Test of Discrete Event Systems - 08.02.2019

Student: ____

Please mark with × the type of test you are willing to take:

i) Endterm (A&QS): Exercises 1 and 2 *[1 hour and 40 minutes]*

ii) Endterm (DES): Exercises 1, 2 and 4 *[2 hours and 20 minutes]*

iii) Full (A&QS): Exercises 1, 2 and 3 *[2 hours and 20 minutes]*

iv) Full (DES): Exercises 1, 2, 3 and 4 *[3 hours]*

Exercise 1

Consider an activity with a single operator. Service requests are always possible, and accepted only if the operator is idle. During the main service, the customer may require an extra-service, which is accomplished after the main one. The operator is initially idle.

1. Assume that service requests arrive at times 2.0, 9.0, 16.0, 21.0, 30.0, 36.0, 41.0, 46.0, 50.0 and 57.0 min. Durations of the main service are all equal to 12.0 min. The third, fourth, sixth and eighth customer would require an extra-service, whose duration would be 5.0, 3.5, 6.0 and 4.5 min, respectively. Determine the number of rejected service requests, and the average total service time (including the extra-service for those customers requiring it).

In the following, assume that arrivals of service requests occur with interarrival times following a uniform distribution over the interval [5.0,15.0] min. Durations of the main service are all equal to 12.0 min. A customer requires an extra-service with probability $q = 0.2$. Durations of the extra-service follow a uniform distribution over the interval [1.0,4.0] min.

2. Model the system through a stochastic timed automaton $(\mathcal{E}, \mathcal{X}, \Gamma, p, x_0, F)$.

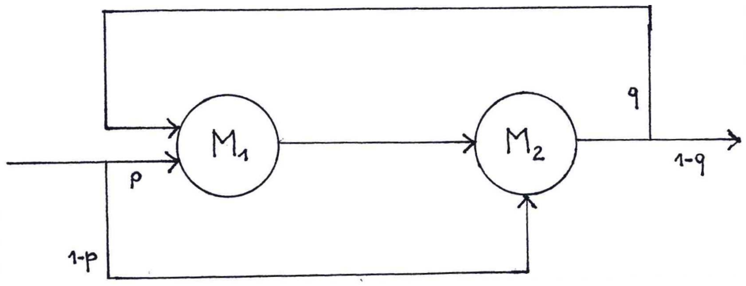
3. Compute the probability that after the third event the operator is accomplishing an extra-service.

4. Compute the probability that exactly two service requests are rejected over the interval [0.0,18.0] min.

5. Determine the cumulative distribution function of the total service time.

Exercise 2

Consider the manufacturing cell in the figure, composed of two machines M_1 and M_2 connected in series. Raw parts arrive as generated by a Poisson process with average interarrival time equal to 10 min. An arriving part requires first processing in M_1 with probability $p = 1/3$, otherwise it is routed directly to M_2. An arriving part is rejected if the corresponding machine is busy. When M_1 terminates a job, it keeps the part until M_2 is available. A part processed by M_2 may need to be sent back to M_1 with probability $q = 1/10$. In this case, if M_1 is working, M_2 keeps the part until M_1 terminates its job. Durations of the jobs in M_1 and M_2 follow exponential distributions with expected values 8 and 6 min, respectively. The manufacturing cell is initially empty.



1. Model the system through a stochastic timed automaton $(\mathcal{E}, \mathcal{X}, \Gamma, p, x_0, F)$.

2. Compute the probability that the system stays in a situation with M_1 working and M_2 idle for at least 5 min.

3. At initialization, compute the probability that the system reaches a situation where it is full before a part leaves the system and with M_1 having terminated exactly one job.

4. Assume that both machines are working. Compute the probability that the system is emptied with the minimum number of events and within $T = 12$ min.

Exercise 3

Consider the model of the system of **Exercise 2**. For the following questions, providing **numerical** answers with the help of Matlab will be considered a plus.

1. Compute the probability that both machines are busy 2 min after start.

2. Compute the average number of parts in the system at steady-state.

3. Verify the condition $\lambda_{eff} = \mu_{eff}$ for the system at steady-state.

4. Compute the average time spent by a part in M_2 at steady state.

5. Compute the utilization of the two machines at steady state.

6. Compute the probability that an arriving part is rejected at steady state.

Exercise 4

Three kids with three balls play throwing the ball to each other. If a ball hits a kid, he is eliminated. The three kids have different abilities. The first kid has a probability equal to 0.4 to hit the kid he aims at. The second kid has a probability equal to 0.25, and the third equal to 0.1. The players throw the ball simultaneously. When all the three kids are still in the game, each one aims at the weakest player among the other two. The game terminates when either only one kid remains in the game, or all the three kids are eliminated.

