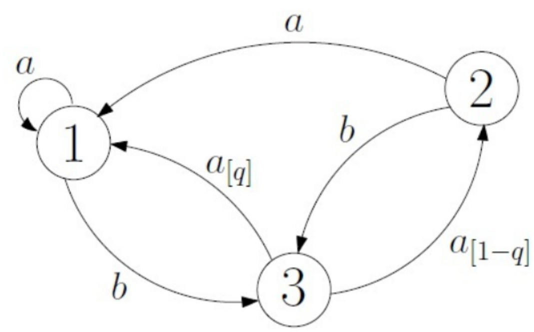
# EXERCISE 1

Consider the stochastic timed automaton in the figure, where $q = 2/5$.




The initial state is uncertain, with pmf $p_{X_0}(1) = \frac{2}{3}$, $p_{X_0}(2) = 0$, and $p_{X_0}(3) = \frac{1}{3}$. Lifetimes of event $a$ have a uniform distribution over the interval $[6, 9]$ min, while lifetimes of event $b$ have an exponential distribution with expected value 5 min.

1. Compute $P(E_2 = a)$.
2. Compute $P(X_2 = 3)$.
3. Compute the probability that event $b$ occurs at least once over the interval $[0, 10]$ min.
4. Compute the cdf of the state holding time in $x = 2$.

## Stochastic clock structure

$V_a \sim U(6,9)$, $V_b \sim Exp(5)$ → $\frac{1}{\lambda} = 5 \Rightarrow \lambda = \frac{1}{5}$

All the lifetimes are expressed in minutes.

$$f_a(u) = \begin{cases} \frac{1}{3} & \text{if } 6 \leq u \leq 9 \\ 0 & \text{otherwise} \end{cases}$$ pdf of $V_{a,i}$

$$f_b(v) = \begin{cases} \frac{1}{5} e^{-\frac{v}{5}} & \text{if } v \geq 0 \\ 0 & \text{otherwise} \end{cases}$$ pdf of $V_{b,i}$

1. Compute $P(E_2 = a)$.

First, we have to identify all the sample paths such that $\{E_2 = a\}$.

We can exclude sample paths starting with $\{X_0 = 2\}$, since $p_{X_0}(2) = 0$.

1) $X_0 = 1 \xrightarrow{E_1 = a} X_1 = 1 \xrightarrow{E_2 = a}$

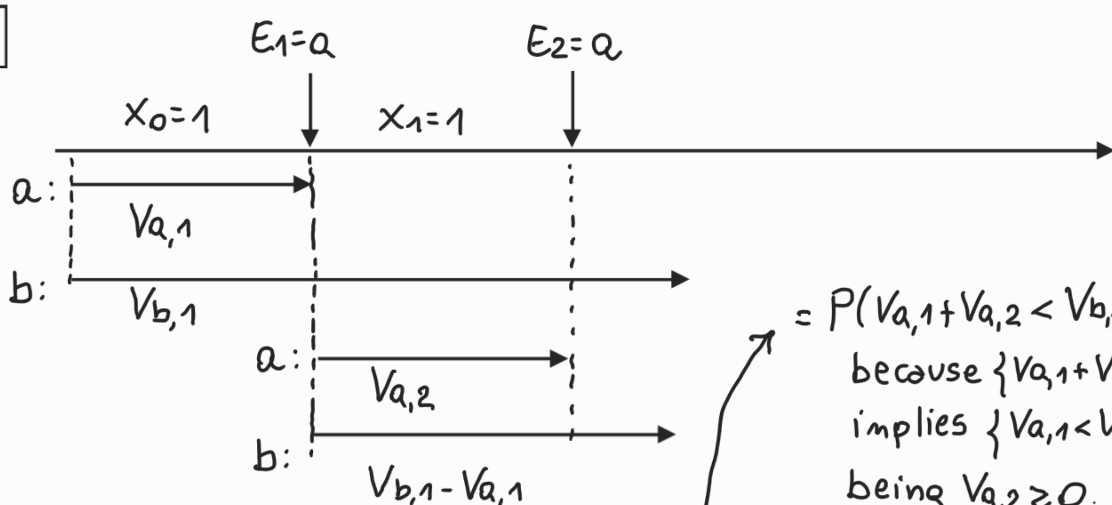
2) $X_0 = 1 \xrightarrow{E_1 = b} X_1 = 3 \xrightarrow{E_2 = a}$

