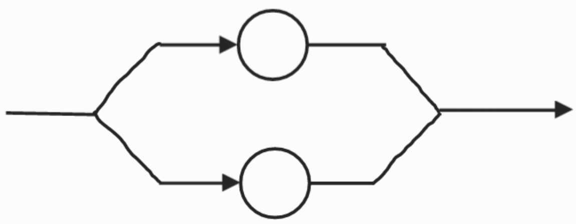
Exercise 1

A small hair salon has two chairs and two hairdressers. Customers arrive according to a Poisson process with rate 3 arrivals/hour. A customer is male with probability $p = 1/3$. The duration of a hair-cut is independent of the hairdresser, but depends on the gender of the customer. It is exponentially distributed with expected value 20 minutes for men, and 45 minutes for women. Since the hair salon does not have a waiting room, customers arriving when both chairs are busy, decide to give up hair cutting. The hair salon is empty at the opening.

1. Model the hair salon through a stochastic timed automaton $(\mathcal{E}, \mathcal{X}, \Gamma, p, x_0, F)$.
2. Assume that one hairdresser is serving a man and the other is serving a woman. Compute the probability that the next event is the arrival of a new customer.
3. Assume that both hairdressers are busy with male customers. Compute the probability that the next event is the termination of a hair cut.
4. Assume that both hairdressers are busy with male customers of different age. Compute the probability that the next event is the termination of the hair cut of the youngest man.
5. Assume that one hairdresser is serving a man and the other is serving a woman. Compute the probability that the hair cut of the man terminates before the hair cut of the woman.
6. Compute the probability that the third customer arriving after the opening has to give up hair cutting.
7. Assume that one hairdresser is serving a man and the other is serving a woman. Compute the probability that both hair cuts terminate:
   (a) before another customer arrives;
   (b) before another customer sits for a hair cut (*i.e.* "is accepted").
8. Compute the probability that:
   (a) at least three customers arrive in the next hour;
   (b) at most two <u>male customers</u> arrive in the next hour.
9. Assume that one hairdresser is serving a man and the other is serving a woman. Compute the probability that, in the next hour:
   (a) both hair cuts are terminated and no customer arrives;
   (b) both hair cuts are terminated and no <u>female customer</u> arrives.
10. Compute the average state holding time when:
    (a) one hairdresser is serving a man and the other is idle;
    (b) one hairdresser is serving a man and the other is serving a woman.

1. Model

Graphical representation of the system:



≈ two parallel servers

Events $\mathcal{E} = \{ a, d_M, d_F \}$

a: arrival of a new customer

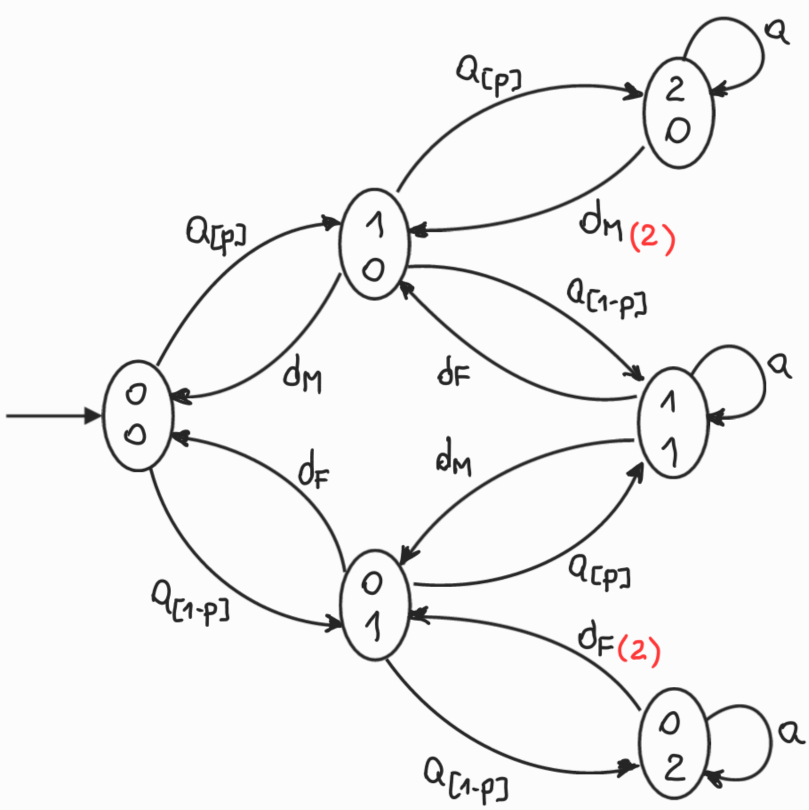
d_M: termination of a man's haircut

d_F: termination of a woman's haircut

State $x = \begin{bmatrix} x_M \\ x_F \end{bmatrix}$

x_M: # male customers $\in \{0,1,2\}$

x_F: # female customers $\in \{0,1,2\}$



Stochastic clock structure: $F = \{ F_a, F_{d_M}, F_{d_F} \}$

$F_a(t) = 1 - e^{-\lambda t}, \; t \geq 0 \qquad \lambda = 3$ arrivals/hour

$F_{d_M}(t) = 1 - e^{-\mu_M t}, \; t \geq 0 \qquad \frac{1}{\mu_M} = 20\,\text{min} = \frac{1}{3}$ hours

$\Rightarrow \mu_M = 3$ services/hour

$F_{d_F}(t) = 1 - e^{-\mu_F t}, \; t \geq 0 \qquad \frac{1}{\mu_F} = 45\,\text{min} = \frac{3}{4}$ hours

$\Rightarrow \mu_F = \frac{4}{3}$ services/hour

All the events have exponential distributions of the lifetimes

$\Rightarrow$ Stochastic timed automaton with Poisson clock structure

$\Rightarrow$ WE CAN USE FORMULAS!

For instance...

2. Current state $X_k = \binom{1}{1}$

$$\Rightarrow P\left(E_{k+1} = a \,\middle|\, X_k = \binom{1}{1}\right) = \frac{\lambda}{\lambda + \mu_M + \mu_F}$$

(immediate 😉)

Recall that, if the stochastic timed automaton were <u>not</u> Poisson, we should compute:

$$P\left(E_{k+1} = a \,\middle|\, X_k = \binom{1}{1}\right) = P\left(Y_{a,k} < \min\{Y_{d_M,k}, Y_{d_F,k}\}\right)$$

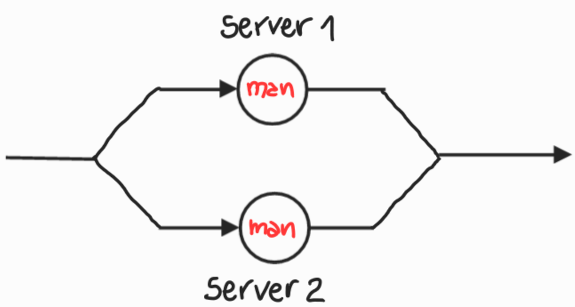
but then we get stuck, because we do not know in general the distributions of the residual lifetimes. We should thus compute all cases starting from initialization...

$\Rightarrow$ Computationally heavy!

3. Current state $X_k = \binom{2}{0}$

$$\Rightarrow P\left(E_{k+1} = d_M \,\middle|\, X_k = \binom{2}{0}\right) = \frac{2\mu_M}{\lambda + 2\mu_M}$$

Why "2" here?