1. Model the game through a discrete-time homogeneous Markov chain.

2. Compute the probability that exactly three plays are needed to eliminate at least one player.

3. Compute the probability that there is no winner in the game.

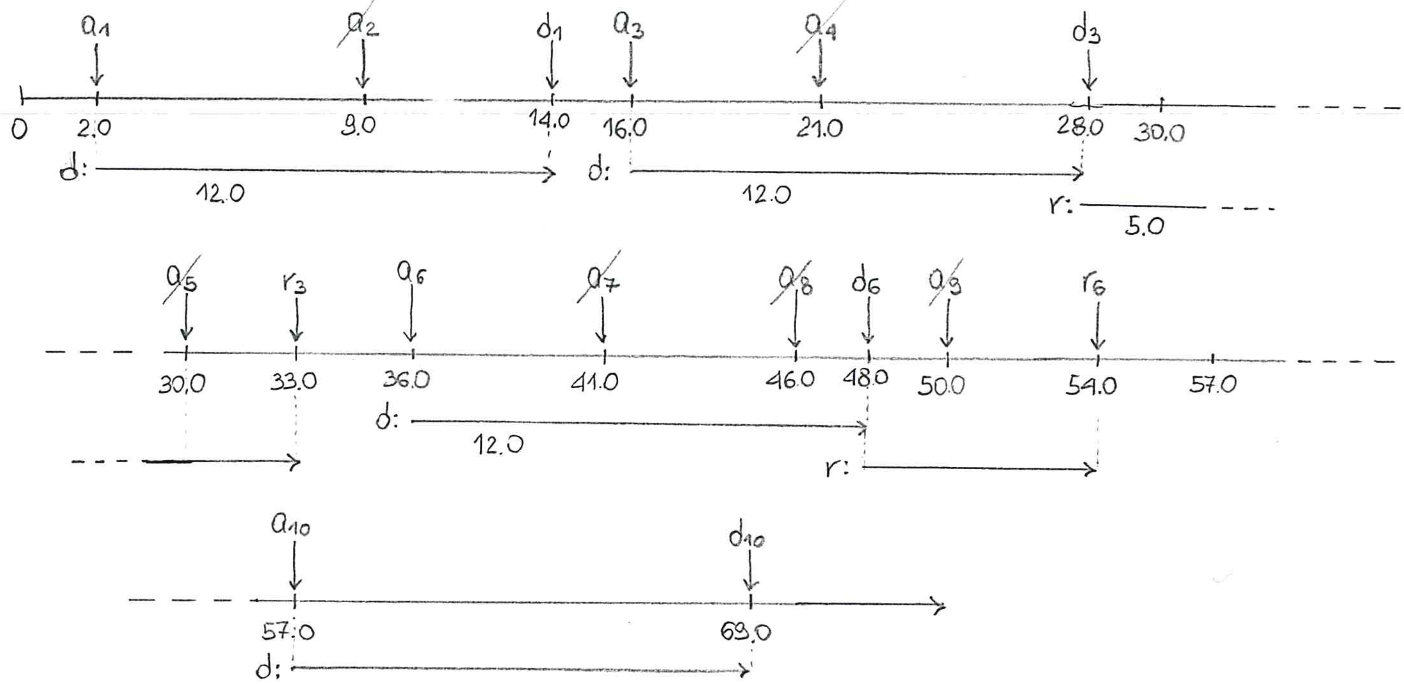
4. Compute the average duration of the game.

Exercise 1

1. $\mathcal{E} = \{a, d, r\}$

a: arrival of a service request
d: termination of the main service
r: termination of the extra-service

$$\left.\begin{array}{l} a_i : \text{event } a \\ d_i : \text{event } d \\ r_i : \text{event } r \end{array}\right\} \text{corresponding to the } i\text{-th customer}$$



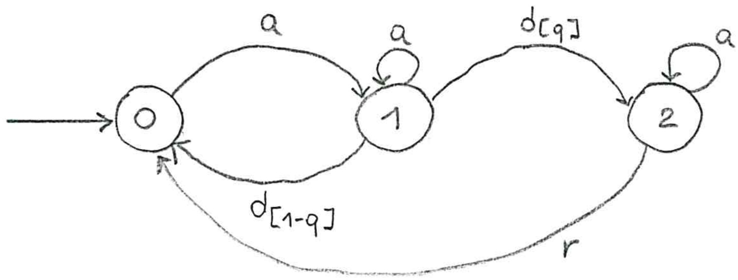
The number of rejected requests is 6 $(a_2, a_4, a_5, a_7, a_8, a_9)$.

$$\text{average total service time} = \frac{(14-2)+(33-16)+(54-36)+(69-57)}{4} = \frac{59}{4} \simeq 14.75 \text{ min}$$

