Midterm test of Automata and Queueing Systems - 18.11.2021

Exercise 1

In the middle of nowhere, there is a crossroad with a traffic light. The traffic light turns *green*, *yellow*, and *red* in sequence. When the traffic light is green, arriving cars pass through. When the traffic light is yellow, an arriving car finding the road free, either passes through, or stops. If it finds other cars stopped, it stops as well. When the traffic light is red, arriving cars stop. All queued cars restart when the traffic light turns green again: assume that they pass through instantaneously. The duration of green and red intervals is 1 min, and the duration of yellow intervals is 10 s. Assume that time $t = 0$ is when the traffic light has just turned green.

1. Assume that the first car arrives at time $t = 28$ s, and then other cars arrive with interarrival times 15, 8, 12, 5, 18, 10, 22, 16 s. Moreover, assume that all drivers are cautious, and always stop in case they find the road free with the yellow light. Determine the number of queued cars at the end of the first red interval.

Now assume that car interarrival times have a uniform distribution over the interval $[32, 80]$ s, and that an arriving car, finding the road free when the traffic light is yellow, decides to pass through with probability $q = 3/5$.

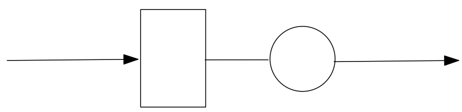
2. Model the system using a stochastic timed automaton $(\mathcal{E}, \mathcal{X}, \Gamma, p, x_0, F)$.
   Suggestions: The state of the system should include the traffic light status, and the number of queued cars. Show only the portion of the state transition diagram that can be actually visited with the given interarrival time distribution.

In the next questions, limit the number of cases using the knowledge of the interarrival time distribution.

3. Compute the probability that there is exactly one stopped car at the end of the first yellow interval.
4. Compute the probability that there are no queued cars at the end of the first red interval.

Exercise 2

A simple production station is shown in the figure. It is composed of a machine, preceded by a one-place buffer. Arrivals of raw parts are generated by a Poisson process with average interarrival time 15 min. Raw parts arriving when the system is full, are rejected.



A part processed for the first time, may either leave the system (with probability $q = 5/8$), or need a second processing, for which the machine has to be preliminarily reconfigured. Durations of first and second processing have exponential distributions with rates 0.1 and 0.0625 min^{-1}, respectively. Reconfiguration times have an exponential distribution with expected value 45 s.

1. Model the production station using a stochastic timed automaton $(\mathcal{E}, \mathcal{X}, \Gamma, p, x_0, F)$, assuming that the system is initially empty.
2. Compute the average number of raw parts arriving while the machine is under reconfiguration.
3. Assume that the machine is processing a part for the first time, and the buffer is full. Compute the probability that the part in the machine leaves the system before the arrival of the next raw part.
4. Compute again the probability of the previous point, adding the constraint that the part in the machine leaves the system not before $T = 8$ min.

EXERCISE 1

1. Arrivals of cars occur at times

28

$28 + 15 = 43$

$43 + 8 = 51$

$51 + 12 = 63$

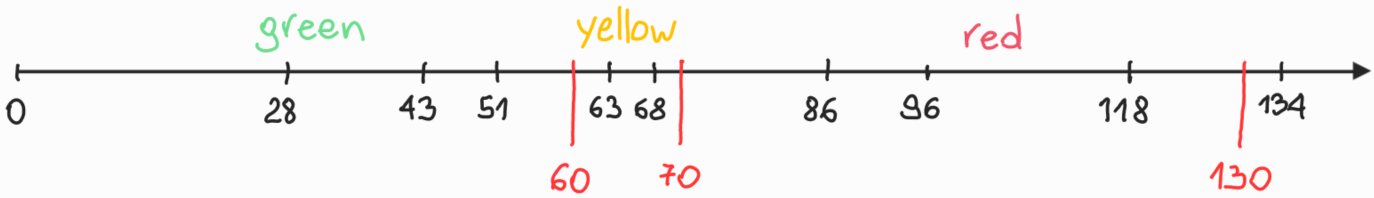
$63 + 5 = 68$

$68 + 18 = 86$

$86 + 10 = 96$

$96 + 22 = 118$

$118 + 16 = 134$



Number of queued cars at the end of the first red interval : 5

(those arriving at times 63, 68, 86, 96, 118)