3) $X_0 = 3 \xrightarrow{E_1 = a} X_1 = 1 \xrightarrow{E_2 = a}$

4) $X_0 = 3 \xrightarrow{E_1 = a} X_1 = 2 \xrightarrow{E_2 = a}$

Then, we have to compute the probability of each sample path.

[1]

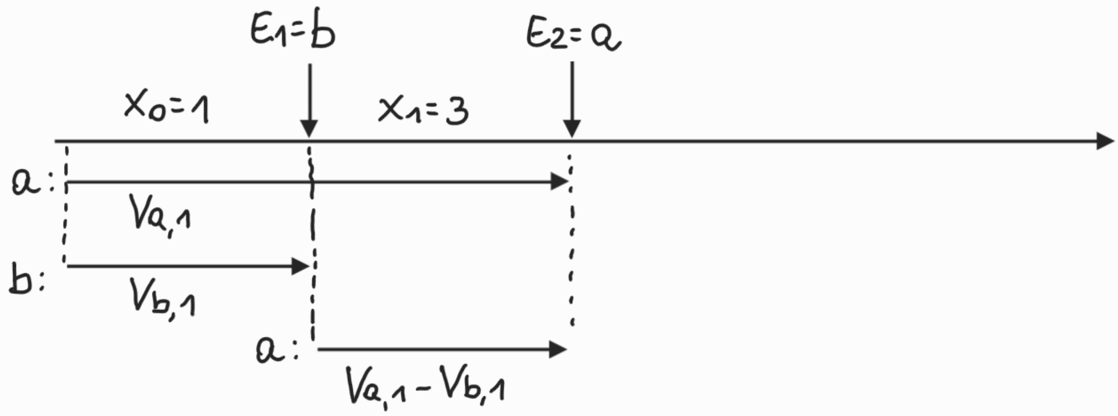



$= P(V_{a,1} + V_{a,2} < V_{b,1})$

because $\{V_{a,1} + V_{a,2} < V_{b,1}\}$ implies $\{V_{a,1} < V_{b,1}\}$, being $V_{a,2} \geq 0$.

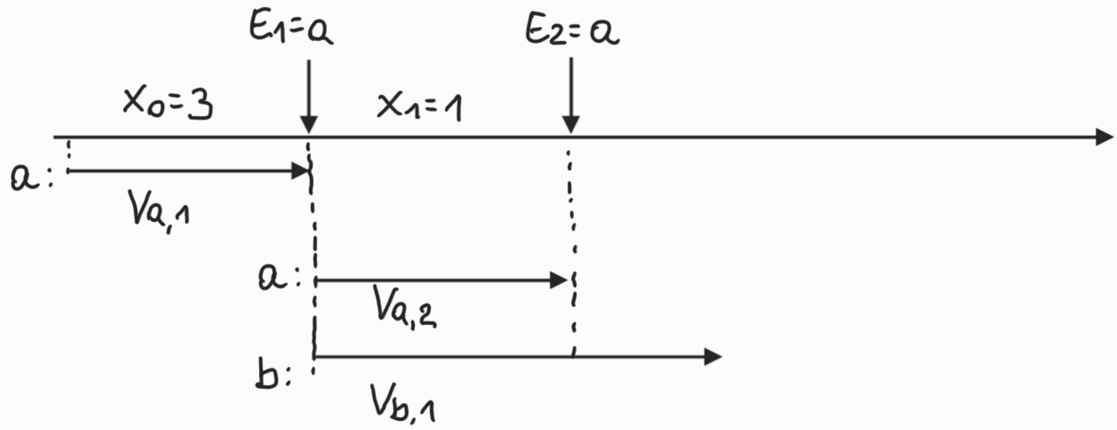
$$\Rightarrow P(\boxed{1}) = p_{X_0}(1) \cdot \underbrace{p(1|1,a)}_{1} \cdot P(V_{a,1} < V_{b,1},\ V_{a,2} < V_{b,1} - V_{a,1})$$