2. event set $\mathcal{E}$ as above

state set $\mathcal{X} = \{0, 1, 2\}$

0: idle
1: main service
2: extra-service



$q = 0.2$

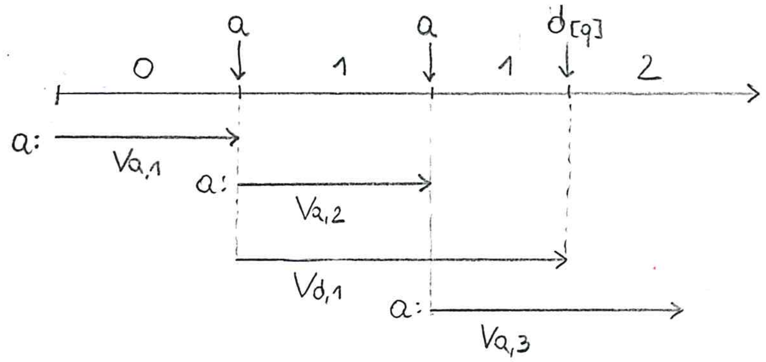
$V_a \sim U(5, 15)$ min

$V_d = 12$ min

$V_r \sim U(1, 4)$ min

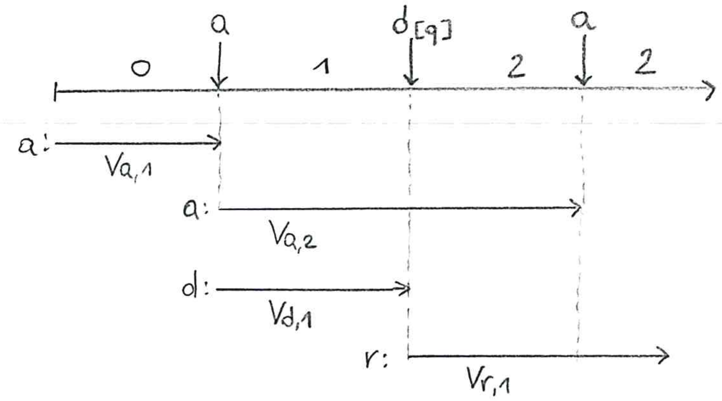
3. There are two possible cases:

$\boxed{1}$



$$\Rightarrow P(\boxed{1}) = q \cdot P(V_{a,2} < V_{d,1} < V_{a,2} + V_{a,3}) = q \cdot P(V_{a,2} < 12,\ V_{a,2} + V_{a,3} > 12) \simeq 0.1360$$

$\boxed{2}$



estimated with Matlab

$$\Rightarrow P(\boxed{2}) = q \cdot P(V_{d,1} < V_{a,2} < V_{d,1} + V_{r,1}) = q \cdot P(V_{a,2} > 12,\ V_{a,2} < V_{r,1} + 12) \simeq 0.0467$$

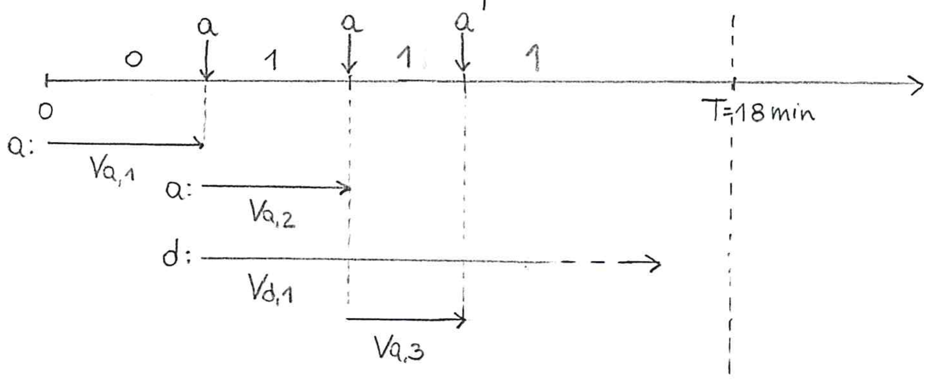
$$\Rightarrow P(\ldots) = P(\boxed{1}) + P(\boxed{2}) \simeq 0.1827$$

4. With the considered distributions of the lifetimes of the events, the following hold over the interval $[0, 18]$ min:

- a maximum of three service requests may arrive;
- the first service request is always accepted.

