Endterm test of Discrete Event Systems - 22.12.2021

Exercise 1

A computer system is composed of a server, preceded by a buffer memory, which may host up to two processes waiting to be executed. If the memory is full, arriving processes are rejected, unless the arrival is simultaneous with the termination of the execution of a process. In that case, the arrival is still accepted. Processes arrive with constant interarrival times equal to 2 s. The computer system is initially empty.

Assume that execution times are deterministic, and all equal to 4 s.

1. Compute the percentage of processes rejected at steady state.
2. Compute the average waiting time of a process at steady state.

Then, assume that execution times have a uniform distribution over the interval $[3.5, 5]$ s.

3. Compute the probability that the third arriving process finds the memory empty.
4. Compute the probability that exactly three processes arrive during the execution of the first two processes.

Exercise 2

The police organize a checkpoint where cars are stopped for inspection. Cars transit the checkpoint as generated by a Poisson process with average interarrival time equal to 90 s. During the inspection of a car, no other car is stopped. After a stopped car has been released, the first car arriving next is stopped with probability $p = 3/5$. If the first car is not stopped, the second car is stopped with probability p. If the second car is not stopped, the third car is definitely stopped. The inspection of a car can be of two types: partial (only documents) and complete (documents and vehicle). Partial inspection is decided with probability $q = 5/8$. The duration of a partial inspection has an exponential distribution with expected value 5 min, whereas the duration of a complete inspection has an exponential distribution with expected value 15 min. Assume that, when the checkpoint is opened, the first car will be stopped.

1. Model the checkpoint by a stochastic timed automaton $(\mathcal{E}, \mathcal{X}, \Gamma, p, x_0, F)$.
2. Compute the average number of cars transiting while a car is under complete inspection.
3. Assume that a stopped car has just been released. Compute the probability to return in the same state after completing a partial inspection, and with at most three events.
4. Assume that a stopped car has just been released. Compute the probability to return in the same state within 10 min, with the stopped car inspected partially, and no car transiting while the car is under inspection.

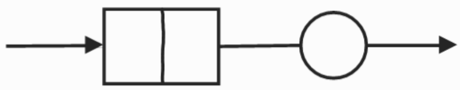
Exercise 3

A production station is composed of a machine, preceded by a bin where only one item can be stored. The machine takes exactly 2 min to process one item. Items arrive in a deterministic fashion at a rate of one per minute. Arriving items are discarded if the system is full and the processing of an item is ongoing. If an arrival and a departure occur simultaneously, the departure is assumed to occur first, in order to free space and accept the arrival. Each accepted item is tested. If an item is found defective (this occurs with probability $p = 1/8$), it is discarded. Assume that the test is instantaneous. Moreover, assume that the system is initially empty.

1. Model the system with a discrete-time homogeneous Markov chain $(\mathcal{X}, P, \pi_0)$.
2. Compute the probability that no arriving item is rejected due to lack of space in the first 20 min.
3. Compute the average number of items in the system at steady state.
4. Compute the average time to the first termination of the processing of an item. Can you provide two alternative ways to compute this quantity?

Exercise 1

The system can be represented as:

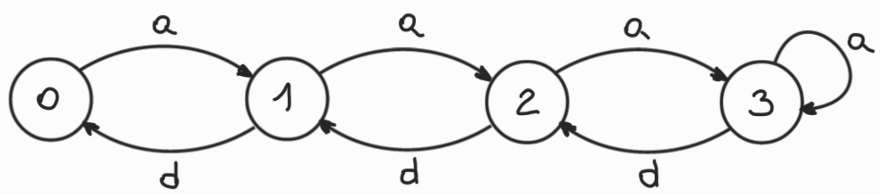


Events $\mathcal{E} = \{a, d\}$

a: arrival of a process

d: termination of the execution of a process

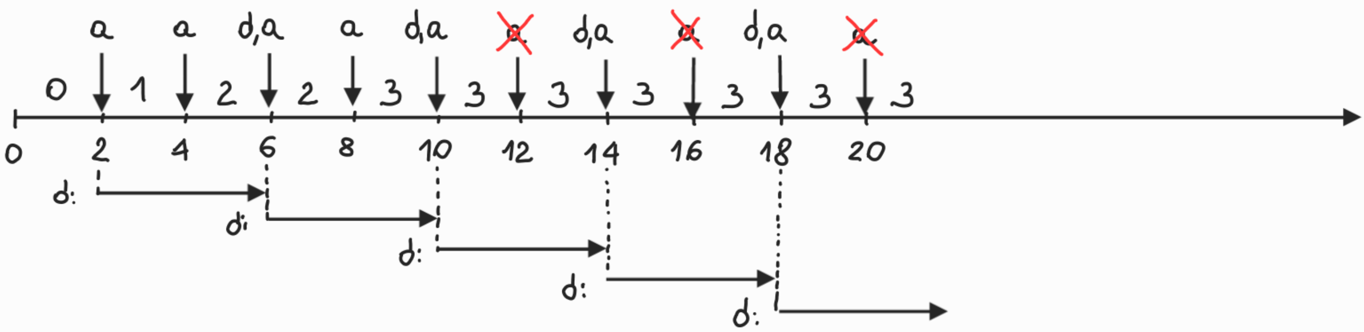
State $x = \#$ processes in the system $\in \{0, 1, 2, 3\}$



1. Event lifetimes

$V_a = 2\ s$

$V_d = 4\ s$



The system is at steady state from time $t = 10\ s$.

At steady state, 50% of the arriving processes are rejected.

2. At steady state, the waiting time of a process is 8 s (sum of the execution times of the two processes ahead of it).

3. Event lifetimes

$V_a = 2$ s

$V_d \sim U(3.5, 5)$

When the third arrival occurs, the state must be either 0 or 1 (=> empty memory). This means that at least one event d must occur before the third arrival. The complement of this event is when the first three events are arrivals. Hence,

$$P(\ldots) = 1 - P(E_1 = a, E_2 = a, E_3 = a \mid X_0 = 0)$$

$$= 1 - P(V_{d,1} > V_{a,2} + V_{a,3})$$

$$= 1 - P(V_{d,1} > 4)$$

$$= P(V_{d,1} \leq 4) = \frac{4 - 3.5}{5 - 3.5} = \frac{1}{3} \simeq 0.3333$$

4. Notice that the second arrival occurs before the termination of the execution of the first process. Indeed, $V_{a,2} = 2$ s, while $V_{d,1} \geqslant 3.5$ s. Hence, the execution of the second process starts immediately when the execution of the first process terminates.

We have:

$$P(\dots) = P\Big(V_{a,2}+V_{a,3}+V_{a,4} < \underbrace{V_{d,1}+V_{d,2}}_{\text{execution time of the first two processes}} < V_{a,2}+V_{a,3}+V_{a,4}+V_{a,5}\Big)$$

the second, the third and the fourth process arrive during the execution time of the first two processes

the fifth process arrives after the execution of the first two processes

$$= P\left(6 < V_{d,1}+V_{d,2} < 8\right) = \frac{\frac{1}{2}}{\frac{9}{4}} = \frac{2}{9} \simeq 0.2222$$

Exercise 2

1. Events $\mathcal{E} = \{ a, d_1, d_2 \}$

a: arrival of a car

d_1: termination of partial inspection

d_2: termination of complete inspection

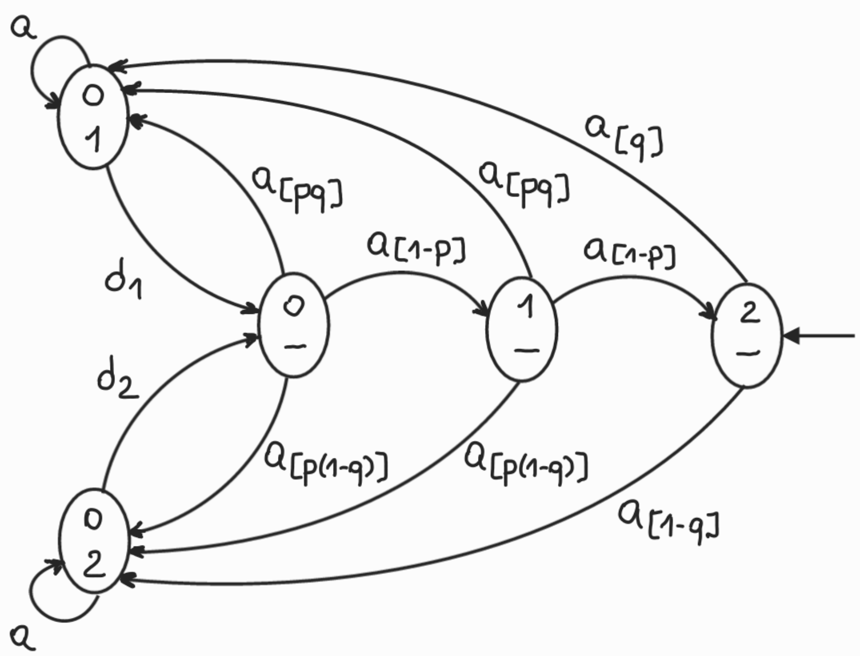
State $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$x_1 \rightarrow$ counter of non-stopped cars $\in \{0,1,2\}$