[2]



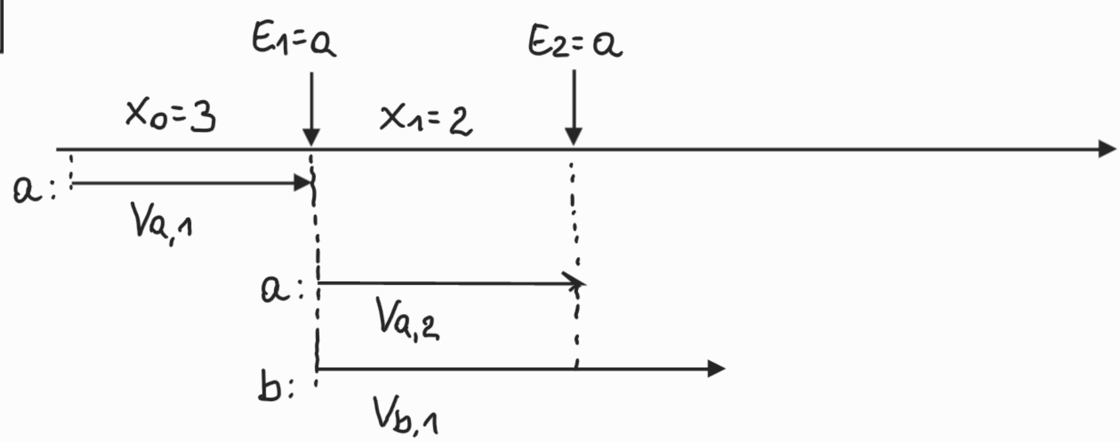
$$\Rightarrow P(\boxed{2}) = p_{x_0}(1) \cdot \underbrace{p(3|1,b)}_{1} \cdot P(V_{b,1} < V_{a,1})$$

$\boxed{3}$




$$\Rightarrow P(\boxed{3}) = p_{x_0}(3) \cdot \underbrace{p(1|3,a)}_{q} \cdot P(V_{a,2} < V_{b,1})$$

$\boxed{4}$




$$\Rightarrow P(\boxed{4}) = p_{x_0}(3) \cdot \underbrace{p(2|3,a)}_{1-q} \cdot P(V_{a,2} < V_{b,1})$$

Finally:

$$P(E_2=a) = \sum_{i=1}^{4} P(\boxed{i})$$

$$= p_{x_0}(1)\, P(V_{a,1} + V_{a,2} < V_{b,1}) + p_{x_0}(1)\, P(V_{b,1} < V_{a,1}) + p_{x_0}(3)\, P(V_{a,2} < V_{b,1})$$

$\simeq 0.6254$ ← computed numericelly with Matlab

# REMARK

Computing a probability involving two or more random variables requires the evaluation of a multiple integral.

In the exam this computation is typically <u>not requested</u>.

In the second part of the course we will see how to evaluate these probabilities numerically.

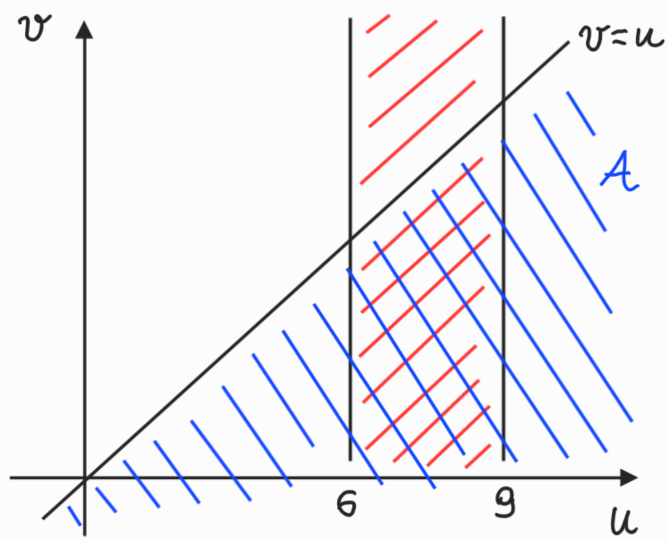
For illustrative purposes only, the computation of $P(V_{b,1} < V_{a,1})$ is shown next.

