Test of Discrete Event Systems - 05.11.2013

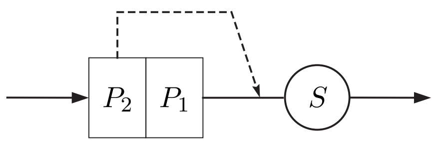
Exercise 1

A small hair salon has two chairs and two hairdressers. Customers arrive according to a Poisson process with rate 3 arrivals/hour. A customer is male with probability $p = 1/3$. The duration of a hair-cut is independent of the hairdresser, but depends on the sex of the customer. It is exponentially distributed with expected value 20 minutes for men, and 45 minutes for women. Since the hair salon does not have a waiting room, customers arriving when both chairs are busy decide to abandon the hair salon.

1. Model the hair salon through a stochastic timed automaton $(\mathcal{E}, \mathcal{X}, \Gamma, p, x_0, F)$, assuming the salon empty at the opening.
2. Compute the probability that the third customer arriving after the opening abandons the hair salon.
3. Compute the probability that exactly three men arrive during one hour (the number of women is not specified).

Exercise 2

Consider the queueing system depicted in the figure, and formed by a server S preceded by a queue with two places P_1 and P_2. When the queue is full and the server S completes the service, with probability $p = 1/4$ the customer waiting in P_2 is admitted to the service before the customer waiting in P_1 ("overtaking"). A customer arriving when the queue in full, is not admitted to the system. Assume that the customers arrive according to a Poisson process with rate $\lambda = 0.1$ arrivals/min, and the service times in S are exponentially distributed with rate $\mu = 0.2$ services/min, respectively.



1. Model the queueing system through a stochastic timed automaton $(\mathcal{E}, \mathcal{X}, \Gamma, p, x_0, F)$, assuming the system initially empty.
2. Assume that the queue is full and the server S is busy. Compute the probability that the customer waiting in P_1 is overtaken exactly twice before being admitted to the service.
3. Assume that the queue is full and the server S is serving a customer who did not overtake. Compute the probability that, after the next event, the server S is serving a customer who did not overtake.
4. Determine the probability distribution function of the holding time in a state where the server S is busy and only one customer is waiting.

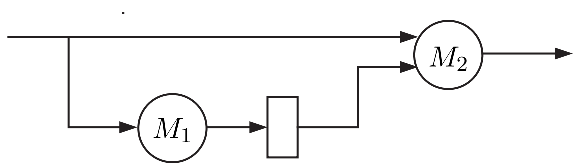
Exercise 3

Consider a queueing system formed by a server and a queue with only one place. Customers arrive according to a Poisson process with rate $\lambda = 1$ arrivals/min, while service times have constant duration equal to $t_s = 3$ min. The queueing system is initially empty.

1. Compute the probabiliy that the queueing system is empty at time $t = 4$ min.

Exercise 4

Consider the queueing system depicted in the figure.



Arriving parts may need preprocessing in M_1 with probability $p = 1/3$, otherwise they go directly to M_2. When a part arrives and the corresponding server is not available, the part is rejected. Between M_1 and M_2 there is a buffer with only one place. When M_1 terminates preprocessing of a part and M_2 is busy, the part is moved to the buffer, if it is empty. Otherwise, the part is kept in M_1, that therefore remains unavailable for a new job until M_2 terminates its job. Parts arrive according to a Poisson process with expected interarrival time equal to 5 min, whereas service times in M_1 and M_2 follow exponential distributions with rates $\mu_1 = 0.5$ services/min and $\mu_2 = 0.8$ services/min, respectively.

1. Model the queueing system through a stochastic timed automaton $(\mathcal{E}, \mathcal{X}, \Gamma, p, x_0, F)$, assuming the system initially empty.

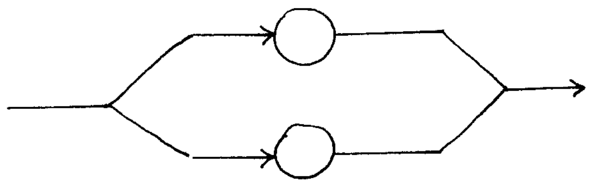
2. Assume that M_1 is idle, the buffer is full, and M_2 is busy. Compute the probability that the two parts in the system exit the system before the acceptance of other parts.

3. Assume that M_1 is busy, the buffer is empty and M_2 is idle. Compute the expected time to the start of a job in M_2.

4. Compute the probability that the first arriving part needs preprocessing in M_1 and no arriving parts go directly to M_2 while the first is being preprocessed.

