Endterm test of Discrete Event Systems - 10.12.2019

Exercise 1

The professor of Italian literature of the University of Nowhere does not have fixed office hours. Students can knock on his door and, if he is available, he receives them. The professor receives only one student at a time. If he is busy with a student, he gently rejects the requests of other students. A student received by the professor can be asked to sit in the office and wait for the professor to terminate his previous task before he answers the student's questions.

1. The professor arrives at 9AM. Assume that students come to the professor's office at 9:45AM, 10:03AM, 10:18AM, 10:50AM, 11:20AM, 11:37AM, 12:08PM, 1:58PM, 2:23PM and 2:40PM, and the professor receives only the first, third and fourth student among those coming when no student is in the office. The second and third student received by the professor wait respectively 10 and 12 min in his office. The time for answering all the student's questions is 50 min, 65 min and 42 min for the three students. The professor goes home after he has answered all the questions of the last student. What time does he leave his office?

In the following, assume that students come to the professor's office as generated by a Poisson process with average interarrival time equal to 45 min. If a student comes when the professor is not busy with another student, the probability to be received is $p = 5/8$. Moreover, if the student is received, the probability that he has to wait in the office is $q = 1/3$. The waiting time in the office is uniformly distributed over the interval [5.0,12.0] min. The time that the professor dedicates to answer the questions is fixed for all the students and equal to $T = 30$ min. No student is initially in the professor's office.

2. Model the system through a stochastic timed automaton $(\mathcal{E}, \mathcal{X}, \Gamma, p, x_0, F)$.

3. The professor arrives at 9AM. Compute the probability that only one student comes between 9AM and 10AM, he is received by the professor, and he leaves the professor's office no later than 10:15AM.

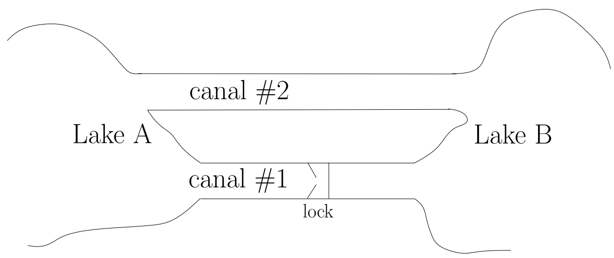
4. If a student is received by the professor, compute the probability that the student arriving next is also received by the professor.

5. Compute the average time spent by a student in the professor's office.

Exercise 2

Two lakes are connected by two canals, as shown in the figure. Boats go from Lake A to Lake B. Canal #1 is divided into two parts by a lock (a device used for raising the boats from the level of Lake A to the level of Lake B). Canal #2 is not interrupted by locks (only one part). Travel time of the first part of canal #1 is exponentially distributed with expected value 12 min. It is assumed that raising the boat in the lock takes a negligible time, as well as stopping and accelerating the boat before and after the lock, respectively. Travel time of the second part of canal #1 is exponentially distributed with expected value 6 min. Travel time of canal #2 is exponentially distributed with expected value 48 min. It is forbidden that two or more boats use simultaneously the same canal. There are always boats waiting in Lake A for transit to Lake B.

Normally, canal #2 is not used, unless the automatic mechanism of the lock of canal #1 fails. In that case, manual maneuvering of the lock is activated and canal #2 is opened. When a boat in transit in canal #2 reaches Lake B, the navigation in canal #2 is closed again if the navigation in canal #1 is regular (that is, manual maneuvering is not activated). Let the automatic mechanism of the lock of canal #1 fail with probability $p = 1/6$. Moreover, assume that the duration of the manual maneuvering of the lock is exponentially distributed with expected value 30 min.



1. Model the system through a stochastic timed automaton $(\mathcal{E}, \mathcal{X}, \Gamma, p, x_0, F)$, leaving the initial state unspecified.