Let:

- $d_M^{(1)}$: event d_M in server 1
- $d_M^{(2)}$: event d_M in server 2

union $\equiv$ or

generic

$$\Rightarrow P\left(E_{k+1} = d_M \,\middle|\, X_k = \binom{2}{0}\right) = P\left(\{E_{k+1} = d_M^{(1)}\} \cup \{E_{k+1} = d_M^{(2)}\} \,\middle|\, X_k = \binom{2}{0}\right)$$

$$= P\left(E_{k+1} = d_M^{(1)} \,\middle|\, X_k = \binom{2}{0}\right) + P\left(E_{k+1} = d_M^{(2)} \,\middle|\, X_k = \binom{2}{0}\right)$$

disjoint

For $d_M^{(1)}$ For $d_M^{(2)}$

$$= \frac{\mu_M}{\lambda + \mu_M + \mu_M} + \frac{\mu_M}{\lambda + \mu_M + \mu_M} = \frac{2\mu_M}{\lambda + 2\mu_M}$$

For $d_M^{(1)}$ For $d_M^{(2)}$ For $d_M^{(1)}$ For $d_M^{(2)}$

4. Current state $X_k = \binom{2}{0}$

Let:

- $d_M^{(y)}$: termination of the haircut of the youngest man
- $d_M^{(o)}$: termination of the haircut of the oldest man

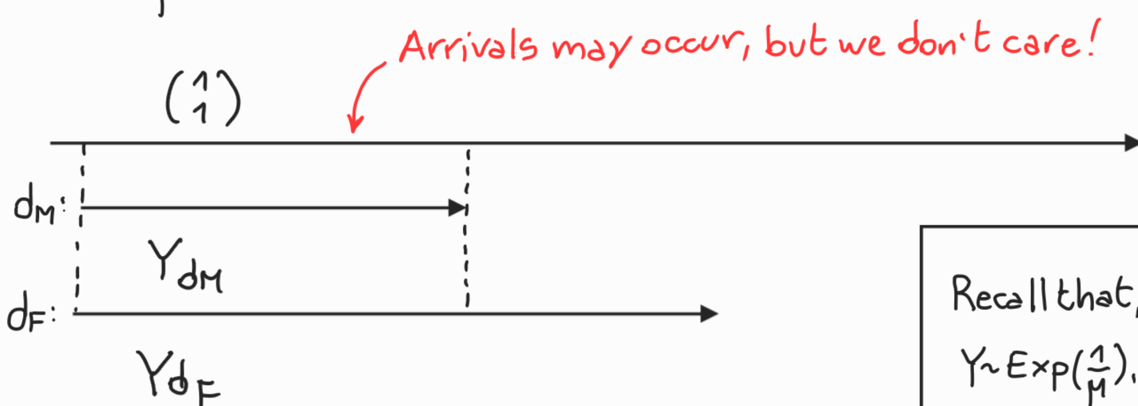
$$\Rightarrow P\left(E_{k+1} = d_M^{(y)} \,\middle|\, X_k = \binom{2}{0}\right) = \frac{\overbrace{\mu_M}^{\text{For } d_M^{(y)}}}{\lambda + \underbrace{\mu_M}_{\text{For } d_M^{(y)}} + \underbrace{\mu_M}_{\text{For } d_M^{(o)}}} = \frac{\mu_M}{\lambda + 2\mu_M}$$

5. Current state $X_k = \binom{1}{1}$.

The probability we are asked to compute <u>is not</u> :

$$P\left(E_{k+1} = d_M \,\middle|\, X_k = \binom{1}{1}\right) = \frac{\mu_M}{\lambda + \mu_M + \mu_F}$$

(This is the probability that event d_M occurs before both event d_F $\underbrace{\text{and event } a}_{\text{not required!}}$)



$$\Rightarrow P\left(Y_{d_M} < Y_{d_F} \,\middle|\, X_k = \binom{1}{1}\right) = \frac{\mu_M}{\mu_M + \mu_F}$$

Recall that, if $X \sim Exp(\frac{1}{\lambda})$, $Y \sim Exp(\frac{1}{\mu})$, independent, then

$$P(X<Y) = \frac{\lambda}{\lambda+\mu}$$

6. We have to compute:

P(the system is full when the 3^{rd} customer arrives)

=> Identify all the sample paths such that the system is full when the third customer arrives.

