Test of Discrete Event Systems - 12.11.2013

Exercise 1

A simple manufacturing system is composed by a buffer with unitary storage capacity and a machine. Parts processed in the machine turn out to be defective with probability $p = \frac{1}{8}$. Any defective part after the first processing, is immediately reprocessed. If a part is still defective after the second processing, it is rejected. Parts are delivered one at a time to the manufacturing system by a mobile robot equipped with a mechanical arm. Executing this task takes a random duration following an exponential distribution with expected value $\frac{1}{\lambda} = 4$ minutes. Arrival of new parts is suspended while the manufacturing system is full. Processing of one part takes a random duration following an exponential distribution with expected value $\frac{1}{\mu} = 2$ minutes.

1. Model the manufacturing system through a stochastic timed automaton $(\mathcal{E}, \mathcal{X}, \Gamma, p, x_0, F)$, assuming that the system is initially empty.
2. Assume that only one part is in the system, and this part is being processed for the first time. Compute the probability that the part is rejected before the arrival of a new part.
3. Compute the probability distribution function of the total time Z which a generic part spends in the machine (i.e. including the possible reprocessing).

Exercise 2

In a post office there are two desks. Desk 1 serves with highest priority customers needing to perform payments (type A), whereas desk 2 serves with highest priority customers requiring mail services (type B). However, if desk 1 is busy and there are no type B waiting customers, desk 2 can serve type A customers; vice versa, if desk 2 is busy and there are no type A waiting customers, desk 1 can serve type B customers. Both interarrival and service times follow exponential distributions. For the arrivals of type A customers, the average rate is 14 customers/hour, whereas for the arrivals of type B customers, the average rate is 8 customers/hour. Average service times do not depend on the desk, and are equal to 3 minutes for type A customers and 5 minutes for type B customers.

1. Model the post office through a stochastic timed automaton $(\mathcal{E}, \mathcal{X}, \Gamma, f, x_0, F)$, assuming that the system is initially empty.
2. Compute the average holding time in a state where there are two type A customers in the system, and none of type B.
3. Assume that there are two type A customers in the system, and none of type B. Compute the probability that the next arriving customer is of type B and finds desk 2 available.
4. Assume that there are one type A customer and one type B customer in the system. Compute the probability that there are exactly 6 arrivals of customers in 4 minutes, and no one of them is admitted to the service in the same time interval.

Exercise 3

The call-center of an insurance company is equipped with 10 phone lines, but only three operators. This means that calling customers exceeding the number of three, are put on hold. Phone calls arrive at the call-center as generated by a Poisson process with average rate equal to 2 calls/minute. Time needed to answer a query is independent of the operator, and follows an exponential distribution with average 5 minutes. The call-center is open 24 hours a day.

1. Model the call-center through a stochastic timed automaton $(\mathcal{E}, \mathcal{X}, \Gamma, f, x_0, F)$, assuming that the three operators are initially idle.

2. Assume that the three operators are idle. Compute the probability that the fourth calling customer is put on hold and, in such a case, the average waiting time to be served.

Exercise 4

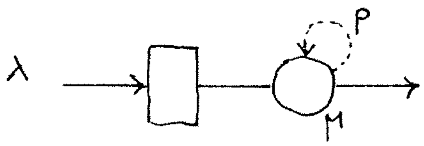
A small bank office has only one desk and a waiting room with 5 chairs. The desk opens at 9 AM, but customers may enter the waiting room starting from 8:30 AM. Customers arrive as generated by a Poisson process with average rate $\lambda = 4$ arrivals/hour, whereas the service time at the desk follows an exponential distribution with expected value 20 minutes.

1. Model the bank office starting from the opening of the desk through a stochastic timed automaton $(\mathcal{E}, \mathcal{X}, \Gamma, f, p_0, F)$.

2. Compute the probability that the second customer arriving after the opening of the desk finds at least one available place in the waiting room.