$x_2 \rightarrow$ type of inspection $\in \{ -, 1, 2 \}$

$-$: none, 1: partial, 2: complete

State transition diagram



$$\boxed{\begin{aligned} p &= 3/5 \\ q &= 5/8 \end{aligned}}$$

Stochastic clock structure

$V_a \sim Exp\left(\frac{1}{\lambda}\right) \qquad \frac{1}{\lambda} = 90\,s = \frac{3}{2}\,min \Rightarrow \lambda = \frac{2}{3}\,min^{-1}$

$$V_{d_1} \sim Exp\left(\frac{1}{\mu_1}\right) \qquad \frac{1}{\mu_1} = 5\,min \quad \Rightarrow \quad \mu_1 = \frac{1}{5}\,min^{-1}$$

$$V_{d_2} \sim Exp\left(\frac{1}{\mu_2}\right) \qquad \frac{1}{\mu_2} = 15\,min \quad \Rightarrow \quad \mu_2 = \frac{1}{15}\,min^{-1}$$

2. $$E[\ldots] = \lambda \cdot \frac{1}{\mu_2} = \frac{\lambda}{\mu_2} = \frac{\frac{2}{3}}{\frac{1}{15}} = 10 \text{ cars}$$

rate of arriving cars

average duration of complete inspection

3. The current state is $X_k = \begin{bmatrix} 0 \\ - \end{bmatrix}$.

Possible cases:

$\boxed{1}$ $\begin{pmatrix} 0 \\ - \end{pmatrix} \xrightarrow{a[pq]} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \xrightarrow{d_1} \begin{pmatrix} 0 \\ - \end{pmatrix}$

$\boxed{2}$ $\begin{pmatrix} 0 \\ - \end{pmatrix} \xrightarrow{a[pq]} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \xrightarrow{a} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \xrightarrow{d_1} \begin{pmatrix} 0 \\ - \end{pmatrix}$

$\boxed{3}$ $\begin{pmatrix} 0 \\ - \end{pmatrix} \xrightarrow{a[1-p]} \begin{pmatrix} 1 \\ - \end{pmatrix} \xrightarrow{a[pq]} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \xrightarrow{d_1} \begin{pmatrix} 0 \\ - \end{pmatrix}$

We have:

$$P(\boxed{1}) = pq \cdot \frac{\mu_1}{\lambda + \mu_1}$$

$$P(\boxed{2}) = pq \cdot \frac{\lambda}{\lambda+\mu_1} \cdot \frac{\mu_1}{\lambda+\mu_1}$$

$$P(\boxed{3}) = (1-p) \cdot pq \cdot \frac{\mu_1}{\lambda+\mu_1}$$

$$\Rightarrow P(\ldots) = P(\boxed{1}) + P(\boxed{2}) + P(\boxed{3}) \simeq 0.1877$$

4. The current state is $X_k = \begin{bmatrix} 0 \\ - \end{bmatrix}$.

Possible cases:

$\boxed{1}$ $\begin{pmatrix} 0 \\ - \end{pmatrix} \xrightarrow{a[pq]} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \xrightarrow{d_1} \begin{pmatrix} 0 \\ - \end{pmatrix}$