Exercise 1

1. The system can be represented as a queueing system with two servers and no queueing space:



Definition of state: $x = \begin{bmatrix} x_M \\ x_F \end{bmatrix}$ $x_M \rightarrow$ number of male customers $\in \{0,1,2\}$, $x_F \rightarrow$ number of female customers $\in \{0,1,2\}$

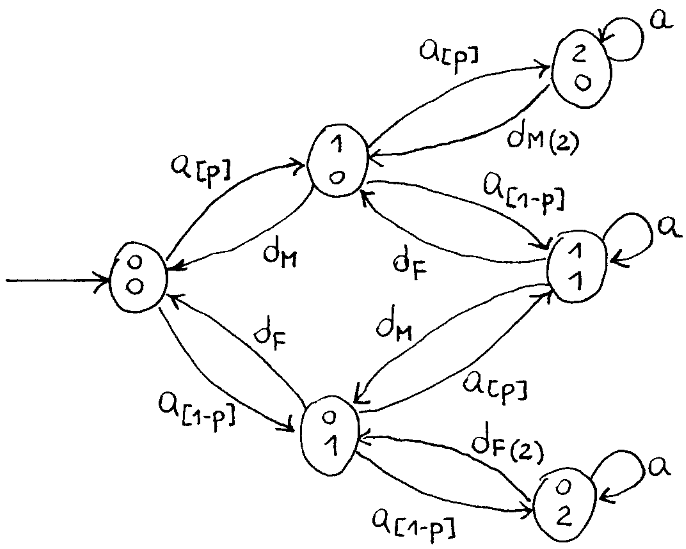
State space: $\mathcal{X} = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \end{bmatrix} \right\}$ => 6 states

Definition of events: $\mathcal{E} = \{a, d_M, d_F\}$

a → arrival of a customer

d_M → termination of the service of a male customer

d_F → termination of the service of a female customer



$F_a(t) = 1 - e^{-\lambda t}, \; t \geq 0$

where $\lambda = 3$ arrivals/hour

$F_{d_M}(t) = 1 - e^{-\mu_M t}, \; t \geq 0$

where $\frac{1}{\mu_M} = 20$ minutes $= \frac{1}{3}$ hours

=> $\mu_M = 3$ services/hour

$F_{d_F}(t) = 1 - e^{-\mu_F t}, \; t \geq 0$

where $\frac{1}{\mu_F} = 45$ minutes $= \frac{3}{4}$ hours

=> $\mu_F = \frac{4}{3}$ services/hour

2. When the third customer arrives, the state must be one of $\binom{2}{0}, \binom{1}{1}$ and $\binom{0}{2}$

=> no termination of any service before the third arrival.

Four possible paths:

$\binom{0}{0} \xrightarrow{a} \binom{1}{0} \xrightarrow{a} \binom{2}{0} \xrightarrow{a}$ => probability $p \cdot \frac{\lambda}{\lambda+\mu_M} \cdot p \cdot \frac{\lambda}{\lambda+2\mu_M}$

$\binom{0}{0} \xrightarrow{a} \binom{1}{0} \xrightarrow{a} \binom{1}{1} \xrightarrow{a}$ => probability $p \cdot \frac{\lambda}{\lambda+\mu_M} \cdot (1-p) \cdot \frac{\lambda}{\lambda+\mu_M+\mu_F}$

$\binom{0}{0} \xrightarrow{a} \binom{0}{1} \xrightarrow{a} \binom{1}{1} \xrightarrow{a}$ => probability $(1-p) \cdot \frac{\lambda}{\lambda+\mu_F} \cdot p \cdot \frac{\lambda}{\lambda+\mu_M+\mu_F}$

$\binom{0}{0} \xrightarrow{a} \binom{0}{1} \xrightarrow{a} \binom{0}{2} \xrightarrow{a}$ => probability $(1-p) \cdot \frac{\lambda}{\lambda+\mu_F} \cdot (1-p) \cdot \frac{\lambda}{\lambda+2\mu_F}$

$$\Rightarrow P(\ldots) = \frac{(\lambda p)^2}{(\lambda+\mu_M)(\lambda+2\mu_M)} + \frac{\lambda^2 p(1-p)}{(\lambda+\mu_M)(\lambda+\mu_M+\mu_F)} + \frac{\lambda^2 p(1-p)}{(\lambda+\mu_F)(\lambda+\mu_M+\mu_F)} + \frac{[\lambda(1-p)]^2}{(\lambda+\mu_F)(\lambda+2\mu_F)} =$$

$$= \frac{344}{1187} \simeq 0.2898$$

3. Arrivals of male customers occur according to a Poisson process with rate $\lambda_M = \lambda p$.

$$\Rightarrow P(N_{a_M}(T)=3) = \frac{(\lambda_M T)^3}{3!} e^{-\lambda_M T} \underbrace{=}_{T=1\text{ hour}} \frac{(\lambda p)^3}{6} e^{-\lambda p} \simeq 0.0613$$