Exercise 1

1. The system can be represented as follows:



$p = \frac{1}{8}$

$\frac{1}{\lambda} = 4$ minutes $\Rightarrow$ $\lambda = 0.25$ arrivals/min

$\frac{1}{\mu} = 2$ minutes $\Rightarrow$ $\mu = 0.5$ services/min

Definition of state:

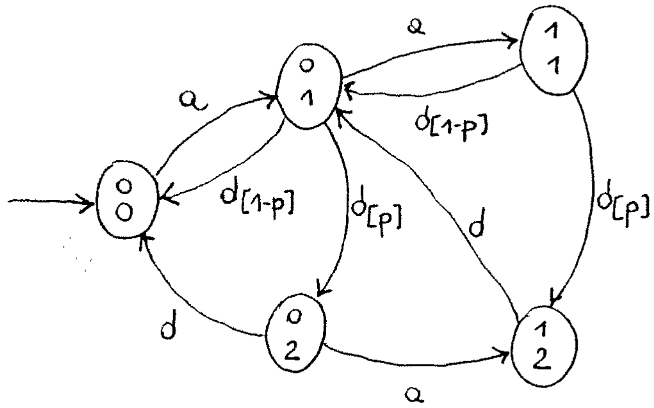
$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \begin{array}{l} \rightarrow \text{buffer: empty (0), full (1)} \\ \rightarrow \text{machine: idle (0), working - first processing (1), working - second processing (2)} \end{array}$$

$$\Rightarrow \mathcal{X} = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$$

Events: $\mathcal{E} = \{a, d\}$

a: arrival of a new part

d: termination of a service in the machine



$F_a(t) = 1 - e^{-\lambda t}, \quad t \geq 0$

$F_b(t) = 1 - e^{-\mu t}, \quad t \geq 0$

2. The current state is $X_k = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. There is only one possible case:

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \xrightarrow{d[p]} \begin{bmatrix} 0 \\ 2 \end{bmatrix} \xrightarrow{d[p]}$$

$$\Rightarrow P(\ldots) = \frac{\mu}{\lambda+\mu} \cdot p \cdot \frac{\mu}{\lambda+\mu} \cdot p = \left(\frac{\mu p}{\lambda+\mu} \right)^2 = \frac{1}{144} \simeq 0.069$$

3. We need to compute $P(Z \leq t)$, $t \geq 0$.

probability distribution function of the random variable Z

There are two possible cases:

- If the part is not defective after the first processing:

$Z = Z_1$ where Z_1 is the duration of the first processing

- If the part is defective after the first processing:

$Z = Z_1 + Z_2$ where Z_i is the duration of the i-th processing, $i=1,2$.

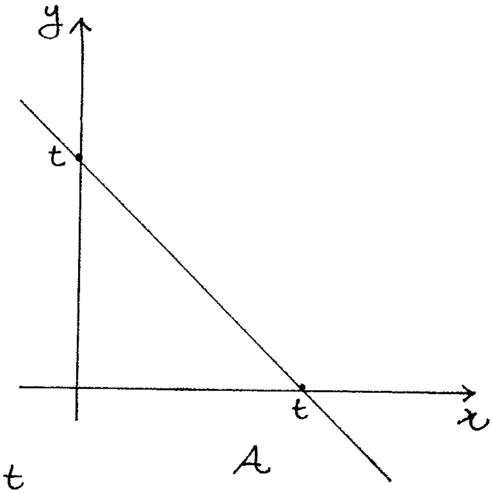
$$\Rightarrow P(Z \leq t) = P(Z \leq t \mid \text{part not defective}) P(\text{part not defective}) +$$
$$+ P(Z \leq t \mid \text{part defective}) P(\text{part defective}) =$$
$$= P(Z_1 \leq t) \cdot (1-p) + P(Z_1 + Z_2 \leq t) \cdot p$$

Recall that Z_1 and Z_2 are independent and exponentially distributed with rate μ.

$$\Rightarrow P(Z_1 \leq t) = 1 - e^{-\mu t}, \quad t \geq 0$$

$$\Rightarrow P(Z_1 + Z_2 \leq t) = \iint_A f_{Z_1}(x) f_{Z_2}(y) \, dx \, dy$$

where $f_{Z_i}(z) = \begin{cases} \mu e^{-\mu z} & \text{if } z \geq 0 \\ 0 & \text{otherwise} \end{cases}$, $i=1,2$ and $A = \{(x,y) \in \mathbb{R}^2 : x + y \leq t\}$



$$P(Z_1 + Z_2 \leq t) = \int_0^t \mu e^{-\mu x} \int_0^{-x+t} \mu e^{-\mu y} \, dy \, dx$$

$$= \int_0^t \mu e^{-\mu x} \left[-e^{-\mu y} \right]_0^{-x+t} dx$$

$$= \int_0^t \mu e^{-\mu x} \cdot \left(1 - e^{\mu x} \cdot e^{-\mu t}\right) dx$$