$\boxed{1}$ $\begin{pmatrix}0\\0\end{pmatrix} \xrightarrow{a} \begin{pmatrix}1\\0\end{pmatrix} \xrightarrow{a} \begin{pmatrix}2\\0\end{pmatrix} \xrightarrow{a}$

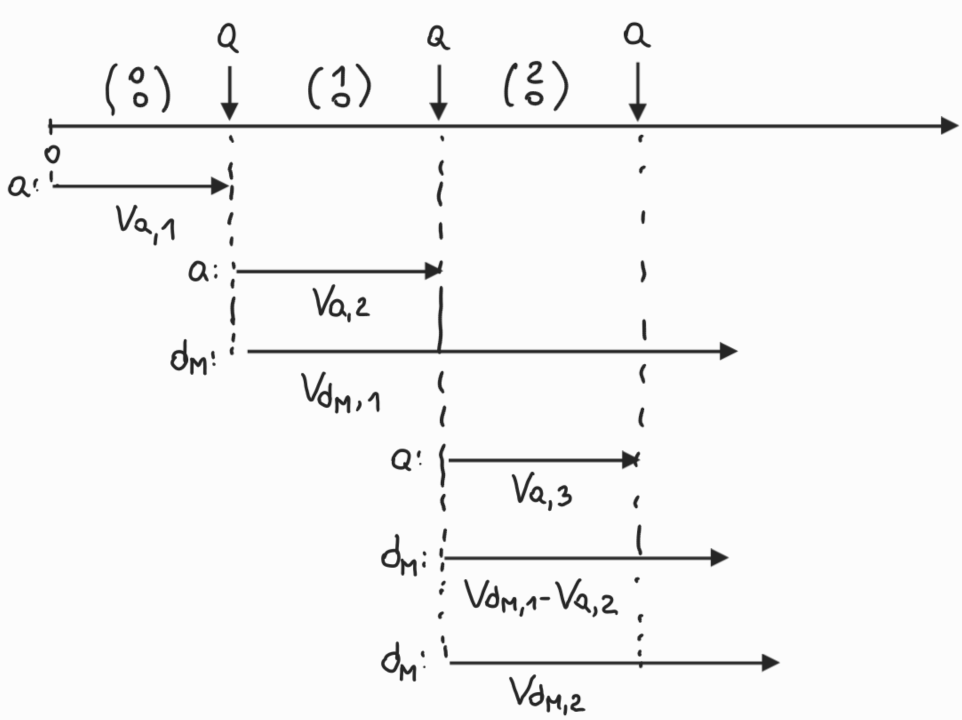
$\boxed{2}$ $\begin{pmatrix}0\\0\end{pmatrix} \xrightarrow{a} \begin{pmatrix}1\\0\end{pmatrix} \xrightarrow{a} \begin{pmatrix}1\\1\end{pmatrix} \xrightarrow{a}$

$\boxed{3}$ $\begin{pmatrix}0\\0\end{pmatrix} \xrightarrow{a} \begin{pmatrix}0\\1\end{pmatrix} \xrightarrow{a} \begin{pmatrix}1\\1\end{pmatrix} \xrightarrow{a}$

$\boxed{4}$ $\begin{pmatrix}0\\0\end{pmatrix} \xrightarrow{a} \begin{pmatrix}0\\1\end{pmatrix} \xrightarrow{a} \begin{pmatrix}0\\2\end{pmatrix} \xrightarrow{a}$

Case $\boxed{1}$

Recall that the general approach would be the following:



$$\Rightarrow P(\boxed{1}) = \overbrace{p_{X_0}\binom{0}{0}}^{1} \cdot \overbrace{p\left(\binom{1}{0} \middle| \binom{0}{0}, a\right)}^{p=1/3} \cdot \overbrace{p\left(\binom{2}{0} \middle| \binom{1}{0}, a\right)}^{p=1/3} \cdot$$

$$\cdot P\left(V_{a,2} < V_{d_M,1},\ V_{a,3} < V_{d_M,1} - V_{a,2},\ V_{a,3} < V_{d_M,2}\right)$$

Integral in 4 dimensions...