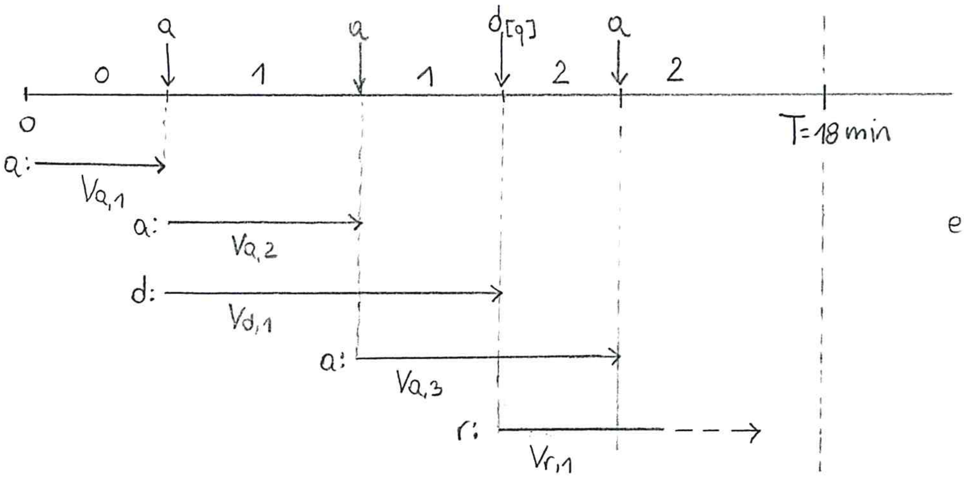
It turns out that there are two possible cases:

$\boxed{1}$



$$\Rightarrow P(\boxed{1}) = P\left(V_{a,2}+V_{a,3}<V_{d,1},\ V_{a,1}+V_{a,2}+V_{a,3}<T\right) \simeq 0.0034$$

$\boxed{2}$



estimated with Matlab

$$\Rightarrow P(\boxed{2}) = q\cdot P\left(V_{a,2}<V_{d,1}<V_{a,2}+V_{a,3}<V_{d,1}+V_{r,1}\ ,\ V_{a,1}+V_{a,2}+V_{a,3}<T\right) \simeq 0.0002$$

$$\Rightarrow P(\ldots) = P(\boxed{1}) + P(\boxed{2}) \simeq 0.0036$$

5. Let Z be the total service time.
Then,

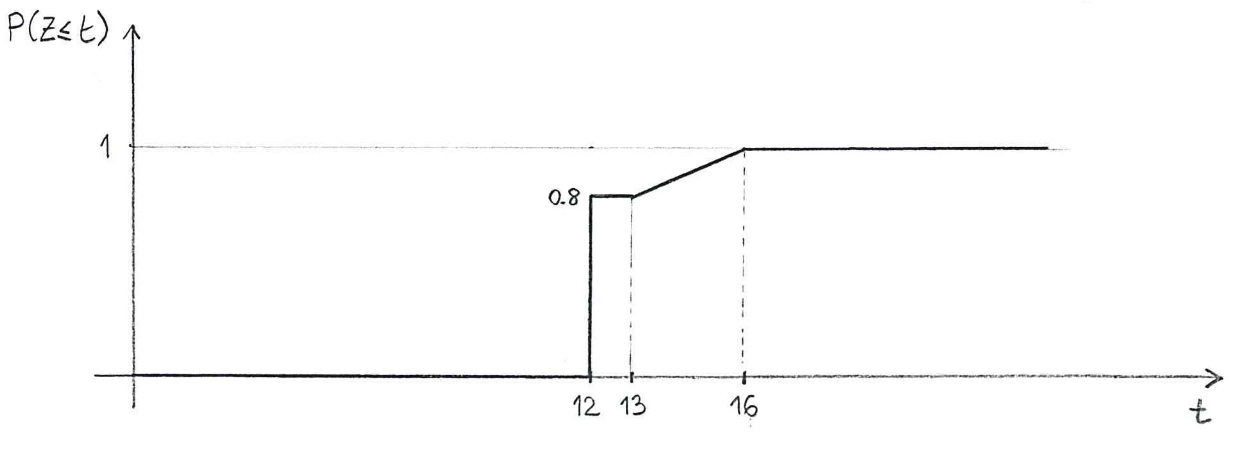
$$P(Z\leq t) = P(\text{extra-service})P(Z\leq t \mid \text{extra-service}) + $$
$$+ P(\text{no extra-service})P(Z\leq t \mid \text{no extra-service})$$

$$= q\cdot P(V_d+V_r\leq t) + (1-q)\cdot P(V_d\leq t) = qP(V_r\leq t-12) + (1-q)P(t\geq 12)$$

Recall that $V_r \sim U(1,4)$, hence

$$P(V_r\leq v) = \begin{cases} 0 & \text{if } v<1 \\ \frac{v-1}{3} & \text{if } 1\leq v<4 \\ 1 & \text{otherwise} \end{cases}$$

$$\Rightarrow P(Z\leq t) = \begin{cases} 0 & \text{if } t<12 \\ (1-q) & \text{if } 12\leq t<13 \\ q\cdot\frac{t-13}{3}+(1-q) & \text{if } 13\leq t<16 \\ 1 & \text{otherwise} \end{cases}$$