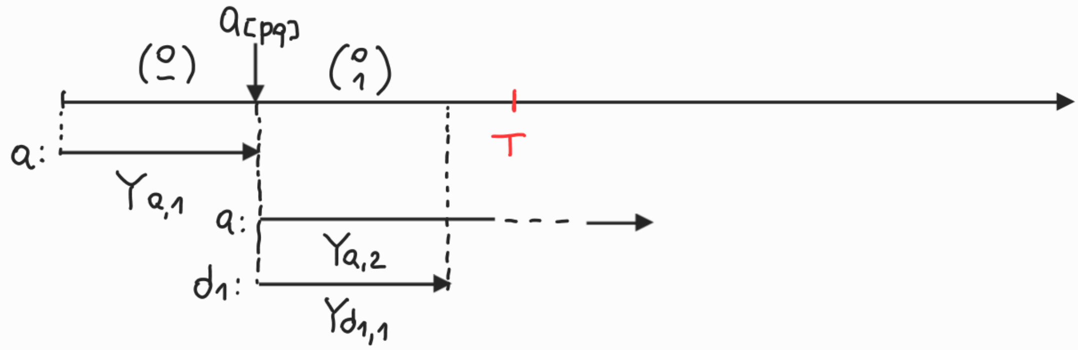
$\boxed{2}$ $\begin{pmatrix} 0 \\ - \end{pmatrix} \xrightarrow{a[1-p]} \begin{pmatrix} 1 \\ - \end{pmatrix} \xrightarrow{a[pq]} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \xrightarrow{d_1} \begin{pmatrix} 0 \\ - \end{pmatrix}$

$\boxed{3}$ $\begin{pmatrix} 0 \\ - \end{pmatrix} \xrightarrow{a[1-p]} \begin{pmatrix} 1 \\ - \end{pmatrix} \xrightarrow{a[1-p]} \begin{pmatrix} 2 \\ - \end{pmatrix} \xrightarrow{a[q]} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \xrightarrow{d_1} \begin{pmatrix} 0 \\ - \end{pmatrix}$

+ impose the constraint that d_1 occurs before $T = 10$ min

We use the sample paths.

$\boxed{1}$ $\quad \begin{pmatrix} 0 \\ - \end{pmatrix} \xrightarrow{a[pq]} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \xrightarrow{d_1} \begin{pmatrix} 0 \\ - \end{pmatrix}$



$$\Rightarrow P(\boxed{1}) = pq \cdot P\left(Y_{d_1,1} < Y_{a,2},\ Y_{a,1} + Y_{d_1,1} < T\right) \simeq 0.0861$$

Similarly, in the other two cases we have:

$$P(\boxed{2}) = (1-p) \cdot pq \cdot P\left(Y_{d_1,1} < Y_{a,3},\ Y_{a,1} + Y_{a,2} + Y_{d_1,1} < T\right)$$

$$\simeq 0.0337$$

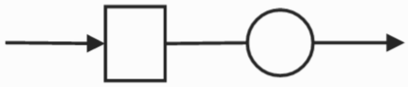
$$P(\boxed{3}) = (1-p)^2 \cdot q \cdot P\left(Y_{d_1,1} < Y_{a,4},\ Y_{a,1} + Y_{a,2} + Y_{a,3} + Y_{d_1,1} < T\right)$$

$$\simeq 0.0213$$

$$\Rightarrow P(\ldots) = P(\boxed{1}) + P(\boxed{2}) + P(\boxed{3}) \simeq 0.1411$$

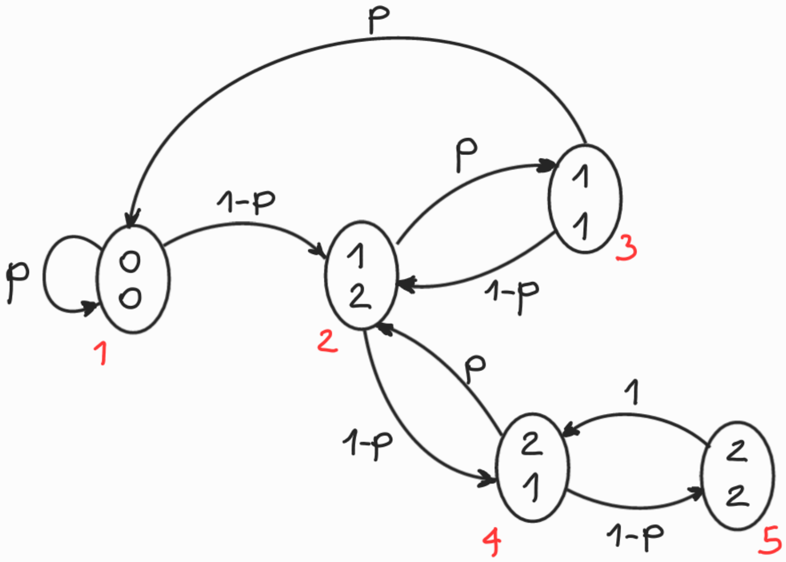
EXERCISE 3

The system can be represented as:



We consider a sampling time $T_s = 1$ min, and we model the system in discrete time.