2. Compute the average duration of the transit of a boat in canal #1.

3. Assume that a boat is in the first part of canal #1 and canal #2 is closed. Compute the probability that the automatic mechanism of the lock fails, but the boat navigating canal #1 reaches Lake B before the one navigating canal #2.

4. Assume that a boat is in the first part of canal #1 and canal #2 is closed. Compute the probability that another boat may enter canal #1 before one hour and, if applicable, before that a boat navigating canal #2 reaches Lake B.

Exercise 3

$N = 3$ mobile robots move randomly in an environment composed of two rooms (say room A and room B) connected by an open door. During a sampling time, each robot can change room at most once. This occurs with probability $p = 2/5$ from room A to room B, and with probability $q = 5/8$ from room B to room A.

1. Model the system through a discrete-time homogeneous Markov chain, assuming that all the robots start from room A.

2. Compute the average number of robots in room A at steady state. Could you guess this value in terms of N, p and q?

3. Compute the probability to find all the robots simultaneously in room B for the first time after five sampling times.

4. Compute the average number of sampling times to find all the robots simultaneously in room B.

5. Compute the probability that all the robots visit room B at least once during the first five sampling times.

EXERCISE 1

state $x = \begin{cases} 1: \text{ no student in the professor's office} \\ 2: \text{ student waiting in the professor's office} \\ 3: \text{ professor answering the student's questions} \end{cases}$

events $\mathcal{E} = \{a, w, d\}$

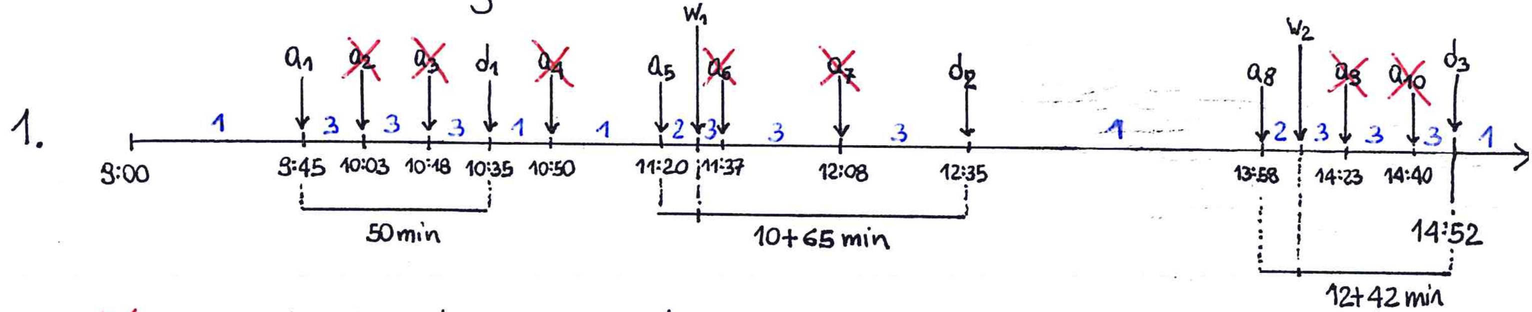
a: arrival of a student; w: termination of waiting; d: termination of answering

Arrival times are given.

$V_w = \{10, 12, \dots\}$

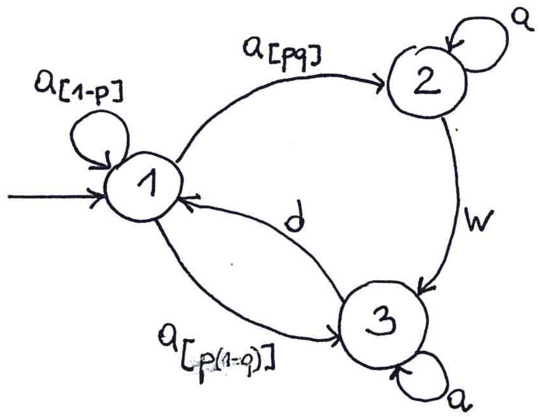
$V_d = \{50, 65, 42, \dots\}$

1.