EXERCISE 2

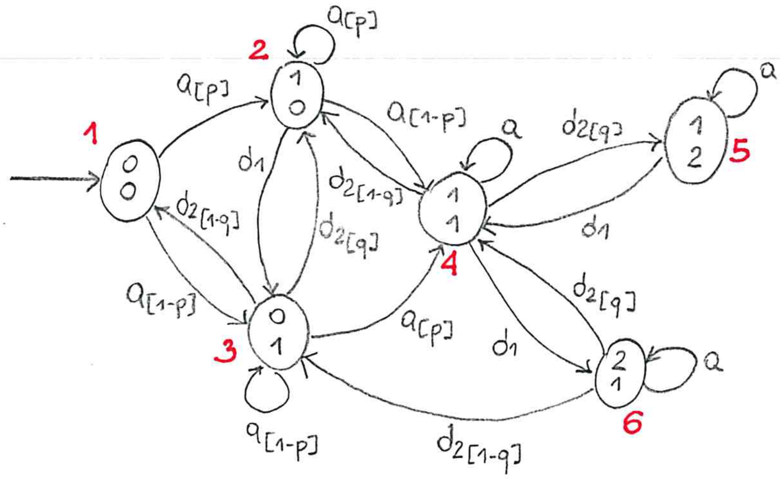
1. state $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ $\begin{array}{l} \rightarrow M_1: 0 \text{(idle)}, 1 \text{(working)}, 2 \text{(blocked)} \\ \rightarrow M_2: \quad '' \quad , \quad '' \quad , \quad '' \end{array}$

events $\mathcal{E} = \{ a, d_1, d_2 \}$

a: arrival of a raw part

d_1: termination of a job in M_1

d_2: termination of a job in M_2



- $p = \frac{1}{3}$
- $q = \frac{1}{10}$
- $V_a \sim Exp\left(\frac{1}{\lambda}\right)$

  $\frac{1}{\lambda} = 10 \text{ min} \Rightarrow \lambda = \frac{1}{10}$ arrivals/min
- $V_{d_1} \sim Exp\left(\frac{1}{\mu_1}\right)$

  $\frac{1}{\mu_1} = 8 \text{ min} \Rightarrow \mu_1 = \frac{1}{8}$ services/min
- $V_{d_2} \sim Exp\left(\frac{1}{\mu_2}\right)$

  $\frac{1}{\mu_2} = 6 \text{ min} \Rightarrow \mu_2 = \frac{1}{6}$ services/min

2. $X_k = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $T = 5 \text{min}$

$$\Rightarrow P(\underbrace{V(X_k)}_{\text{state holding time}} > T) = 1 - P(V(X_k) \leq T) = 1 - \left(1 - e^{-[\lambda(1-p)+\mu_1]T}\right) = e^{-[\lambda(1-p)+\mu_1]T}$$

$$\simeq 0.3835$$

3. $X_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

We want to reach state $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ before an event $d_{2[1-q]}$ and with exactly one event d_1.
There are four possible cases.

$\boxed{1}$ $\begin{pmatrix} 0 \\ 0 \end{pmatrix} \xrightarrow{a[p]} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \xrightarrow[\text{(ignore } a[p])]{d_1} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \xrightarrow[\text{(ignore } a[1-p])]{a[p]} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$P(\boxed{1}) = p \cdot \frac{\mu_1}{\lambda(1-p)+\mu_1} \cdot \frac{\lambda p}{\lambda p + \mu_2}$$

$\boxed{2}$

$$\binom{0}{0} \xrightarrow{a[p]} \binom{1}{0} \xrightarrow[\text{(ignore } a[p])]{d_1} \binom{0}{1} \xrightarrow[\text{(ignore } a[1-p])]{d_2[q]} \binom{1}{0} \xrightarrow[\text{(ignore } a[p]]{a[1-p]} \binom{1}{1}$$

$$P(\boxed{2}) = p \cdot \frac{\mu_1}{\lambda(1-p)+\mu_1} \cdot \frac{\mu_2 q}{\lambda p + \mu_2} \cdot \frac{\lambda(1-p)}{\lambda(1-p)+\mu_1}$$

$\boxed{3}$

$$\binom{0}{0} \xrightarrow{a[1-p]} \binom{0}{1} \xrightarrow[\text{(ignore } a[1-p])]{d_2[q]} \binom{1}{0} \xrightarrow[\text{(ignore } a[p])]{d_1} \binom{0}{1} \xrightarrow[\text{(ignore } a[1-p])]{a[p]} \binom{1}{1}$$

$$P(\boxed{3}) = (1-p) \cdot \frac{\mu_2 q}{\lambda p + \mu_2} \cdot \frac{\mu_1}{\lambda(1-p)+\mu_1} \cdot \frac{\lambda p}{\lambda p + \mu_2}$$