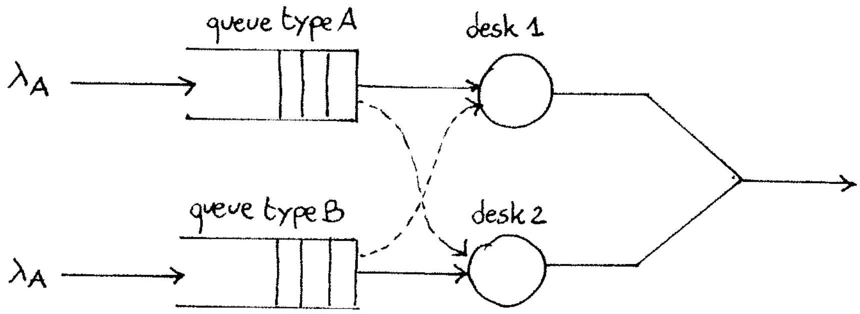
$$= \int_0^t \left(\mu e^{-\mu x} - \mu e^{-\mu t}\right) dx = \left[-e^{-\mu x} - \mu e^{-\mu t} \cdot x \right]_0^t$$

$$= 1 - e^{-\mu t} - \mu e^{-\mu t} \cdot t = 1 - e^{-\mu t}(1 + \mu t), \quad t \geq 0$$

$$\Rightarrow P(Z \leq t) = (1 - e^{-\mu t}) \cdot (1-p) + \left[1 - e^{-\mu t}(1+\mu t)\right] \cdot p = 1 - e^{-\mu t}(1 + p\mu t), \quad t \geq 0$$

Exercise 2

1. The system can be represented as follows:



Definition of state:

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \begin{array}{l} \to \text{desk 1: idle (0), serving A (A), serving B (B)} \\ \to \text{desk 2: idle (0), serving A (A), serving B (B)} \\ \to \text{length of queue type A} \in \{0,1,2,\dots\} \\ \to \text{length of queue type B} \in \{0,1,2,\dots\} \end{array}$$

Events: $\mathcal{E} = \{a_A, a_B, d_{A1}, d_{A2}, d_{B1}, d_{B2}\}$

a_A: arrival type A

a_B: arrival type B

d_{A1}: termination of a type A service in desk 1

...

Since in this case the cardinality of the state space is infinite, it is more convenient to define the state automaton in a formal way.

- function Γ:

$$\Gamma\left(\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right) = \{a_A, a_B\}$$

$$\Gamma\left(\begin{bmatrix} A \\ 0 \\ 0 \\ 0 \end{bmatrix}\right) = \{a_A, a_B, d_{A1}\}, \quad \Gamma\left(\begin{bmatrix} B \\ 0 \\ 0 \\ 0 \end{bmatrix}\right) = \{a_A, a_B, d_{B1}\},$$

$$\Gamma\left(\begin{bmatrix} 0 \\ A \\ 0 \\ 0 \end{bmatrix}\right) = \{a_A, a_B, d_{A2}\}, \quad \Gamma\left(\begin{bmatrix} 0 \\ B \\ 0 \\ 0 \end{bmatrix}\right) = \{a_A, a_B, d_{B2}\}$$

$$\Gamma\left(\begin{bmatrix} A \\ A \\ x_3 \\ x_4 \end{bmatrix}\right) = \{a_A, a_B, d_{A1}, d_{A2}\}, \quad \Gamma\left(\begin{bmatrix} A \\ B \\ x_3 \\ x_4 \end{bmatrix}\right) = \{a_A, a_B, d_{A1}, d_{B2}\},$$

$$\Gamma\left(\begin{bmatrix} B \\ A \\ x_3 \\ x_4 \end{bmatrix}\right) = \{a_A, a_B, d_{B1}, d_{A2}\}, \quad \Gamma\left(\begin{bmatrix} B \\ B \\ x_3 \\ x_4 \end{bmatrix}\right) = \{a_A, a_B, d_{B1}, d_{B2}\}, \quad x_3 \geq 0, \ x_4 \geq 0$$

