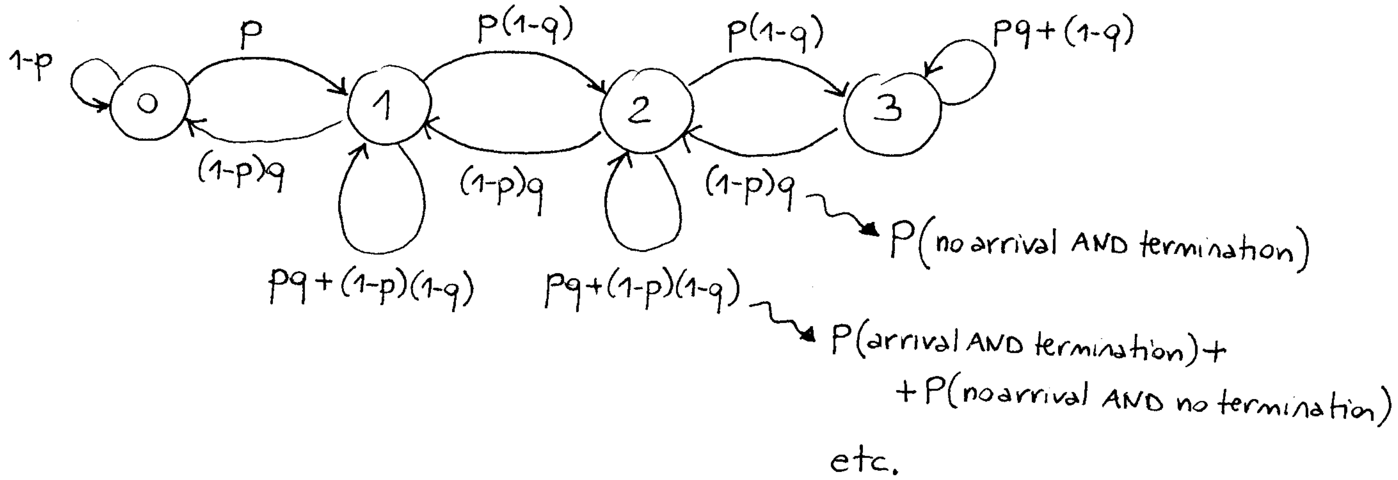


During each time interval of length T:

- the probability that one new message arrives is $p = \frac{3}{5}$
- the probability that the transmission of a message terminates (provided that the transmission line is busy) is $q = \frac{2}{3}$.

The length L of a message is expressed in Kbytes and follows a geometric distribution with parameter q. Therefore, the transmission of a message can be seen as a Bernoulli process with probability of success q during each time interval of length T.

The graph of the Markov chain looks as follows, by defining the state as the number of messages in the system ($X \in \{0, 1, 2, 3\}$):



The transition probability matrix P is:

$$P = \begin{bmatrix} \frac{2}{5} & \frac{3}{5} & 0 & 0 \\ \frac{4}{15} & \frac{8}{15} & \frac{1}{5} & 0 \\ 0 & \frac{4}{15} & \frac{8}{15} & \frac{1}{5} \\ 0 & 0 & \frac{4}{15} & \frac{11}{15} \end{bmatrix}$$

, and the initial state probability vector:

$$\pi_0 = [1 \quad 0 \quad 0 \quad 0]$$

2. $5\text{ ms} = 5T \Rightarrow$ discrete-time $t = 5$

The answer is:

$$P(X(5)=2) = \pi_2(5) = \frac{198}{893} \simeq 0.2217$$

taken from $\pi(5) = \pi_0 P^5 = \left[\frac{369}{1595} \quad \frac{1341}{2870} \quad \frac{198}{893} \quad \frac{249}{3125} \right]$

$\pi_2(5)$

3. The answer is:

$$E[X] = 0 \cdot \pi_0 + 1 \cdot \pi_1 + 2 \cdot \pi_2 + 3 \cdot \pi_3 = \frac{603}{397} \simeq 1.5189$$

where $\pi = \left[\frac{64}{397} \quad \frac{144}{397} \quad \frac{108}{397} \quad \frac{81}{397} \right]$ is computed by solving $\begin{cases} \pi = \pi P \\ \sum_{i=0}^{3} \pi_i = 1 \end{cases}$

(the Markov chain is irreducible, aperiodic and finite).