2. Model $(\mathcal{E}, \mathcal{X}, \Gamma, p, x_0, F)$

Events $\mathcal{E} = \{ a, d_G, d_Y, d_R \}$

a: Arrival of a car

d_G: Termination of green light

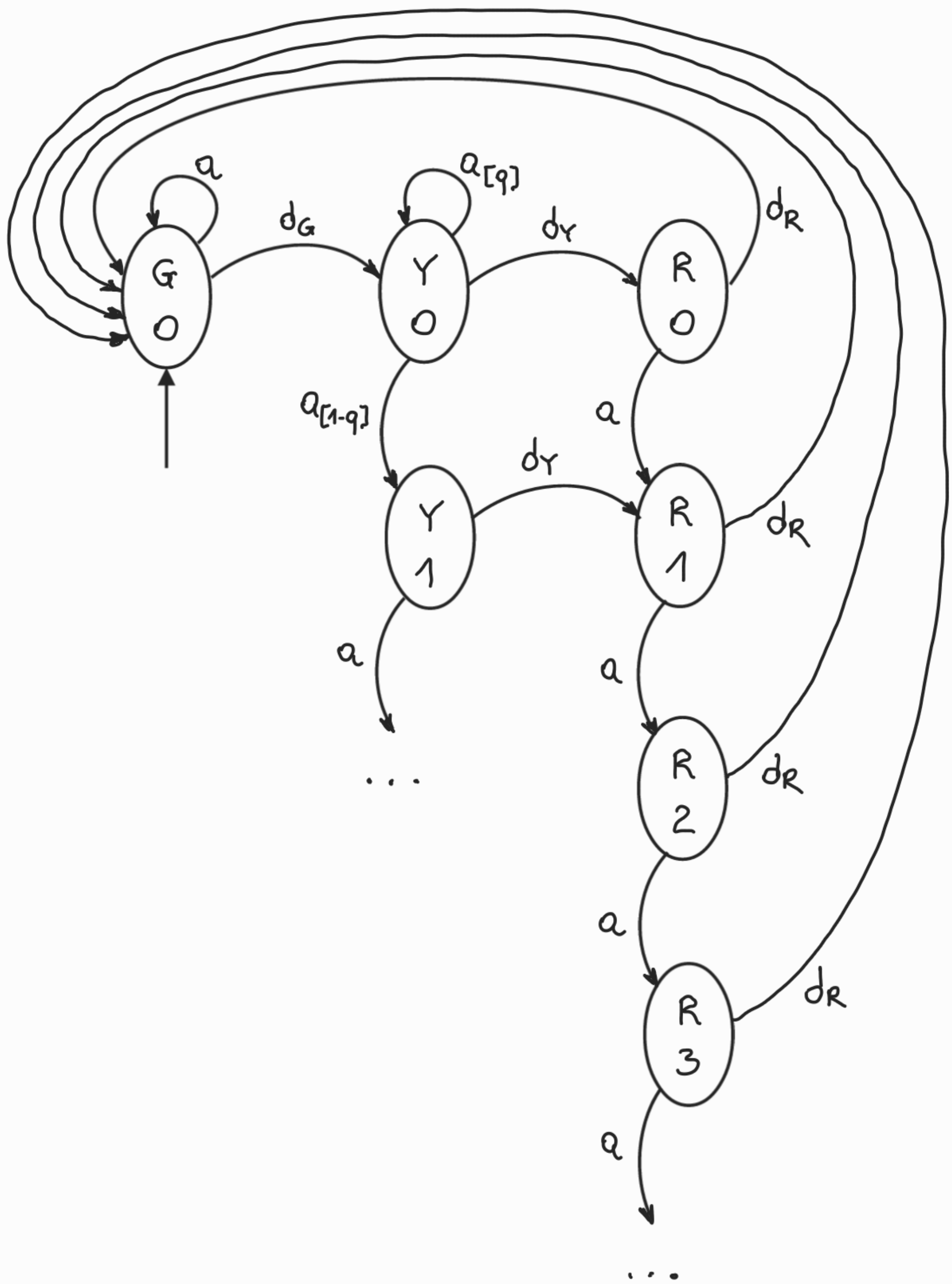
d_Y: Termination of yellow light

d_R: Termination of red light

State $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$x_1 \rightarrow$ Trafficlight: G (green), Y (yellow), R (red)

$x_2 \rightarrow$ # queued cars $\in \{0, 1, 2, 3, \dots\}$



$\boxed{q = \frac{3}{5}}$

Stochastic clock structure:

$V_a \sim U(32, 80)$ s

$V_{d_G} = 60$ s

$V_{d_Y} = 10$ s

$V_{d_R} = 60$ s

REMARK: Notice that, with this stochastic clock structure, at most one arrival is possible when the traffic light is yellow, and at most three arrivals are possible when the traffic light is either yellow or red. This implies that the states represented above, are those that can be actually visited.

3. We have to compute the probability that the state is $\binom{Y}{1}$ when d_Y occurs the first time.

   We consider that:

   - d_Y occurs the first time at time 70
   - over the interval $[0,70)$, at most two arrivals may occur
   - one arrival must be over $[60,70)$, to comply with the requirement that the state is $\binom{Y}{1}$ when d_Y occurs the first time.