$$P(V_{b,1} < V_{a,1}) = \iint_A f_a(u) f_b(v)\,du\,dv$$

($V_{b,1} \to v$, $V_{a,1} \to u$)

$\searrow$ $V_{a,1}$ and $V_{b,1}$ are independent, therefore their joint pdf is the product of the marginal pdfs.

$$A = \{(u,v) \in \mathbb{R}^2 : v < u\}$$




The product $f_a(u) f_b(v)$ is nonzero only over the red region.

Hence, the integral is to be computed only over the blue-and-red region.

$$P(V_{b,1} < V_{a,1}) = \int_6^9 \int_0^u \frac{1}{3} \cdot \frac{1}{5} e^{-\frac{v}{5}}\,du\,dv = \int_6^9 \frac{1}{3}\left[-e^{-\frac{v}{5}}\right]_0^u du$$

$$= \int_6^9 \frac{1}{3}\left(1 - e^{-\frac{u}{5}}\right) du = \frac{1}{3}\left[u + 5e^{-\frac{u}{5}}\right]_6^9 = \frac{1}{3}\left(9 + 5e^{-\frac{9}{5}} - 6 - 5e^{-\frac{6}{5}}\right)$$

$$= 1 - \frac{5}{3}\left(e^{-\frac{6}{5}} - e^{-\frac{9}{5}}\right) \simeq 0.7735$$

2. Compute $P(X_2=3)$

First, we have to identify all the sample paths such that $\{X_2=3\}$.
As before, we can exclude sample paths starting with $\{X_0=2\}$.

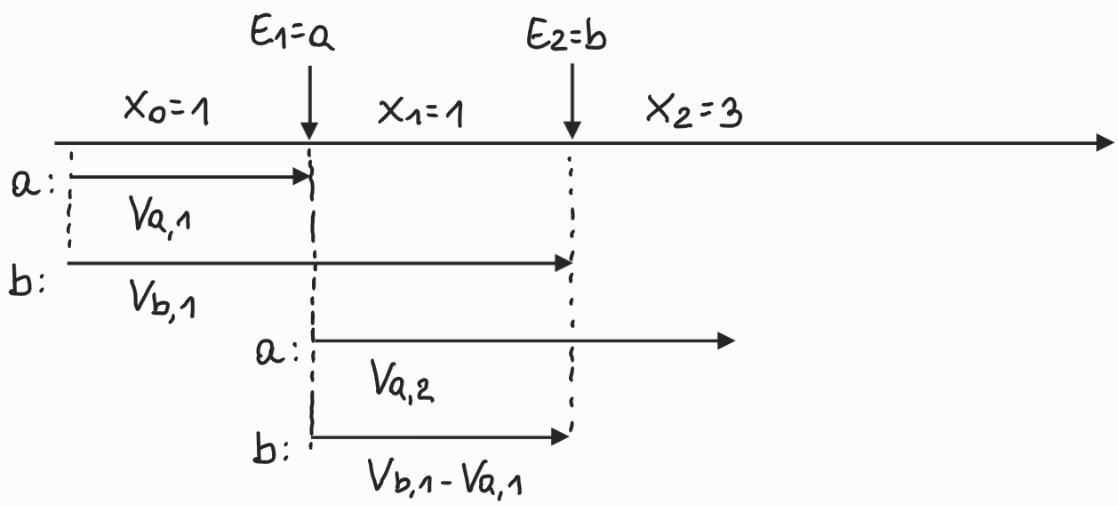
1) $X_0=1 \xrightarrow{E_1=a} X_1=1 \xrightarrow{E_2=b} X_2=3$