Exercise 2

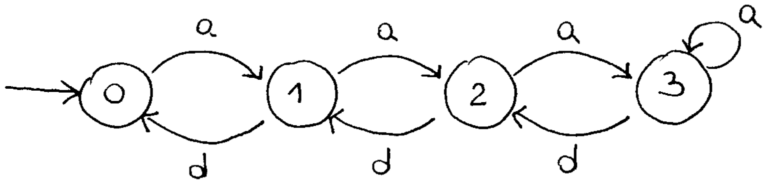
1. Definition of state: x = number of customers in the system $\in \{0,1,2,3\}$

$\Rightarrow$ state space $\mathcal{X} = \{0,1,2,3\}$ 4 states

Events: $\mathcal{E} = \{a, d\}$

a: arrival of a customer

d: termination of a service



$F_a(t) = 1 - e^{-\lambda t}, \; t \geq 0$

where $\lambda = 0.1$ arrivals/min

$F_d(t) = 1 - e^{-\mu t}, \; t \geq 0$

where $\mu = 0.2$ services/min

2. The current state is $X_k = 3$.

There are two possible cases:

$3 \xrightarrow[\text{d with overtaking}]{} 2 \xrightarrow[a]{} 3 \xrightarrow[\text{d with overtaking}]{} 2 \xrightarrow[d]{}$

(ignore event a in state 3) (ignore event a in state 3)

probability: $p \cdot \dfrac{\lambda}{\lambda+\mu} \cdot p \cdot \dfrac{\mu}{\lambda+\mu}$

$3 \xrightarrow[\text{d with overtaking}]{} 2 \xrightarrow[a]{} 3 \xrightarrow[\text{d with overtaking}]{} 2 \xrightarrow[a]{} 3 \xrightarrow[\text{d without overtaking}]{}$

(ignore event a in state 3) (ignore event a in state 3) (ignore event a in state 3)

probability: $p \cdot \dfrac{\lambda}{\lambda+\mu} \cdot p \cdot \dfrac{\lambda}{\lambda+\mu} \cdot (1-p)$

$$\Rightarrow P(\ldots) = \frac{\lambda p^2}{\lambda+\mu}\left[\frac{\mu}{\lambda+\mu} + \frac{\lambda(1-p)}{\lambda+\mu}\right] = \frac{11}{576} \simeq 0.0191$$

3. The current state is $X_k = 3$.

There are two possible cases:

$3 \xrightarrow{a} 3$ probability: $\dfrac{\lambda}{\lambda+\mu}$

$3 \xrightarrow[\text{}]{\text{d without overtaking}} 2$ probability: $\dfrac{\mu}{\lambda+\mu} \cdot (1-p)$

$$\Rightarrow P(\ldots) = \frac{\lambda}{\lambda+\mu} + \frac{\mu(1-p)}{\lambda+\mu} = \frac{5}{6} \simeq 0.8333$$

4. The current state is $X_k = 2$.

The holding time is the time spent by the system in $X_k = 2$.

If we denote by $V(2)$ the holding time, we have that $V(2)$ is the superposition of the residual lifetimes of all events that take the system away from state 2.

In this case, $V(2) = \min\{\underbrace{Y_{a,k}}_{\text{exponential with rate } \lambda}, \underbrace{Y_{b,k}}_{\text{exponential with rate } \mu}\}$ $\Rightarrow$ $V(2)$ is exponential with rate $\lambda+\mu$

$$\Rightarrow P(V(2) \leq t) = 1 - e^{-(\lambda+\mu)t} = 1 - e^{-0.3t}, \quad t \geq 0.$$