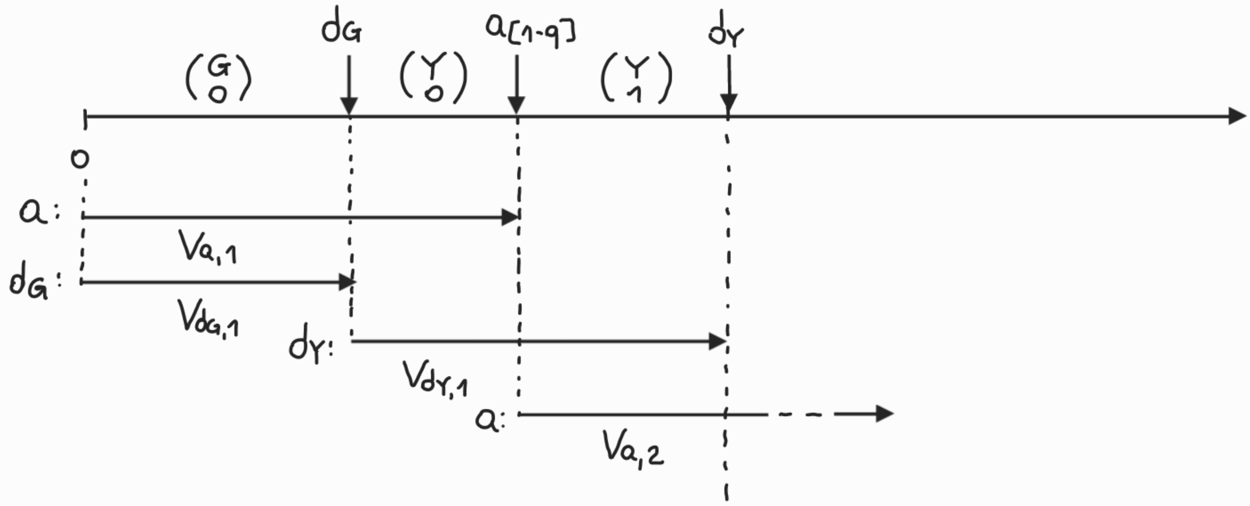
   Hence, we have only two cases:

   $\boxed{1}$ $\binom{G}{0} \xrightarrow{d_G} \binom{Y}{0} \xrightarrow{a_{[1-9]}} \binom{Y}{1} \xrightarrow{d_Y}$

   $\boxed{2}$ $\binom{G}{0} \xrightarrow{a} \binom{G}{0} \xrightarrow{d_G} \binom{Y}{0} \xrightarrow{a_{[1-9]}} \binom{Y}{1} \xrightarrow{d_Y}$

   We use the sample paths to compute the probabilities of these two cases.

$\boxed{1}$ $\quad \binom{G}{0} \xrightarrow{d_G} \binom{Y}{0} \xrightarrow{a[1-q]} \binom{Y}{1} \xrightarrow{d_Y}$