X represents students that are not received by the professor.

2. model $(\mathcal{E}, \mathcal{X}, \Gamma, p, x_0, F)$



$V_a \sim Exp\left(\frac{1}{\lambda}\right)$ [min]

$\frac{1}{\lambda} = 45 \text{ min} \Rightarrow \lambda = \frac{1}{45} \text{ min}^{-1}$

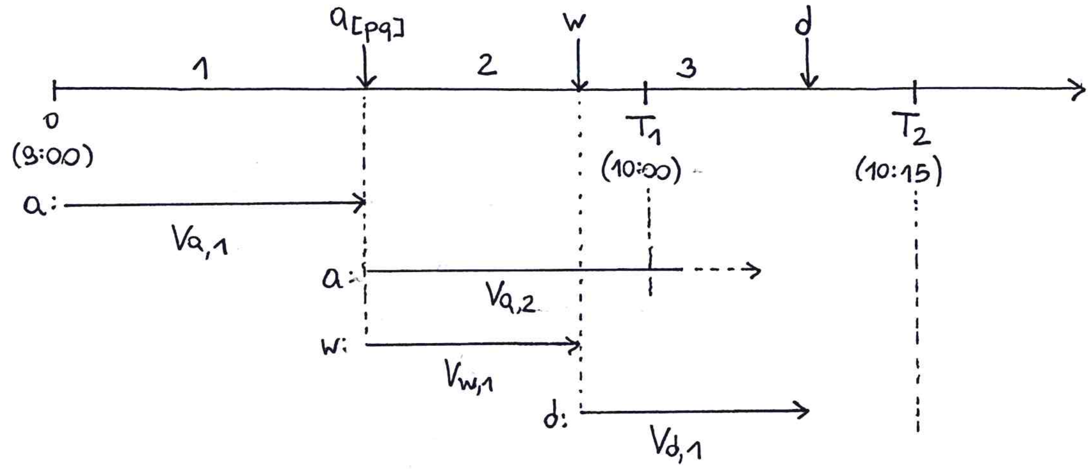
$V_w \sim U(5, 12)$ [min]

$V_d = T = 30$ [min]

$p = \frac{5}{8}, \; q = \frac{1}{3}$

3. There are two cases. Let $T_1 = 60$ [min] and $T_2 = 75$ [min].

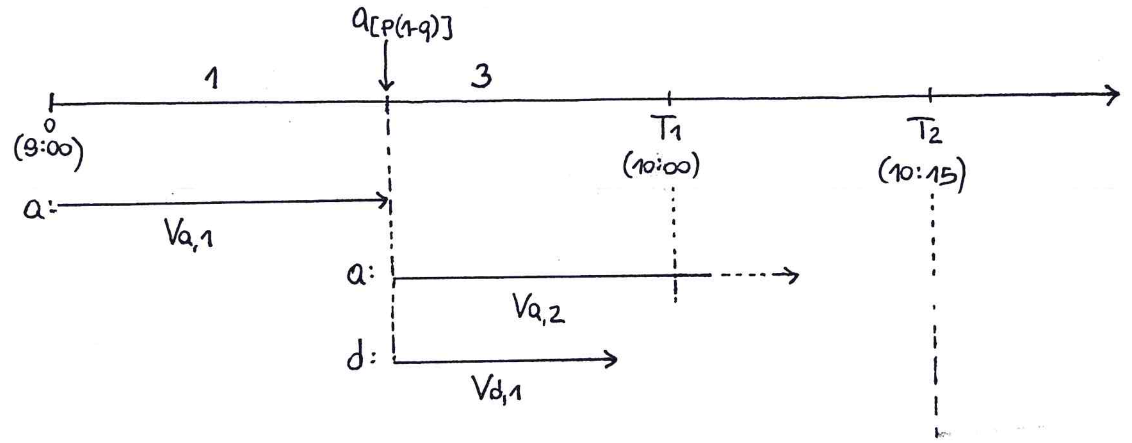
$\boxed{1}$



$$\Rightarrow P(\boxed{1}) = pq\, P\left(V_{a,1} < T_1,\ V_{a,1} + V_{a,2} > T_1,\ V_{a,1} + V_{w,1} + V_{d,1} < T_2\right) \simeq 0.0446$$