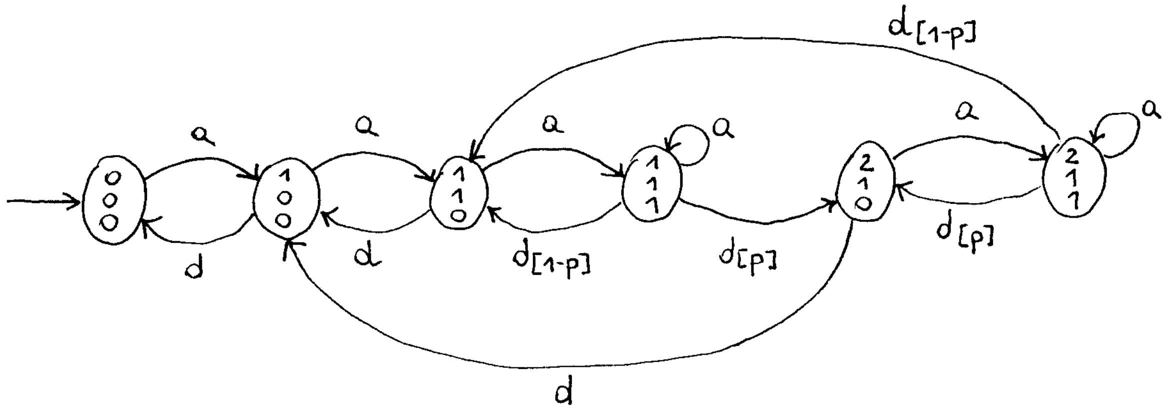
<u>Remark</u>: a definition of state which takes into account the "type" of the customer being served, leads to a model with more states.

$x = \begin{bmatrix} x_S \\ x_{P1} \\ x_{P2} \end{bmatrix}$

$x_S \rightarrow$ server: idle (0), busy - customer did not overtake (1), busy - customer did overtake (2)

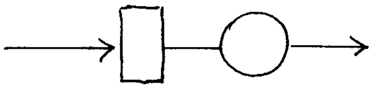
$x_{P1} \rightarrow$ Place P1: free (0), busy (1)

$x_{P2} \rightarrow$ Place P2: free (0), busy (1)



Exercise 3

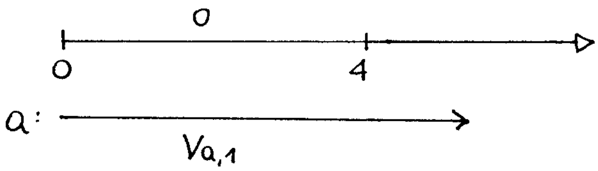
1. The system can be represented as follows:



If we define the state X as the number of customers in the system, the problem is to compute $P(X(4)=0)$.

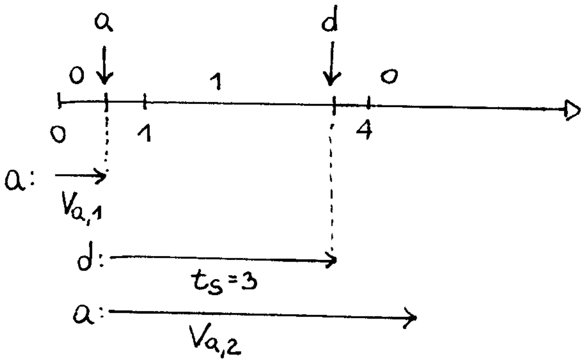
There are two possible cases.

Case 1: there are no arrivals in $[0,4]$



$\Rightarrow P(V_{a,1}>4)$

Case 2: there is one arrival in $[0,1]$ and no arrivals in $(1,4]$.



$\Rightarrow P(V_{a,1}\leq 1, V_{a,1}+V_{a,2}>4)$

$$\Rightarrow P(X(4)=0) = P(V_{a,1}>4) + P(V_{a,1}\leq 1,\ V_{a,1}+V_{a,2}>4) = \ldots =$$

$$= P(N_a(4)=0) + P(N_a(1)=1)\,P(N_a(3)=0) =$$

$$= e^{-4\lambda} + \lambda e^{-\lambda}\cdot e^{-3\lambda} = e^{-4\lambda}(1+\lambda) \simeq 0.0366$$

$\lambda = 1$ arrival/min

Exercise 4

1. Definition of state:

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \begin{array}{l} \to M_1: \text{idle (0), working (1), blocked (2)} \\ \to M_2: \text{idle (0), working (1)} \\ \to \text{buffer: empty (0), full (1)} \end{array}$$