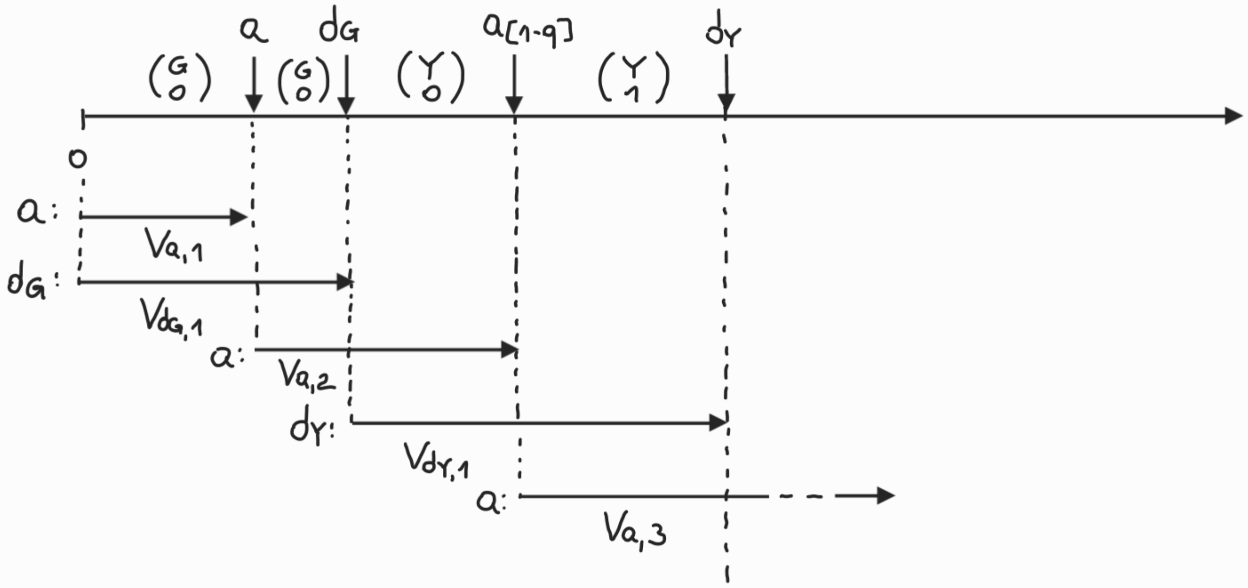


$$\Rightarrow P(\boxed{1}) = (1-q)P\left(V_{d_G,1} < V_{a,1} < V_{d_G,1} + V_{d_Y,1} < V_{a,1} + V_{a,2}\right)$$

$$= (1-q)P\left(60 < V_{a,1} < \underbrace{70 < V_{a,1} + V_{a,2}}_{\substack{\text{always true given} \\ 60 < V_{a,1} < 70}}\right)$$

$$= (1-q)P\left(60 < V_{a,1} < 70\right) = \frac{2}{5} \, \frac{70-60}{80-32} = \frac{1}{12} \simeq 0.0833$$

$\boxed{2}$ $\quad \binom{G}{0} \xrightarrow{a} \binom{G}{0} \xrightarrow{d_G} \binom{Y}{0} \xrightarrow{a[1-q]} \binom{Y}{1} \xrightarrow{d_Y}$



$$\Rightarrow P(\boxed{2}) = (1-q)P\left(V_{a,1} < V_{dG,1} < V_{a,1}+V_{a,2} < V_{dG,1}+V_{dY,1} < V_{a,1}+V_{a,2}+V_{a,3}\right)$$

$$= (1-q)P\Big(\underbrace{V_{a,1} < 60 <}_{\substack{\text{always true given}\\ V_{a,1}+V_{a,2}<70}} V_{a,1}+V_{a,2} < \underbrace{70 < V_{a,1}+V_{a,2}+V_{a,3}}_{\substack{\text{always true given}\\ V_{a,1}+V_{a,2}<70}}\Big)$$

$$= (1-q)P\left(V_{a,1}+V_{a,2} < 70\right) = \frac{2}{5}\,\frac{\frac{6\cdot 6}{2}}{48\cdot 48} = \frac{1}{320} \simeq 0.0031$$

$$\Rightarrow P(\ldots) = P(\boxed{1}) + P(\boxed{2}) = \frac{83}{960} \simeq 0.0865$$

4. We have to compute the probability that the state is $\binom{R}{0}$ when d_R occurs the first time. To comply with this requirement, we need no arrivals over $[70, 130)$, when the traffic light is red. Hence, since $V_{a,1} \sim U(32, 80)$, at least one arrival must occur over $[0, 70)$.

Recall that at most two arrivals may occur over the interval $[0, 70)$. If two arrivals occur, one is over $[0, 60)$, and one over $[60, 70)$.

Finally, if there is an arrival over $[60, 70)$, the yellow interval, it should pass through.

Hence, we have only three cases:

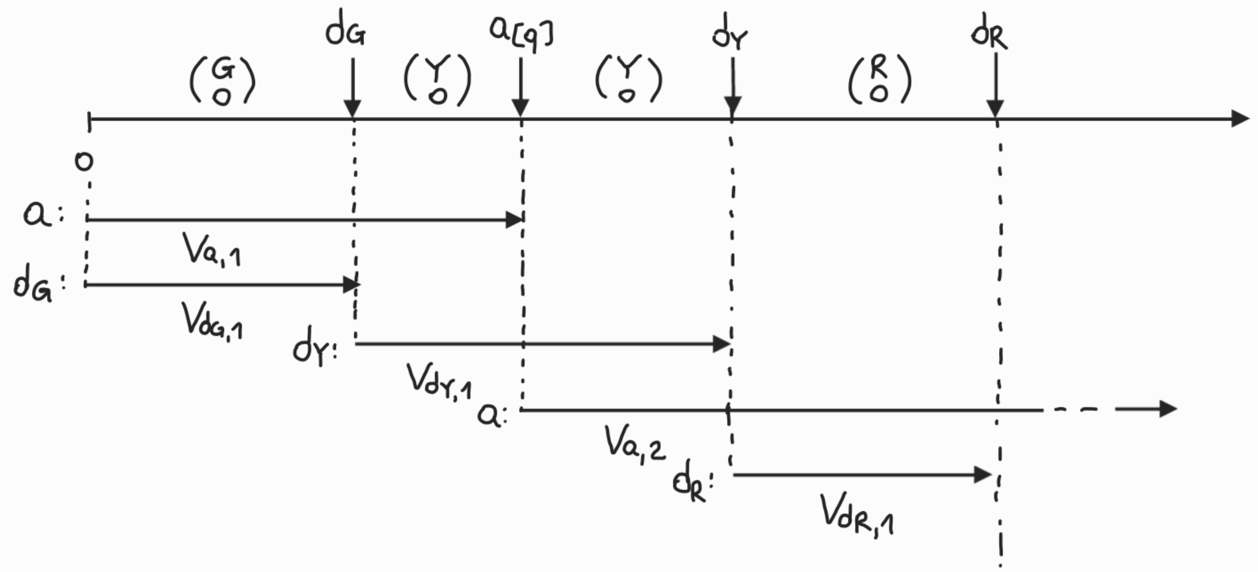
$$\boxed{1}\quad \binom{G}{0} \xrightarrow{d_G} \binom{Y}{0} \xrightarrow{a[q]} \binom{Y}{0} \xrightarrow{d_Y} \binom{R}{0} \xrightarrow{d_R}$$