2) $X_0=3 \xrightarrow{E_1=a} X_1=1 \xrightarrow{E_2=b} X_2=3$

3) $X_0=3 \xrightarrow{E_1=a} X_1=2 \xrightarrow{E_2=b} X_2=3$

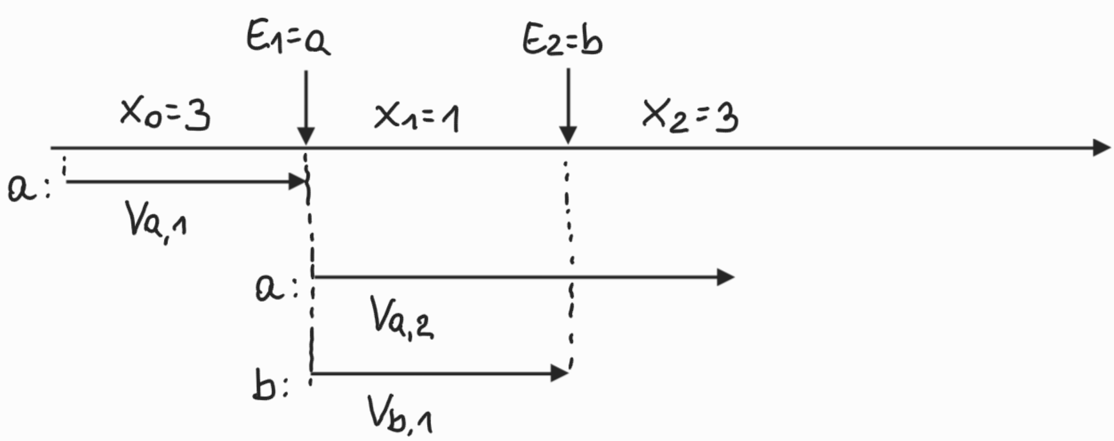
Then, we have to compute the probability of each sample path.

[1]



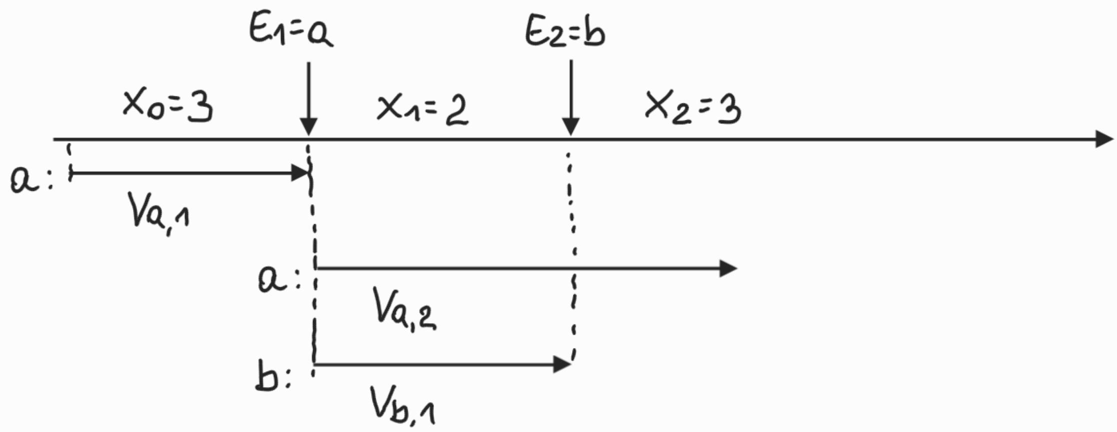

$$\Rightarrow P(\boxed{1}) = p_{X_0}(1)\cdot \underbrace{p(1|1,a)}_{1}\cdot \underbrace{p(3|1,b)}_{1}\cdot P(V_{a,1}<V_{b,1},\ V_{b,1}-V_{a,1}<V_{a,2})$$

[2]



$$\Rightarrow \quad P(\boxed{2}) = p_{x_0}(3)\cdot \underbrace{p(1|3,a)}_{q}\cdot \underbrace{p(3|1,b)}_{1}\cdot P(V_{b,1} < V_{a,2})$$