$\boxed{4}$

$$\binom{0}{0} \xrightarrow{a[1-p]} \binom{0}{1} \xrightarrow[\text{(ignore } a[1-p])]{d_2[q]} \binom{1}{0} \xrightarrow[\text{(ignore } a[p])]{d_1} \binom{0}{1} \xrightarrow[\text{(ignore } a[1-p])]{d_2[q]} \binom{1}{0} \xrightarrow[\text{(ignore } a[p]]{a[1-p]} \binom{1}{1}$$

$$P(\boxed{4}) = (1-p) \cdot \frac{\mu_2 q}{\lambda p + \mu_2} \cdot \frac{\mu_1}{\lambda(1-p)+\mu_1} \cdot \frac{\mu_2 q}{\lambda p + \mu_2} \cdot \frac{\lambda(1-p)}{\lambda(1-p)+\mu_1}$$

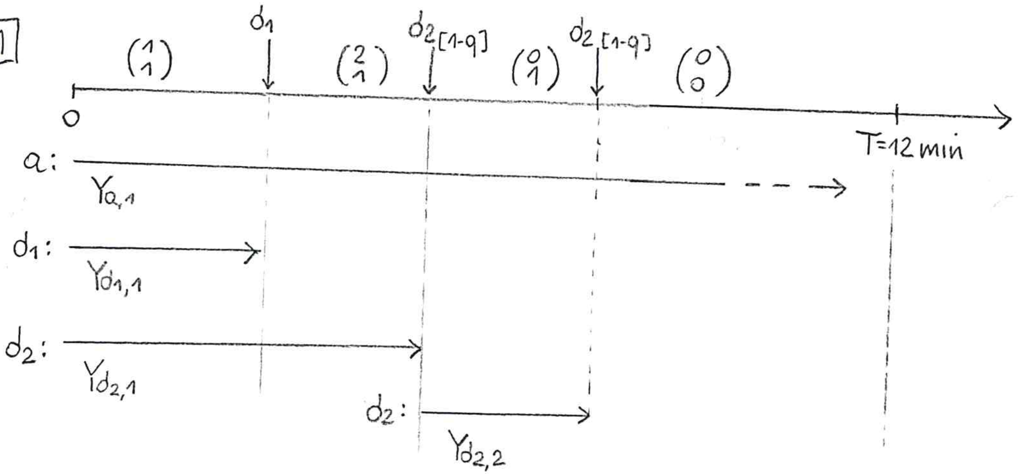
$$\Rightarrow P(\ldots) = P(\boxed{1}) + P(\boxed{2}) + P(\boxed{3}) + P(\boxed{4}) \simeq 0.0496$$

4. $X_k = \binom{1}{1}$

The minimum number of events is three.

There are two possible cases.

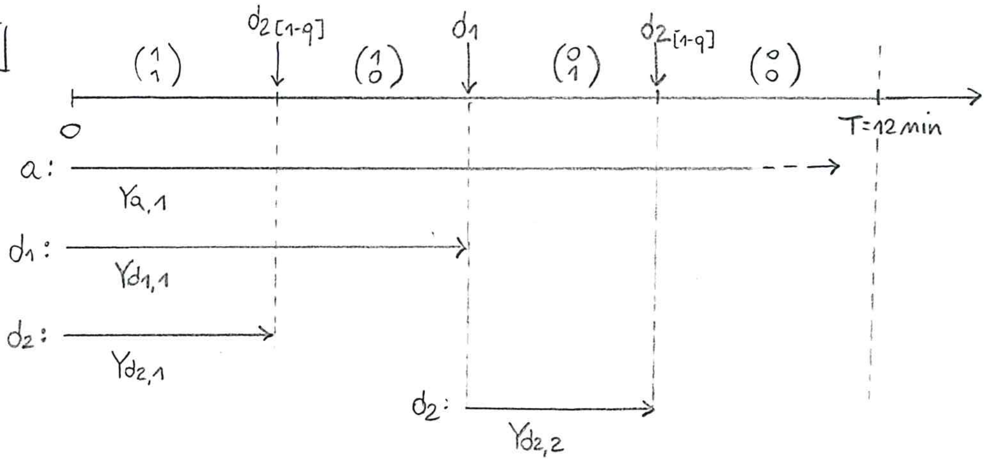
$\boxed{1}$



estimated with Matlab

$$\Rightarrow P(\boxed{1}) = (1-q)^2 \cdot P\left(Y_{d_1,1} < Y_{d_2,1},\ Y_{d_2,1} + Y_{d_2,2} < \min\{Y_{a,1}, T\}\right) \simeq 0.0702$$