1. State $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$
   - $x_1 \rightarrow$ # items in the system $\in \{0,1,2\}$
   - $x_2 \rightarrow$ residual duration of the ongoing job $\in \{0,1,2\}$



$$\boxed{p = \frac{1}{8}}$$

Transition probability matrix:

$$P = \begin{bmatrix} p & 1-p & 0 & 0 & 0 \\ 0 & 0 & p & 1-p & 0 \\ p & 1-p & 0 & 0 & 0 \\ 0 & p & 0 & 0 & 1-p \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Initial state probability vector: $\Pi_0 = [1 \ 0 \ 0 \ 0 \ 0]$

2. The only state where arriving items are rejected due to lack of space, is state 5.

We modify the model of point 1 making state 5 absorbing.

$$\tilde{P} = \begin{bmatrix} p & 1-p & 0 & 0 & 0 \\ 0 & 0 & p & 1-p & 0 \\ p & 1-p & 0 & 0 & 0 \\ 0 & p & 0 & 0 & 1-p \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Using the modified model:

$$P(\ldots) = P(\tilde{X}(20) \neq 5)$$

$$= \pi_0 \tilde{P}^{20} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \simeq 0$$

3. The model of point 1 is irreducible, aperiodic and finite.

Limit probabilities can be computed solving

$$\begin{cases} \pi P = \pi \\ \sum_{i=1}^{5} \pi_i = 1 \end{cases} \Rightarrow \pi \simeq \begin{array}{c} \begin{bmatrix} 0.013 & 0.0701 & 0.0088 & 0.4906 & 0.4293 \end{bmatrix} \\ \begin{array}{ccccc} \pi_1 & \pi_2 & \pi_3 & \pi_4 & \pi_5 \end{array} \end{array}$$

$$\Rightarrow E[\ldots] = 0 \cdot \pi_1 + 1 \cdot (\pi_2 + \pi_3) + 2 \cdot (\pi_4 + \pi_5) \simeq 1.9186 \text{ items}$$

4. Let T_1 be the time to the first termination of the processing of an item.

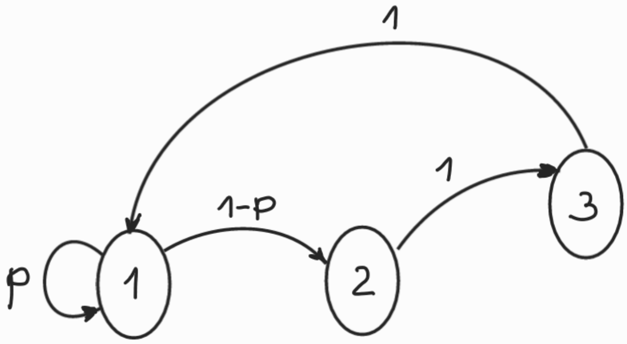
We have:

$$E[T_1] = \underbrace{E[V(1)]}_{\text{state holding time of state 1}} + 2 = \frac{1}{1-p} + 2 = \frac{22}{7} \simeq 3.1429$$

The same quantity can be computed as

$E[T_1] = \dfrac{1}{\bar{\pi}_2}$, or equivalently $E[T_1] = \dfrac{1}{\bar{\pi}_3}$.

using the modified model:



irreducible
aperiodic
finite

$$\bar{P} = \begin{bmatrix} p & 1-p & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\begin{cases} \bar{\pi}\bar{P} = \bar{\pi} \\ \sum_{i=1}^{3} \bar{\pi}_i = 1 \end{cases} \Rightarrow \bar{\pi} \simeq [\underbrace{0.3636}_{\bar{\pi}_1} \;\; \underbrace{0.3182}_{\bar{\pi}_2} \;\; \underbrace{0.3182}_{\bar{\pi}_3}]$$

$$\Rightarrow E[T_1] = \frac{1}{\bar{\pi}_2} \simeq 3.1429$$