$$\boxed{2}\quad \binom{G}{0} \xrightarrow{a} \binom{G}{0} \xrightarrow{d_G} \binom{Y}{0} \xrightarrow{d_Y} \binom{R}{0} \xrightarrow{d_R}$$

$$\boxed{3}\quad \binom{G}{0} \xrightarrow{a} \binom{G}{0} \xrightarrow{d_G} \binom{Y}{0} \xrightarrow{a[q]} \binom{Y}{0} \xrightarrow{d_Y} \binom{R}{0} \xrightarrow{d_R}$$

We use the sample paths to compute the probabilities of these three cases.

$$\boxed{1}\quad \binom{G}{0} \xrightarrow{d_G} \binom{Y}{0} \xrightarrow{a[q]} \binom{Y}{0} \xrightarrow{d_Y} \binom{R}{0} \xrightarrow{d_R}$$

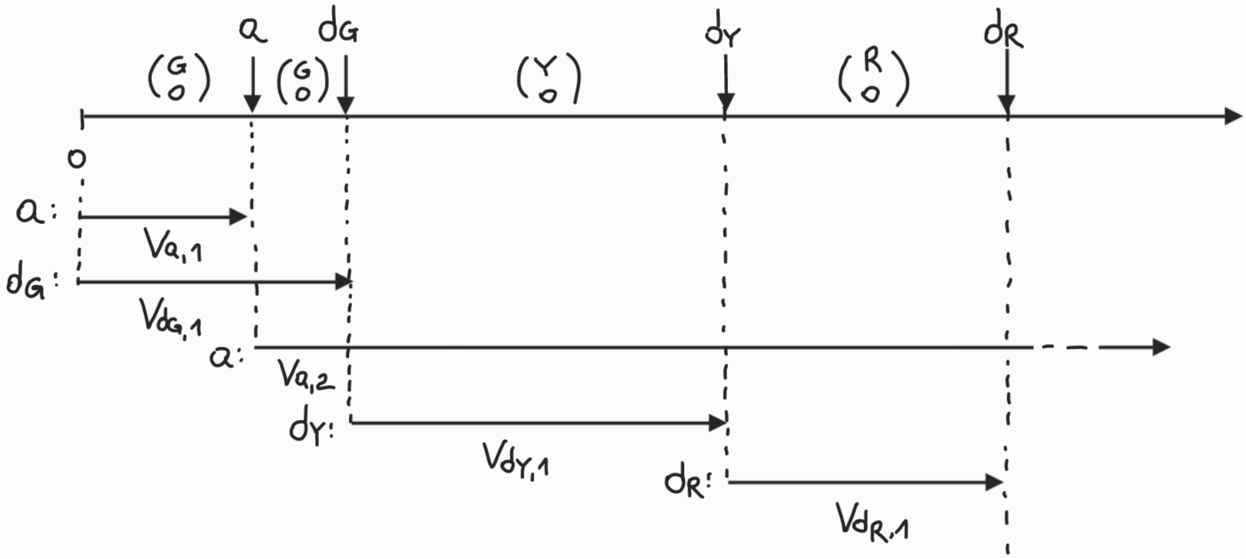


$$\Rightarrow P(\boxed{1}) = q \cdot P\left(V_{d_G,1} < V_{a,1} < V_{d_G,1} + V_{d_Y,1},\ V_{d_G,1} + V_{d_Y,1} + V_{d_R,1} < V_{a,1} + V_{a,2}\right)$$

$$= q \cdot P\left(60 < V_{a,1} < 70,\ V_{a,1} + V_{a,2} > 130\right) \simeq 0.0391$$

estimated with Matlab

$\boxed{2}$ $\quad \binom{G}{0} \xrightarrow{a} \binom{G}{0} \xrightarrow{d_G} \binom{Y}{0} \xrightarrow{d_Y} \binom{R}{0} \xrightarrow{d_R}$