estimated with Matlab

$\boxed{2}$

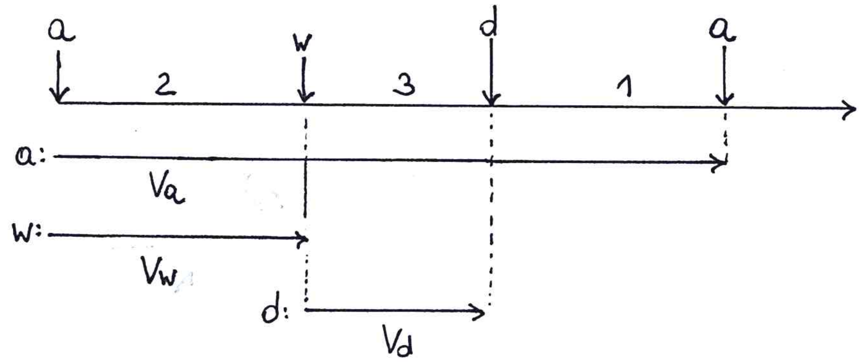


$$\Rightarrow P(\boxed{2}) = p(1-q) P\left(V_{a,1} < T_1,\ V_{a,1} + V_{a,2} > T_1,\ V_{a,1} + V_{d,1} < T_2\right) \simeq 0.1098$$

$$\Rightarrow P(\ldots) = P(\boxed{1}) + P(\boxed{2}) \simeq 0.1544$$

4. There are two cases.

$\boxed{1}$

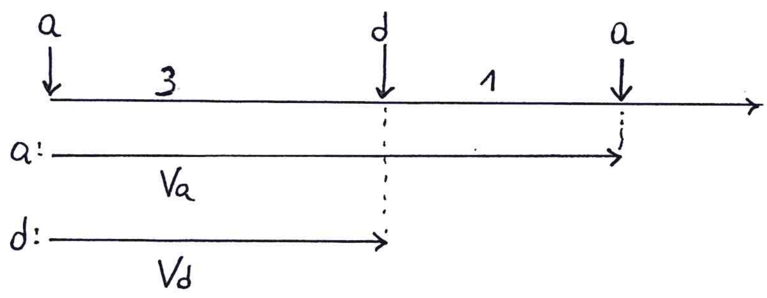


$$\Rightarrow P(\boxed{1}) = q \cdot P(V_w + V_d < V_a) \cdot p \simeq 0.0886$$

prob. that the current student has to wait

prob. that the next student is received

$\boxed{2}$



$$\Rightarrow P(\boxed{2}) = \underset{\substack{\text{prob. that}\\ \text{the current student}\\ \text{has not to wait}}}{(1-q)} \cdot \underbrace{P(V_d < V_a)}_{P(V_a > T)} \cdot \underset{\substack{\text{prob. that the next student}\\ \text{is received}}}{p} = (1-q)\, e^{-\lambda T} \cdot p \simeq 0.2139$$

$$\Rightarrow P(\ldots) = P(\boxed{1}) + P(\boxed{2}) \simeq 0.3026$$

5. The time spent by a student in the professor's office depends on whether he has to wait or not.

$$\Rightarrow E[\ldots] = q\,E[V_w + V_d] + (1-q)E[V_d]$$

$$= q \cdot \underbrace{E[V_w]}_{\frac{12+5}{2}} + \underbrace{E[V_d]}_{30} \simeq 32.8333 \ [\text{min}]$$

EXERCISE 2

1.

state $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ $x_1 \rightarrow$ canal #1: 1 (boat in part 1), 2 (boat in part 2), 3 (manual maneuvering)
$x_2 \rightarrow$ canal #2: 0 (closed), 1 (open)