$\boxed{3}$




$$\Rightarrow \quad P(\boxed{3}) = p_{x_0}(3)\cdot \underbrace{p(2|3,a)}_{1-q}\cdot \underbrace{p(3|2,b)}_{1}\cdot P(V_{b,1} < V_{a,2})$$

Finally:

$$P(X_2=3) = \sum_{i=1}^{3} P(\boxed{i})$$

$$= p_{x_0}(1)\, P(V_{a,1} < V_{b,1} < V_{a,1}+V_{a,2}) + p_{x_0}(3)\, P(V_{b,1} < V_{a,2})$$

$$\simeq 0.3746$$ ← computed numerically with Matlab

3. Compute the probability that event b occurs at least once over the interval [0, 10] min.

If b were always possible, the answer would be trivial:

$$P(V_{b,1} \leq T) = F_b(T) = 1 - e^{-T/5}, \text{ where } T = 10 \text{ min.}$$

Unfortunately, this is not the case, because event b is not possible in state 3.

On the other hand, event a is always possible.

Therefore, we have in principle to consider all the possible sample paths with event b preceded by an undetermined number of events a, while imposing that event b occurs before time T.

Luckily enough, since $V_a \sim U(6, 9)$, we can exploit the fact that event a may occur at most once before time T = 10 min (the second occurrence cannot be before 6 + 6 = 12 min).

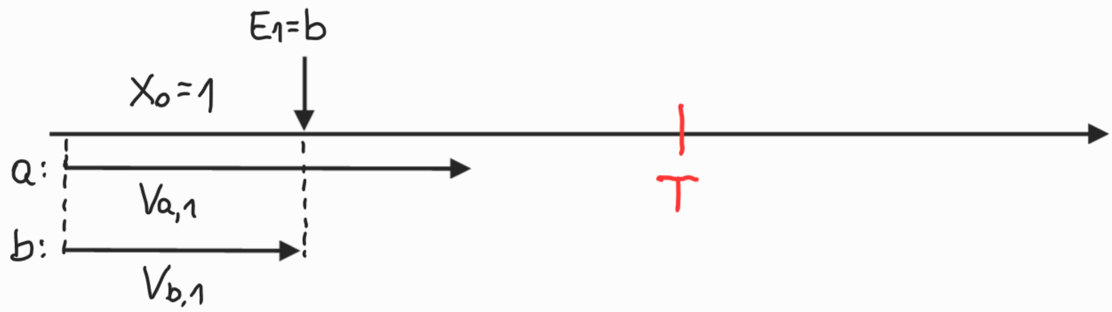
The possible cases are thus the following:

1. $X_0 = 1 \xrightarrow{E_1 = b}$
2. $X_0 = 1 \xrightarrow{E_1 = a} X_1 = 1 \xrightarrow{E_2 = b}$
3. $X_0 = 3 \xrightarrow{E_1 = a} X_1 = 1 \xrightarrow{E_2 = b}$
4. $X_0 = 3 \xrightarrow{E_1 = a} X_1 = 2 \xrightarrow{E_2 = b}$

+ impose that event b occurs before time T

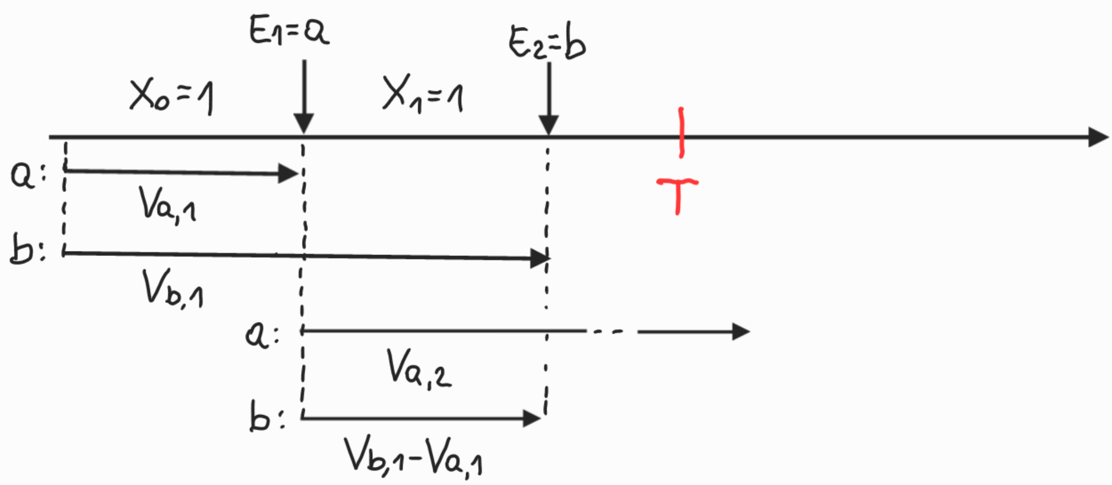
We compute the probability of each case.