- function f:

$$f\left(\begin{bmatrix}0\\0\\0\\0\end{bmatrix}, a_A\right) = \begin{bmatrix}A\\0\\0\\0\end{bmatrix}, \quad f\left(\begin{bmatrix}0\\0\\0\\0\end{bmatrix}, a_B\right) = \begin{bmatrix}0\\B\\0\\0\end{bmatrix}$$

$$f\left(\begin{bmatrix}A\\0\\0\\0\end{bmatrix}, a_A\right) = \begin{bmatrix}A\\A\\0\\0\end{bmatrix}, \quad f\left(\begin{bmatrix}A\\0\\0\\0\end{bmatrix}, a_B\right) = \begin{bmatrix}A\\B\\0\\0\end{bmatrix}, \quad f\left(\begin{bmatrix}A\\0\\0\\0\end{bmatrix}, d_{A1}\right) = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

... (omitted)

$$f\left(\begin{bmatrix}A\\A\\x_3\\x_4\end{bmatrix}, a_A\right) = \begin{bmatrix}A\\A\\x_3+1\\x_4\end{bmatrix}, \quad f\left(\begin{bmatrix}A\\A\\x_3\\x_4\end{bmatrix}, a_B\right) = \begin{bmatrix}A\\A\\x_3\\x_4+1\end{bmatrix}$$

$$f\left(\begin{bmatrix}A\\A\\x_3\\x_4\end{bmatrix}, d_{A1}\right) = \begin{cases}\begin{bmatrix}A\\A\\x_3-1\\x_4\end{bmatrix} & \text{if } x_3 \geq 1\\ \begin{bmatrix}B\\A\\0\\x_4-1\end{bmatrix} & \text{if } x_3=0,\ x_4\geq 1\\ \begin{bmatrix}0\\A\\0\\0\end{bmatrix} & \text{if } x_3=0,\ x_4=0\end{cases}$$

$$f\left(\begin{bmatrix}A\\A\\x_3\\x_4\end{bmatrix}, d_{A_2}\right) = \begin{cases}\begin{bmatrix}A\\B\\x_3\\x_4-1\end{bmatrix} & \text{if } x_4 \geq 1\\ \begin{bmatrix}A\\A\\x_3-1\\0\end{bmatrix} & \text{if } x_3\geq 1,\ x_4=0\\ \begin{bmatrix}A\\0\\0\\0\end{bmatrix} & \text{if } x_3=0,\ x_4=0\end{cases}$$

... (omitted)

- initialstate x_0:

$$x_0 = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

- <u>stochastic clock structure F:</u>

$F_{a_A}(t) = 1 - e^{-\lambda_A t}, \quad t \geq 0, \quad$ where $\lambda_A = 14$ arrivals/hour

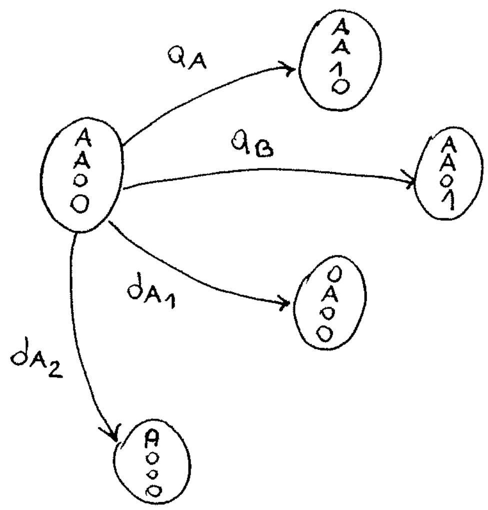
$F_{a_B}(t) = 1 - e^{-\lambda_B t}, \quad t \geq 0, \quad$ where $\lambda_B = 8$ arrivals/hour

$F_{d_{A_1}}(t) = F_{d_{A_2}}(t) = 1 - e^{-\mu_A t}, \quad t \geq 0, \quad$ where $\mu_A = 20$ services/hour

$F_{d_{B_1}}(t) = F_{d_{B_2}}(t) = 1 - e^{-\mu_B t}, \quad t \geq 0, \quad$ where $\mu_B = 12$ services/hour

2. The current state is $X_k = \begin{bmatrix} A \\ A \\ 0 \\ 0 \end{bmatrix}$.

"Local" modeling:



The holding time $V(X_k)$ in state X_k follows an exponential distribution with rate $\lambda_A + \lambda_B + 2\mu_A$:

$$\Rightarrow E[V(X_k)] = \frac{1}{\lambda_A + \lambda_B + 2\mu_A} = \frac{1}{62} \text{ hours} \simeq 58 \text{ seconds}$$