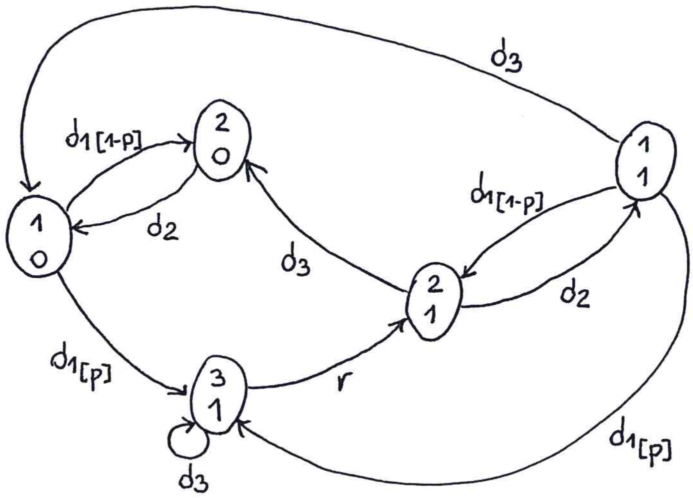
events $\mathcal{E} = \{d_1, d_2, d_3, r\}$

d_1: boat reaches end of part 1 of canal 1

d_2: boat reaches end of part 2 of canal 1

d_3: boat reaches end of canal 2

r: end of manual maneuvering



$$p = \frac{1}{6}$$

$V_{d_1} \sim Exp\left(\frac{1}{\mu_1}\right) \qquad \frac{1}{\mu_1} = 0.2\,h \Rightarrow \mu_1 = 5\,h^{-1}$

$V_{d_2} \sim Exp\left(\frac{1}{\mu_2}\right) \qquad \frac{1}{\mu_2} = 0.1\,h \Rightarrow \mu_2 = 10\,h^{-1}$

$V_{d_3} \sim Exp\left(\frac{1}{\mu_3}\right) \qquad \frac{1}{\mu_3} = 0.8\,h \Rightarrow \mu_3 = \frac{5}{4}\,h^{-1}$

$V_r \sim Exp\left(\frac{1}{\lambda}\right) \qquad \frac{1}{\lambda} = 30\,min = 0.5\,h \Rightarrow \lambda = 2\,h^{-1}$

2. The duration of the transit of a boat in canal #1 depends on whether the lock fails or not.

$$\Rightarrow E[\ldots] = p\,E[V_{d_1} + V_r + V_{d_2}] + (1-p)\,E[V_{d_1} + V_{d_2}]$$

$$= E[V_{d_1}] + pE[V\ ] + E[V_{d_2}]$$

$$= \frac{1}{\mu_1} + p \cdot \frac{1}{\lambda} + \frac{1}{\mu_2} \simeq 0.3833\,h$$

3. $X_k = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

There is only one case:

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \xrightarrow{d_{1[p]}} \begin{pmatrix} 3 \\ 1 \end{pmatrix} \xrightarrow{r} \begin{pmatrix} 2 \\ 1 \end{pmatrix} \xrightarrow{d_2}$$

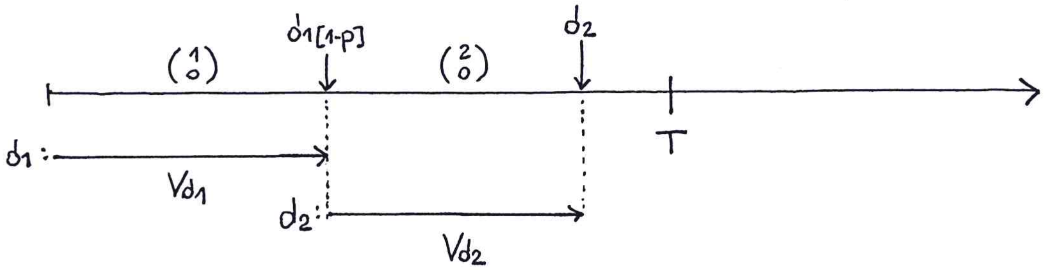
$$\Rightarrow P(\ldots) = p \cdot \frac{\lambda}{\lambda + \mu_3} \cdot \frac{\mu_2}{\mu_2 + \mu_3} \simeq 0.0912$$

4. $X_k = \binom{1}{0}$. Let $T = 1h$.