$\boxed{2}$



$$\Rightarrow P(\boxed{2}) = (1-q)^2 \cdot P\left(Y_{d_2,1} < Y_{d_1,1},\ Y_{d_1,1} + Y_{d_2,2} < \min\{Y_{a,1}, T\}\right) \simeq 0.0781$$

$$\Rightarrow P(\ldots) = P(\boxed{1}) + P(\boxed{2}) \simeq 0.1483$$

EXERCISE 3

Equivalent continuous-time homogeneous Markov chain:

$$Q = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} -\lambda & \lambda p & \lambda(1-p) & 0 & 0 & 0 \\ 0 & -[\lambda(1-p)+\mu_1] & \mu_1 & \lambda(1-p) & 0 & 0 \\ \mu_2(1-q) & \mu_2 q & -(\lambda p+\mu_2) & \lambda p & 0 & 0 \\ 0 & \mu_2(1-q) & 0 & -(\mu_1+\mu_2) & \mu_2 q & \mu_1 \\ 0 & 0 & 0 & \mu_1 & -\mu_1 & 0 \\ 0 & 0 & \mu_2(1-q) & \mu_2 q & 0 & -\mu_2 \end{bmatrix} \end{array}$$

$\pi(0) = [1\ 0\ 0\ 0\ 0\ 0]$

Irreducible, finite

$$\begin{cases} \pi Q = 0 \\ \sum \pi_i = 1 \end{cases}$$

$\Downarrow$

$$\pi = \begin{bmatrix} 0.4208 & 0.1566 & 0.2806 & 0.0754 & 0.0101 & 0.0565 \end{bmatrix}$$
$$\quad\ \ \pi_1 \qquad \pi_2 \qquad \pi_3 \qquad \pi_4 \qquad \pi_5 \qquad \pi_6$$

1. $t = 2\,\text{min}$

$$\Rightarrow P(\ldots) = P(X(t)=4) + P(X(t)=5) + P(X(t)=6)$$

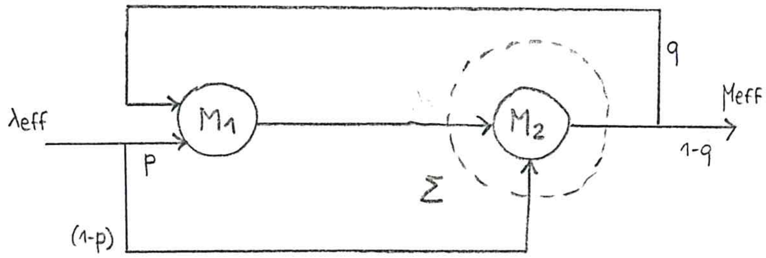
$$= \pi(0)\, e^{Qt} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} \simeq 0.0069$$

2. $E[\#\text{parts}] = 0 \cdot \pi_1 + 1 \cdot (\pi_2 + \pi_3) + 2 \cdot (\pi_4 + \pi_5 + \pi_6) \simeq 0.7212$

3. $\lambda_{eff} = \lambda\pi_1 + \lambda(1-p)\pi_2 + \lambda p\pi_3 \simeq 0.0619$

$\mu_{eff} = \mu_2(1-q)\cdot(\pi_3+\pi_4+\pi_6) \simeq 0.0619$

4.



$$E[X_\Sigma] = 0\cdot(\pi_1+\pi_2) + 1\cdot(\pi_3+\pi_4+\pi_5+\pi_6) \simeq 0.4226$$

$\lambda_\Sigma = \dfrac{\mu_{eff}}{1-q} \simeq 0.0688$ (seen at the output of Σ)

$\Rightarrow$ (Little's law) $E[S_\Sigma] = \dfrac{E[X_\Sigma]}{\lambda_\Sigma} \simeq 6.1462$

Notice that $E[S_\Sigma] > \dfrac{1}{\mu_2} = 6$ due to the blocking condition for M_2

5. $U_1 = \pi_2+\pi_4+\pi_5 \simeq 24.2043\ \%$

$U_2 = \pi_3+\pi_4+\pi_6 \simeq 41.2508\ \%$

6. $P(\ldots) = P(\ldots \mid \text{the part needs } M_1)P(\text{the part needs } M_1) + P(\ldots \mid \text{the part needs } M_2)\,P(\text{the part needs } M_2)$

$= p\cdot(\pi_2+\pi_4+\pi_5+\pi_6) + (1-p)(\pi_3+\pi_4+\pi_5+\pi_6) \simeq 0.3812$