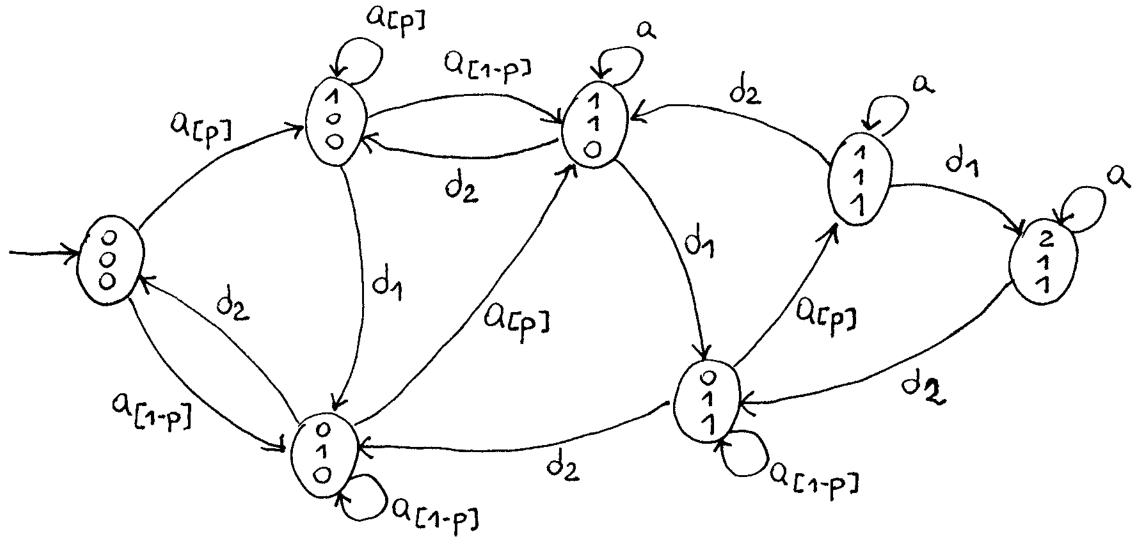
State space: $\mathcal{X} = \left\{ \begin{bmatrix} 0\\0\\0 \end{bmatrix}, \begin{bmatrix} 1\\0\\0 \end{bmatrix}, \begin{bmatrix} 0\\1\\0 \end{bmatrix}, \begin{bmatrix} 1\\1\\0 \end{bmatrix}, \begin{bmatrix} 0\\1\\1 \end{bmatrix}, \begin{bmatrix} 1\\1\\1 \end{bmatrix}, \begin{bmatrix} 2\\1\\1 \end{bmatrix} \right\}$ 7 states

Events: $\mathcal{E} = \{a, d_1, d_2\}$

a → arrival of a new part

d_1 → termination of a job in M_1

d_2 → termination of a job in M_2



$F_a(t) = 1 - e^{-\lambda t}, \; t \geq 0$ where $\frac{1}{\lambda} = 5$ minutes $\Rightarrow \lambda = \frac{1}{5}$ arrivals/min

$F_{d_1}(t) = 1 - e^{-\mu_1 t}, \; t \geq 0$ where $\mu_1 = \frac{1}{2}$ services/min

$F_{d_2}(t) = 1 - e^{-\mu_2 t}, \; t \geq 0$ where $\mu_2 = \frac{4}{5}$ services/min

2. The current state is $X_k = \begin{bmatrix} 0\\1\\1 \end{bmatrix}$.

There is only one possible case:

$$\begin{bmatrix} 0\\1\\1 \end{bmatrix} \xrightarrow[d_2]{} \begin{bmatrix} 0\\1\\0 \end{bmatrix} \xrightarrow[d_2]{} \begin{bmatrix} 0\\0\\0 \end{bmatrix}$$

(ignore $a_{[1-p]}$ in state $\begin{bmatrix} 0\\1\\1 \end{bmatrix}$) (ignore $a_{[1-p]}$ in state $\begin{bmatrix} 0\\1\\0 \end{bmatrix}$)

$\Rightarrow$ event $a_{[1-p]}$ corresponds to a rejected part
We are interested only in accepted parts.

$$\Rightarrow P(\dots) = \frac{\mu_2}{\lambda p + \mu_2} \cdot \frac{\mu_2}{\lambda p + \mu_2} = \frac{144}{169} \simeq 0.8521$$

3. The current state is $X_k = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

It coincides with the expected holding time in state $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

exponential with rate $\lambda(1-p)+\mu_1$

$$\Rightarrow E\left[V\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right)\right] = \frac{1}{\lambda(1-p)+\mu_1} = \frac{30}{19} \simeq 1.5789$$

4. There is only one possible case:

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \xrightarrow[a]{} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \xrightarrow[d_1]{}$$

(ignore $a_{[p]}$ in state $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$)

$$\Rightarrow P(\dots) = p \cdot \frac{\mu_1}{\lambda(1-p)+\mu_1} = \frac{5}{19} \simeq 0.2632$$