Another boat may enter canal #1 only when the boat currently in the first part of canal #1 reaches the end of the second part.

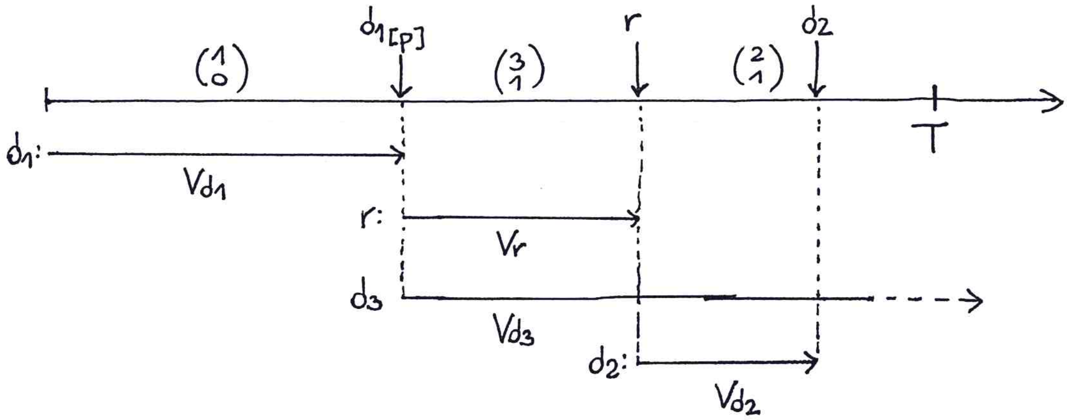
There are two cases.

$\boxed{1}$



$$\Rightarrow P(\boxed{1}) = (1-p)\, P\left(V_{d_1} + V_{d_2} < T\right) \simeq 0.8221$$

$\boxed{2}$



$$\Rightarrow P(\boxed{2}) = p\, P\left(V_{d_1} + V_r + V_{d_2} < T,\ V_r + V_{d_2} < V_{d_3}\right) \simeq 0.0790$$

$$\Rightarrow P(\ldots) = P(\boxed{1}) + P(\boxed{2}) \simeq 0.9012$$

EXERCISE 3

$$\text{room A} \underset{q=5/8}{\overset{p=2/5}{\rightleftarrows}} \text{room B} \qquad N=3 \text{ robots}$$

1. state $X = \#$ robots in room $A \in \{0,1,2,3\}$

$p_{0,0} = (1-q)^3$

$p_{0,1} = 3q(1-q)^2$

$p_{0,2} = 3q^2(1-q)$

$p_{0,3} = q^3$

$p_{1,0} = p(1-q)^2$

$p_{1,1} = (1-p)(1-q)^2 + 2pq(1-q)$

$p_{1,2} = 2(1-p)q(1-q) + pq^2$

$p_{1,3} = (1-p)q^2$

$p_{2,0} = p^2(1-q)$

$p_{2,1} = 2p(1-p)(1-q) + p^2 q$

$p_{2,2} = (1-p)^2(1-q) + 2p(1-p)q$

$p_{2,3} = (1-p)^2 q$

$p_{3,0} = p^3$

$p_{3,1} = 3p^2(1-p)$

$p_{3,2} = 3p(1-p)^2$

$p_{3,3} = (1-p)^3$

$$P = \begin{bmatrix} p_{0,0} & p_{0,1} & p_{0,2} & p_{0,3} \\ p_{1,0} & p_{1,1} & p_{1,2} & p_{1,3} \\ p_{2,0} & p_{2,1} & p_{2,2} & p_{2,3} \\ p_{3,0} & p_{3,1} & p_{3,2} & p_{3,3} \end{bmatrix} \qquad \pi(0) = [0 \quad 0 \quad 0 \quad 1]$$