3. The current state is $X_k = \begin{bmatrix} A \\ A \\ 0 \\ 0 \end{bmatrix}$. There must be an event d_{A_2} before any arrival; then the next arrival must be of type B:

$$\Rightarrow P(\ldots) = \frac{\mu_A}{\lambda_A + \lambda_B + \mu_A} \cdot \frac{\lambda_B}{\lambda_A + \lambda_B} = \frac{40}{231} \simeq 0.1732$$

Note that we ignore event d_{A_1} in state $X_k = \begin{bmatrix} A \\ A \\ 0 \\ 0 \end{bmatrix}$...

4. The current state is either $X_k = \begin{bmatrix} A \\ B \\ 0 \\ 0 \end{bmatrix}$ or $X_k = \begin{bmatrix} B \\ A \\ 0 \\ 0 \end{bmatrix}$.

$$\Rightarrow P(\ldots) = P\left(N_a(T)=6,\ Y_{d_{A,k}} > T,\ Y_{d_{B,k}} > T\right) =$$

events are independent

residual lifetime of event d_B

residual lifetime of event d_A

$$= P(N_a(T)=6)\,P(Y_{d_{A,k}} > T)\,P(Y_{d_{B,k}} > T) =$$

$$= \frac{(\lambda T)^6}{6!} e^{-\lambda T} \cdot e^{-\mu_A T} \cdot e^{-\mu_B T}$$

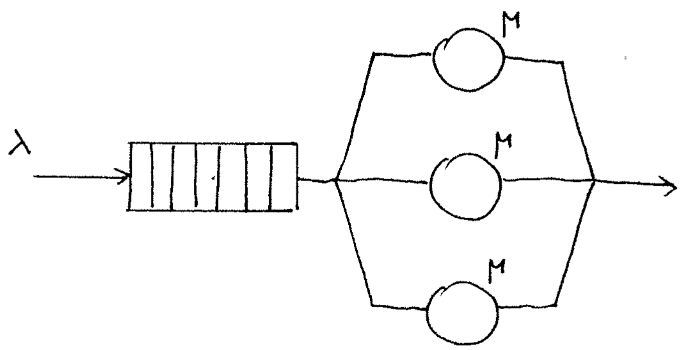
$$\simeq 3.78 \cdot 10^{-4}$$

The arrival process is a Poisson process with rate $\lambda = \lambda_A + \lambda_B$ (superposition of two Poisson processes with rates λ_A and λ_B)

$T = 4$ minutes $= \frac{1}{15}$ hours

Exercise 3

1. The system can be represented as follows:



$\lambda = 2$ arrivals/min

$\frac{1}{\mu} = 5$ minutes $\Rightarrow \mu = 0.2$ services/min

Definition of state:

$x = \#$ calls in the system $\in \{0, 1, 2, \ldots, 10\}$

Note that there is no need to distinguish which server is busy and which not, since all servers are equal.

Events:

$\mathcal{E} = \{a, d\}$

arrival of a new call

termination of a service

4. The current state is either $X_k = \begin{bmatrix} A \\ B \\ 0 \\ 0 \end{bmatrix}$ or $X_k = \begin{bmatrix} B \\ A \\ 0 \\ 0 \end{bmatrix}$.

$$\Rightarrow P(\ldots) = P\left(N_a(T)=6,\ Y_{d_{A,k}} > T,\ Y_{d_{B,k}} > T\right) =$$

events are independent

residual lifetime of event d_B

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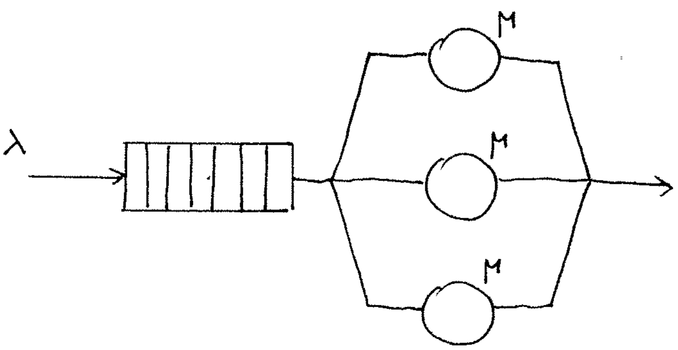

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