Luckily, we are in the Poisson case, hence the computation is dramatically simplified...

$$\binom{0}{0} \xrightarrow{a} \binom{1}{0} \xrightarrow{a} \binom{2}{0} \xrightarrow{a}$$

$$P(\boxed{1}) = 1 \cdot \frac{\lambda}{\lambda} \cdot p \cdot \frac{\lambda}{\lambda + \mu_M} \cdot p \cdot \frac{\lambda}{\lambda + 2\mu_M}$$

$p_{X_0}\binom{0}{0}$

$p\left(E_1 = a \middle| X_0 = \binom{0}{0}\right)$

$p\left(\binom{1}{0} \middle| \binom{0}{0}, a\right)$

$P\left(E_2 = a \middle| X_1 = \binom{1}{0}\right)$

$p\left(\binom{2}{0} \middle| \binom{1}{0}, a\right)$

$P\left(E_3 = a \middle| X_2 = \binom{2}{0}\right)$

$$= \frac{(\lambda p)^2}{(\lambda + \mu_M)(\lambda + 2\mu_M)}$$

Case $\boxed{2}$

$$\begin{pmatrix}0\\0\end{pmatrix} \xrightarrow{a} \begin{pmatrix}1\\0\end{pmatrix} \xrightarrow{a} \begin{pmatrix}1\\1\end{pmatrix} \xrightarrow{a}$$

$$P(\boxed{2}) = 1\frac{\lambda}{\lambda} \cdot p \cdot \frac{\lambda}{\lambda+\mu_M} \cdot (1-p) \cdot \frac{\lambda}{\lambda+\mu_M+\mu_F}$$

Case $\boxed{3}$

$$\begin{pmatrix}0\\0\end{pmatrix} \xrightarrow{a} \begin{pmatrix}0\\1\end{pmatrix} \xrightarrow{a} \begin{pmatrix}1\\1\end{pmatrix} \xrightarrow{a}$$

$$P(\boxed{3}) = 1 \cdot \frac{\lambda}{\lambda} \cdot (1-p) \cdot \frac{\lambda}{\lambda+\mu_F} \cdot p \cdot \frac{\lambda}{\lambda+\mu_M+\mu_F}$$

Case $\boxed{4}$

$$\begin{pmatrix}0\\0\end{pmatrix} \xrightarrow{a} \begin{pmatrix}0\\1\end{pmatrix} \xrightarrow{a} \begin{pmatrix}0\\2\end{pmatrix} \xrightarrow{a}$$

$$P(\boxed{4}) = 1 \cdot \frac{\lambda}{\lambda} \cdot (1-p) \cdot \frac{\lambda}{\lambda+\mu_F} \cdot (1-p) \cdot \frac{\lambda}{\lambda+2\mu_F}$$

$$\Rightarrow P(\ldots) = \sum_{i=1}^{4} P(\boxed{i})$$

7. The current state is $X_k = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

a. $P\left(\text{both haircuts terminate before the next arrival} \mid X_k = \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right)$

Two cases:

$\boxed{1}$ $\quad \begin{pmatrix} 1 \\ 1 \end{pmatrix} \xrightarrow{d_M} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \xrightarrow{d_F}$

$$P(\boxed{1}) \approx \frac{\mu_M}{\lambda + \mu_M + \mu_F} \cdot \frac{\mu_F}{\lambda + \mu_F}$$

$\boxed{2}$ $\quad \begin{pmatrix} 1 \\ 1 \end{pmatrix} \xrightarrow{d_F} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \xrightarrow{d_M}$

$$P(\boxed{2}) \approx \frac{\mu_F}{\lambda + \mu_M + \mu_F} \cdot \frac{\mu_M}{\lambda + \mu_M}$$

$$\Rightarrow P(\ldots) = P(\boxed{1}) + P(\boxed{2})$$

b. $P\left(\text{both haircuts terminate before another customer is accepted in the system} \mid X_k = \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right)$

$\boxed{1}$ $\quad \begin{pmatrix} 1 \\ 1 \end{pmatrix} \xrightarrow{d_M} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \xrightarrow{d_F}$

(ignore a)

$$P(\boxed{1}) \approx \frac{\mu_M}{\mu_M + \mu_F} \cdot \frac{\mu_F}{\lambda + \mu_F}$$

We don't have λ here!