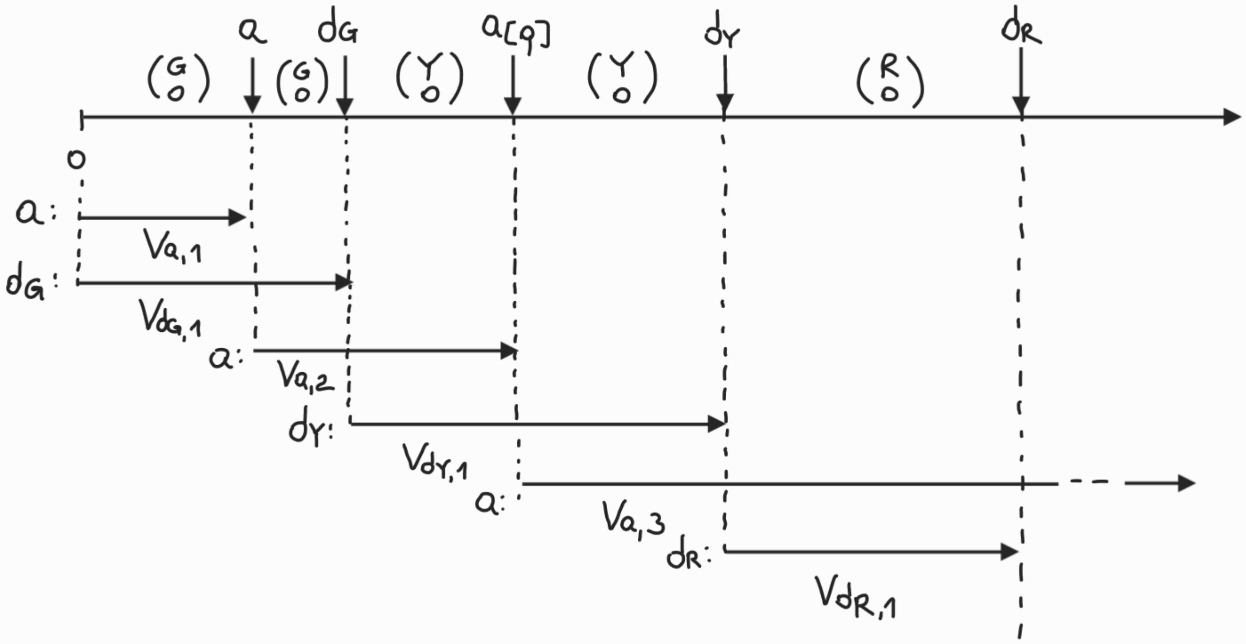


$$\Rightarrow P(\boxed{2}) = P\left(V_{a,1} < V_{d_G,1},\ V_{d_G,1}+V_{d_Y,1}+V_{d_R,1} < V_{a,1}+V_{a,2}\right)$$

$$= P\left(V_{a,1} < 60,\ V_{a,1}+V_{a,2} > 130\right) \simeq 0.0217$$

estimated with Matlab

$\boxed{3}$ $\quad \binom{G}{0} \xrightarrow{a} \binom{G}{0} \xrightarrow{d_G} \binom{Y}{0} \xrightarrow{a[q]} \binom{Y}{0} \xrightarrow{d_Y} \binom{R}{0} \xrightarrow{d_R}$



$$\Rightarrow P(\boxed{3}) = q \cdot P\big(V_{a,1} < V_{d_{G,1}} < V_{a,1} + V_{a,2} < V_{d_{G,1}} + V_{d_{Y,1}},$$

$$V_{d_{G,1}} + V_{d_{Y,1}} + V_{d_{R,1}} < V_{a,1} + V_{a,2} + V_{a,3}\big)$$

$$= q \cdot P\big(V_{a,1} + V_{a,2} < 70,\ V_{a,1} + V_{a,2} + V_{a,3} > 130\big)$$

$$\simeq 0.0018$$

estimated with Matlab

$$\Rightarrow P(\ldots) = P(\boxed{1}) + P(\boxed{2}) + P(\boxed{3}) \simeq 0.0625$$