2. $$\begin{cases} \pi P = \pi \\ \sum_{i=0}^{3} \pi_i = 1 \end{cases} \Rightarrow \quad \pi \simeq [\underbrace{0.0594}_{\pi_0} \quad \underbrace{0.2786}_{\pi_1} \quad \underbrace{0.4353}_{\pi_2} \quad \underbrace{0.2267}_{\pi_3}]$$

the MC is irreducible, aperiodic and finite

$$E[X] = 0 \cdot \pi_0 + 1 \cdot \pi_1 + 2 \cdot \pi_2 + 3 \cdot \pi_3 \simeq 1.893$$

$$N \frac{q}{p+q} \quad !!!!$$

3. We make state 0 absorbing.

$$\tilde{P} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ p_{1,0} & p_{1,1} & p_{1,2} & p_{1,3} \\ p_{2,0} & p_{2,1} & p_{2,2} & p_{2,3} \\ p_{3,0} & p_{3,1} & p_{3,2} & p_{3,3} \end{bmatrix} \qquad \tilde{\pi}(0) = [0 \;\; 0 \;\; 0 \;\; 1]$$

$$\Rightarrow P(\dots) = P(\tilde{X}(5)=0, \tilde{X}(4) \neq 0) = \sum_{i=1}^{3} P(\tilde{X}(5)=0, \tilde{X}(4)=i)$$

$$= \sum_{i=1}^{3} P(\tilde{X}(5)=0 \mid \tilde{X}(4)=i) P(\tilde{X}(4)=i)$$

$$= \sum_{i=1}^{3} p_{i,0}\, \tilde{\pi}_i(4) \simeq 0.0466$$

$$\tilde{\pi}(4) = \tilde{\pi}(0)\tilde{P}^4 \simeq [\underset{\tilde{\pi}_0(4)}{0.2221} \;\; \underset{\tilde{\pi}_1(4)}{0.2313} \;\; \underset{\tilde{\pi}_2(4)}{0.3599} \;\; \underset{\tilde{\pi}_3(4)}{0.1867}]$$

4. We introduce a deterministic transition from state 0 to state 3.

$$\bar{P} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ p_{1,0} & p_{1,1} & p_{1,2} & p_{1,3} \\ p_{2,0} & p_{2,1} & p_{2,2} & p_{2,3} \\ p_{3,0} & p_{3,1} & p_{3,2} & p_{3,3} \end{bmatrix} \qquad \bar{\pi}(0) = [0 \;\; 0 \;\; 0 \;\; 1]$$

$$T_{3,0} = \bar{T}_{0,0} - 1 \quad \Rightarrow \quad E[T_{3,0}] = E[\bar{T}_{0,0}] - 1 = \frac{1}{\bar{\pi}_0} - 1 \simeq 16.6415$$

$$\begin{cases} \bar{\pi} = \bar{\pi}\bar{P} \\ \sum_{i=0}^{3} \bar{\pi}_i = 1 \end{cases} \Rightarrow \bar{\pi} \simeq [\underset{\bar{\pi}_0}{0.0567} \;\; \underset{\bar{\pi}_1}{0.2641} \;\; \underset{\bar{\pi}_2}{0.4102} \;\; \underset{\bar{\pi}_3}{0.2690}]$$

5. The three robots move independently.

The probability that a robot never visits room B during the first five sampling times is $(1-p)^5$.

Hence, the probability that a robot visits room B at least once during the first five sampling times is $1-(1-p)^5$.

It turns out that the probability that all the robots visit room B at least once during the first five sampling times is $\left[1-(1-p)^5\right]^3 \simeq 0.7844$