$\boxed{2}$ $\begin{pmatrix}1\\1\end{pmatrix} \xrightarrow{d_F} \begin{pmatrix}1\\0\end{pmatrix} \xrightarrow{d_M}$

(ignore a)

$$P(\boxed{2}) = \frac{\mu_F}{\mu_M + \mu_F} \cdot \frac{\mu_M}{\lambda + \mu_M}$$

We don't have λ here!

$$\Rightarrow P(\dots) = P(\boxed{1}) + P(\boxed{2})$$

8.a. Arrivals are generated by a Poisson process with rate λ.

$\Rightarrow$ We can use the Poisson distribution.

$$P(N_a(T) \geq 3) = 1 - P(N_a(T)=0) - P(N_a(T)=1) - P(N_a(T)=2)$$

$$= 1 - e^{-\lambda T} - (\lambda T)e^{-\lambda T} - \frac{(\lambda T)^2}{2}e^{-\lambda T} = 1 - e^{-\lambda T}\left[1 + \lambda T + \frac{(\lambda T)^2}{2}\right]$$

$$\boxed{P(N_a(T)=n) = \frac{(\lambda T)^n}{n!}e^{-\lambda T}, \quad n = 0, 1, 2, 3, \dots}$$

8.b. Arrivals of male customers can be seen as generated by a Poisson process with rate λp.

$\Rightarrow$ We can use the Poisson distribution.

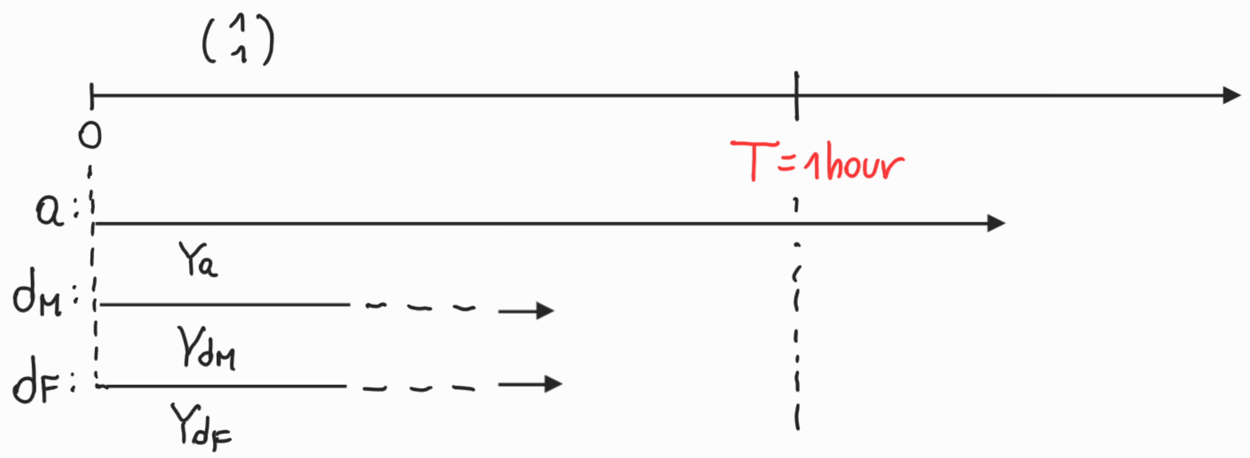
$$P(N_{a_M}(T) \leq 2) = P(N_{a_M}(T)=0) + P(N_{a_M}(T)=1)$$

Event a_M: arrival of a male customer

$$+ P(N_{a_M}(T)=2) = e^{-\lambda p T} + (\lambda p T)e^{-\lambda p T} + \frac{(\lambda p T)^2}{2}e^{-\lambda p T}$$