EXERCISE 2

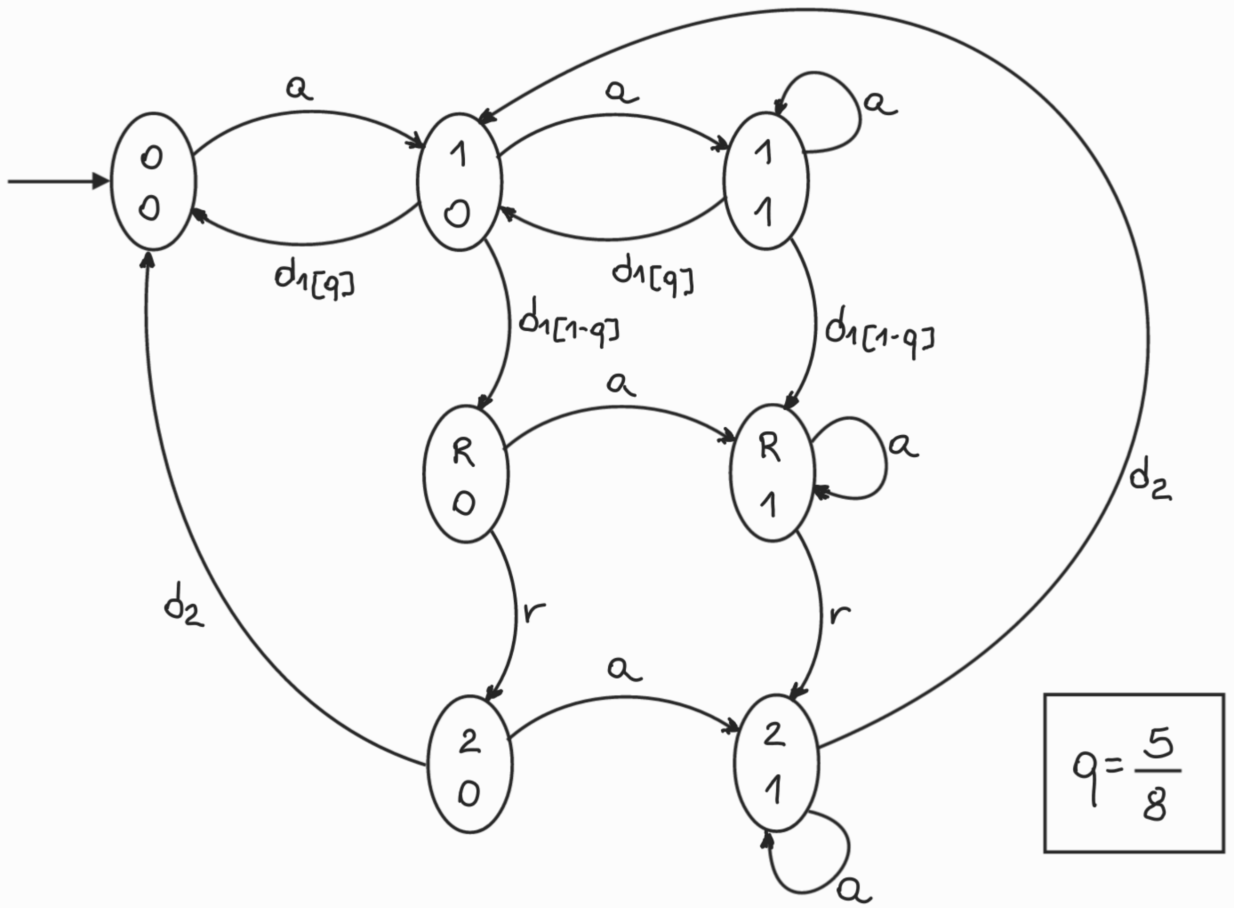
1. Model $(\mathcal{E}, \mathcal{X}, \Gamma, p, x_0, F)$

Events $\mathcal{E} = \{ a,\ d_1,\ d_2,\ r \}$

- a: Arrival of a raw part
- d_1: Termination of 1st processing
- d_2: Termination of 2nd processing
- r: End of reconfiguration

State $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

- $x_1 \to$ M: 0 (idle), 1 (1st proc.), 2 (2nd proc.), R (reconfig.)
- $x_2 \to$ B: 0 (empty), 1 (full)



$\boxed{q = \frac{5}{8}}$

Stochastic clock structure $F = \{F_a, F_{d_1}, F_{d_2}, F_r\}$

$F_a(t) = 1 - e^{-\lambda t}, \; t \geq 0 \qquad \lambda = \frac{1}{15} \simeq 0.0667$ arrivals/min

$F_{d_1}(t) = 1 - e^{-\mu_1 t}, \; t \geq 0 \qquad \mu_1 = 0.1$ jobs/min

$F_{d_2}(t) = 1 - e^{-\mu_2 t}, \; t \geq 0 \qquad \mu_2 = 0.0625$ jobs/min

$F_r(t) = 1 - e^{-\rho t}, \; t \geq 0 \qquad \frac{1}{\rho} = 45\,s = \frac{3}{4}$ min