1



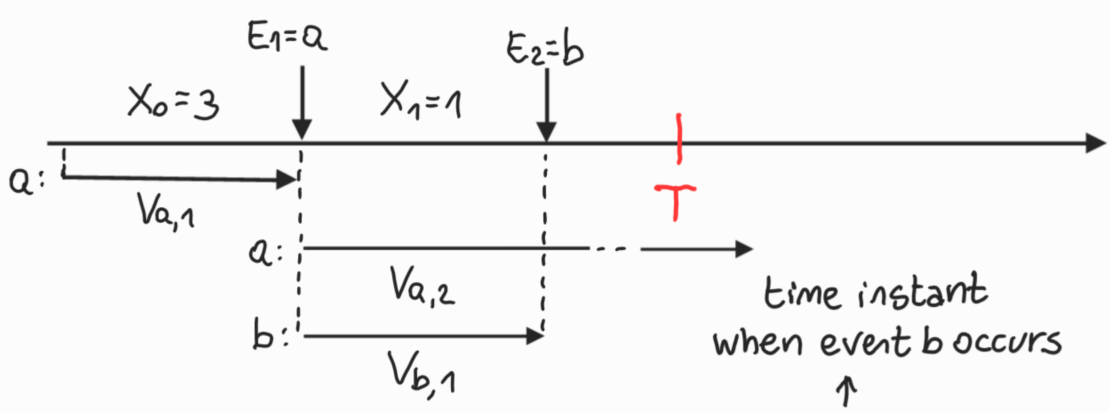

$$\Rightarrow \quad P(\boxed{1}) = p_{X_0}(1)\, P\left(V_{b,1} < V_{a,1},\ V_{b,1} < T\right)$$

2




$$\Rightarrow \quad P(\boxed{2}) = p_{X_0}(1)\, P\left(V_{a,1} < V_{b,1} < V_{a,1}+V_{a,2},\ V_{b,1} < T\right)$$

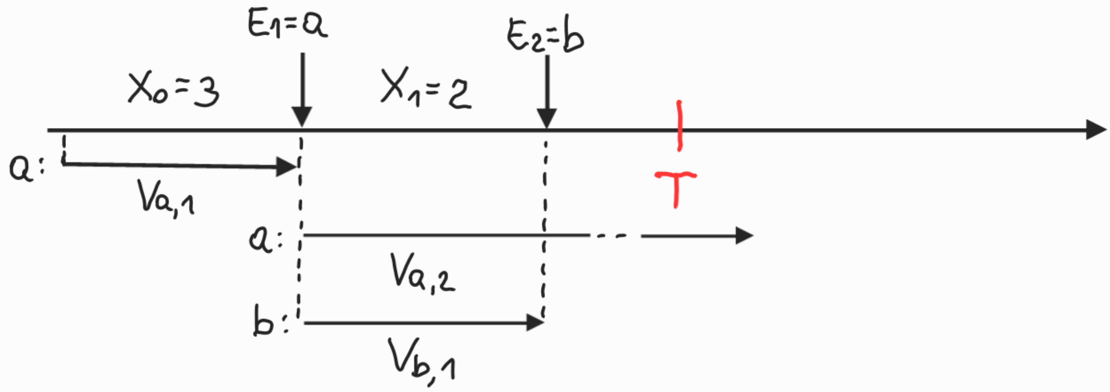
3




time instant when event b occurs

$$\Rightarrow \quad P(\boxed{3}) = p_{X_0}(3)\cdot q \cdot P\left(V_{b,1} < V_{a,2},\ \underbrace{V_{a,1}+V_{b,1}}_{} < T\right)$$