9. The current state is $X_k = \binom{1}{1}$

a. The situation can be illustrated as follows:



$$P(\ldots) = P(Y_a > T, Y_{d_M} < T, Y_{d_F} < T)$$

$$= P(Y_a > T) P(Y_{d_M} < T) P(Y_{d_F} < T)$$

Independent

$$= e^{-\lambda T} \cdot (1 - e^{-\mu_M T}) \cdot (1 - e^{-\mu_F T})$$

b. Let:

- a_M = arrival of a male customer

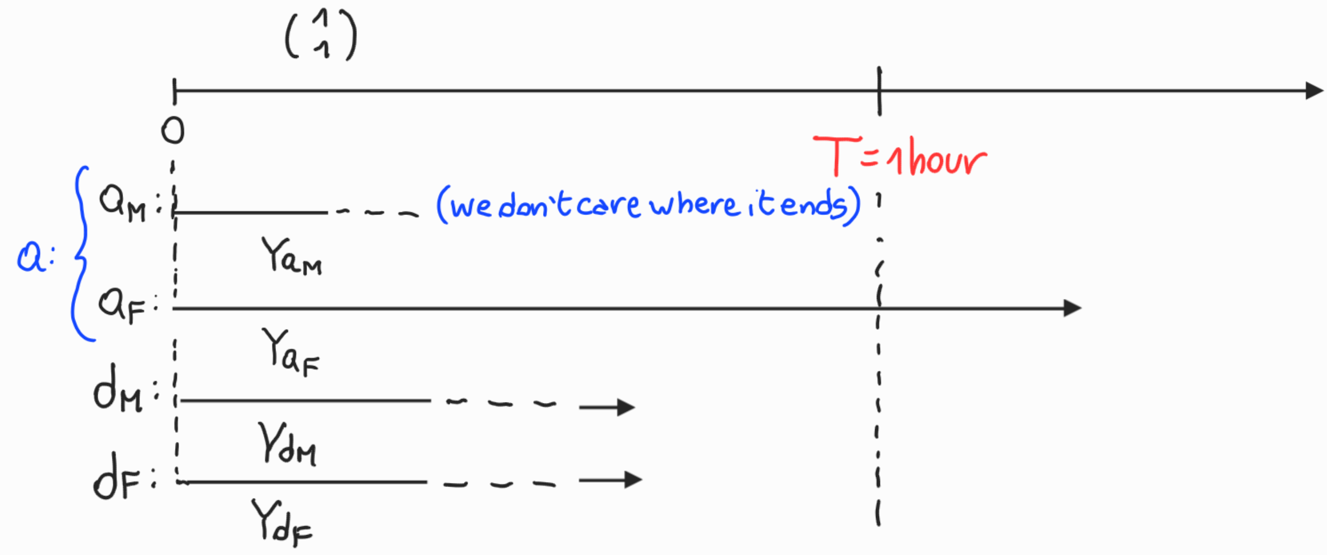
  => Poisson process with rate λp

- a_F = arrival of a female customer

  => Poisson process with rate $\lambda(1-p)$

Event a (Poisson process with rate λ) is the superposition of the Poisson processes of a_M and a_F.

The situation can be illustrated as follows:



$$P(\ldots) = P\left(Y_{a_F} > T,\ Y_{d_M} < T,\ Y_{d_F} < T\right)$$

Independent

$$= P(Y_{a_F} > T)\, P(Y_{d_M} < T)\, P(Y_{d_F} < T)$$

$$= e^{-\lambda(1-p)T} \cdot \left(1 - e^{-\mu_M T}\right) \cdot \left(1 - e^{-\mu_F T}\right)$$

10.a. The current state is $X_k = \binom{1}{0}$.

$$\Rightarrow E\left[V\binom{1}{0}\right] = \frac{1}{\lambda + \mu_M}$$

$$\underbrace{\lambda p + \lambda(1-p)}_{\lambda} + \mu_M$$

10.b. The current state is $X_k = \binom{1}{1}$.

$$\Rightarrow E\left[V\binom{1}{1}\right] = \frac{1}{\mu_M + \mu_F}$$