$\Rightarrow \rho = \frac{4}{3} \simeq 1.3333$ reconf./min

$\Rightarrow$ Poisson stochastic clock structure

2. Let N_a = # raw parts arriving while the machine is under reconfiguration

$$\Rightarrow E[N_a] = \lambda \cdot \frac{1}{\rho} = \frac{\lambda}{\rho} = \frac{1}{20} = 0.05$$

rate of arriving raw parts

average reconfiguration time

3. The current state is $X_k = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. We have two cases:

$\boxed{1}$ $\begin{pmatrix} 1 \\ 1 \end{pmatrix} \xrightarrow{d_1[q]}$

$\boxed{2}$ $\begin{pmatrix} 1 \\ 1 \end{pmatrix} \xrightarrow{d_1[1-q]} \begin{pmatrix} R \\ 1 \end{pmatrix} \xrightarrow{r} \begin{pmatrix} 2 \\ 1 \end{pmatrix} \xrightarrow{d_2}$

$$P(\boxed{1}) = \frac{\mu_1 q}{\lambda + \mu_1} \simeq 0.3750$$

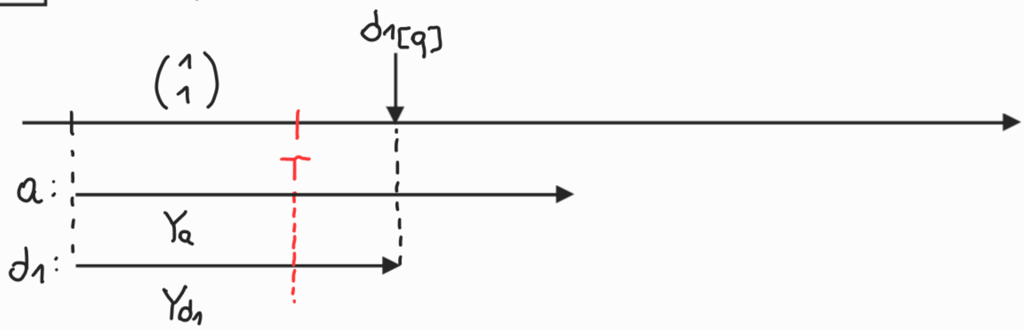
$$P(\boxed{2}) = \frac{\mu_1(1-q)}{\lambda+\mu_1} \frac{\rho}{\lambda+\rho} \frac{\mu_2}{\lambda+\mu_2} \simeq 0.1037$$

$$\Rightarrow \; P(\ldots) = P(\boxed{1}) + P(\boxed{2}) \simeq 0.4787$$

4. We have the same two cases of the previous point, with the additional constraint that the part in the machine leaves the system not before $T = 8$ min.

<u>We use the sample paths.</u>

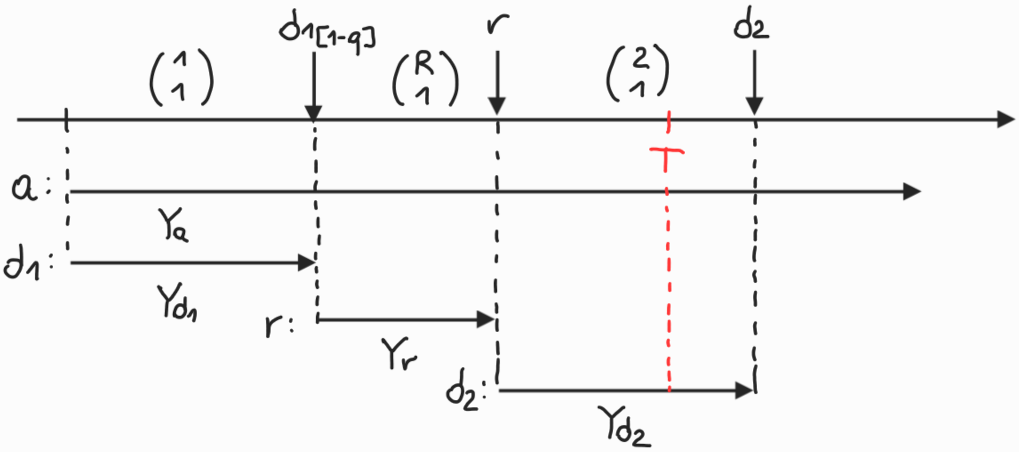
$$\boxed{1} \qquad \begin{pmatrix} 1 \\ 1 \end{pmatrix} \xrightarrow{d_1[q]}$$



$$\Rightarrow P(\boxed{1}) = q\, P(T < Y_{d_1} < Y_a) \simeq 0.0988$$

estimated with Matlab

$$\boxed{2} \qquad \begin{pmatrix}1\\1\end{pmatrix} \xrightarrow{d_{1[1-q]}} \begin{pmatrix}R\\1\end{pmatrix} \xrightarrow{r} \begin{pmatrix}2\\1\end{pmatrix} \xrightarrow{d_2}$$



$$P(\boxed{2}) = (1-q)\,P\left(T < Y_{d_1} + Y_r + Y_{d_2} < Y_a\right) \simeq 0.0738$$

estimated with Matlab

$$\Rightarrow \quad P(\dots) = P(\boxed{1}) + P(\boxed{2}) \simeq 0.1726$$