**4**



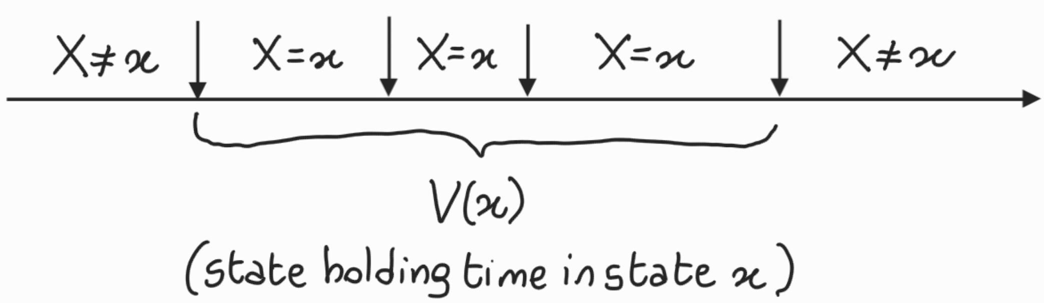

$$P(\boxed{4}) = p_{X_0}(3)\cdot(1-q)\cdot P(V_{b,1} < V_{a,2},\ V_{a,1} + V_{b,1} < T)$$

Finally,

$$P(\text{event } b \text{ occurs before } T=10) = \sum_{i=1}^{4} P(\boxed{i})$$

$\simeq 0.7046$ ← computed numerically with Matlab

4. The **state holding time** is the time that the system remains in a given state.




We have to compute the cdf of $V(2)$.

We observe that:

- the system enters state 2 only from state 3 with event $a$, and in state 3 event $b$ is not possible
  $\Rightarrow$ in state 2 the lifetimes of both events are **total lifetimes**.

- the system leaves state 2 when either of the two events occurs.

Hence, we have:

$$V(2) = \min\{V_a, V_b\}$$

($V_a$: total lifetime of event a; $V_b$: total lifetime of event b)

To compute the cdf of $V(2)$, we start by computing

$$P(V(2)>t) = P(\min\{V_a,V_b\}>t)$$

$$= P(V_a>t, V_b>t) = P(V_a>t)P(V_b>t)$$

($\{\min\{V_a,V_b\}>t\} \iff \{V_a>t, V_b>t\}$; independent)

$$= [1-F_a(t)][1-F_b(t)]$$

$$\Rightarrow P(V(2)\leq t) = 1-P(V(2)>t) = 1-[1-F_a(t)][1-F_b(t)]$$

Since

$$F_a(t) = \begin{cases} 0 & \text{if } t<6 \\ \frac{t-6}{3} & \text{if } 6\leq t\leq 9 \\ 1 & \text{if } t>9 \end{cases} \qquad F_b(t) = \begin{cases} 1-e^{-\frac{t}{5}} & \text{if } t\geq 0 \\ 0 & \text{if } t<0 \end{cases}$$

we have:

$$P(V(2)\leq t) = \begin{cases} 0 & \text{if } t<0 \\ 1-e^{-\frac{t}{5}} & \text{if } 0\leq t<6 \\ 1-\frac{9-t}{3}e^{-t/5} & \text{if } 6\leq t\leq 9 \\ 1 & \text{if } t>9 \end{cases}$$

## REMARK

In state 1 we have:

$$V(1) = Y_b$$ ← residual lifetime of event b

because the system leaves state 1 only with event b, but when the system enters state 1, event b may have a residual lifetime from the previous state. Indeed,

- $Y_b = V_b - V_a$ if the system arrives from state 2,
- $Y_b = V_b$ if the system arrives from state 3.

Computing $P(V(1) \leq t)$ is therefore a non-trivial task...