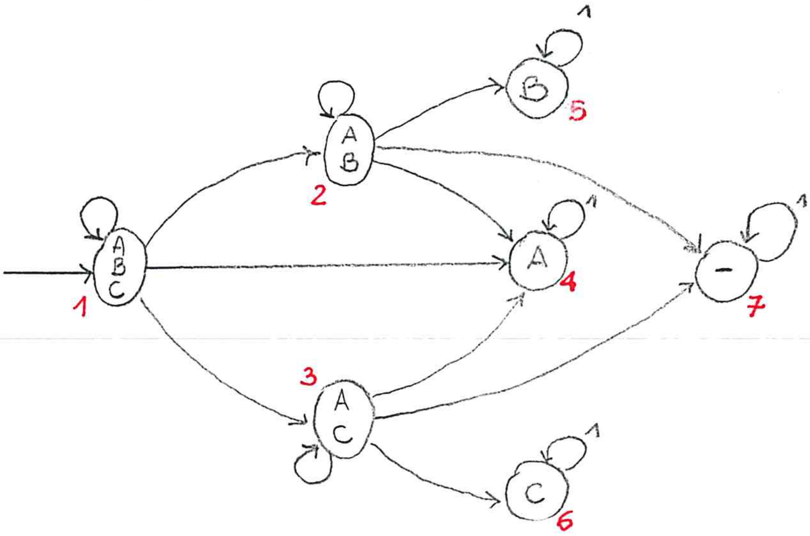
PASTA property

Exercise 4

1. Let A, B and C be the three kids in decreasing order of ability.

State set $\mathcal{X} = \left\{ \begin{pmatrix} A \\ B \\ C \end{pmatrix}, \begin{pmatrix} A \\ B \end{pmatrix}, \begin{pmatrix} A \\ C \end{pmatrix}, (A), (B), (C), (-) \right\}$

↘ each state represents the kids still in the game.



$p_A = 0.4$

$p_B = 0.25$

$p_C = 0.1$

$p_{1,1} = (1-p_A)(1-p_B)(1-p_C)$ ⇝ neither A nor B hits C, C does not hit B

$p_{1,2} = (p_A + p_B - p_A p_B)(1-p_C)$ ⇝ A or B hits C, C does not hit B

$p_{1,3} = (1-p_A)(1-p_B)\, p_C$ ⇝ neither A nor B hits C, C hits B

$p_{1,4} = (p_A + p_B - p_A p_B)\, p_C$ ⇝ A or B hits C, C hits B

$p_{2,2} = (1-p_A)(1-p_B)$ ⇝ A does not hit B, B does not hit A

$p_{2,4} = p_A(1-p_B)$ ⇝ A hits B, B does not hit A

$p_{2,5} = (1-p_A)\, p_B$ ⇝ A does not hit B, B hits A

$p_{2,7} = p_A p_B$ ⇝ A hits B, B hits A

$p_{3,3} = (1-p_A)(1-p_C)$ ⇝ A does not hit C, C does not hit A

$p_{3,4} = p_A(1-p_C)$ ⇝ A hits C, C does not hit A

$p_{3,6} = (1-p_A)\, p_C$ ⇝ A does not hit C, C hits A

$p_{3,7} = p_A p_C$ ⇝ A hits C, C hits A

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array} \begin{array}{c} \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{bmatrix} p_{1,1} & p_{1,2} & p_{1,3} & p_{1,4} & 0 & 0 & 0 \\ 0 & p_{2,2} & 0 & p_{2,4} & p_{2,5} & 0 & p_{2,7} \\ 0 & 0 & p_{3,3} & p_{3,4} & 0 & p_{3,6} & p_{3,7} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

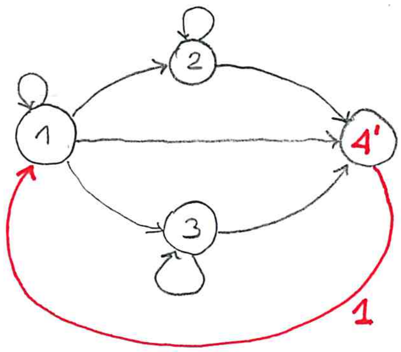
$$\pi(0) = [1\ 0\ 0\ 0\ 0\ 0\ 0]$$

2. $P(\underbrace{V(1)}_{\text{state holding time}} = 3) = p_{1,1}^2 \cdot (1 - p_{1,1}) \simeq 0.0376$

3. $P(\dots) = \lim_{t\to\infty} \pi_7(t) = \lim_{t\to\infty} \pi(0) P^t \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \simeq 0.1578$

4. We merge the states 4, 5, 6 and 7 into a single state 4', then we add a deterministic transition from 4' to 1:

$$\tilde{P} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4' \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} p_{1,1} & p_{1,2} & p_{1,3} & p_{1,4} \\ 0 & p_{2,2} & 0 & p_{2,4}+p_{2,5}+p_{2,7} \\ 0 & 0 & p_{3,3} & p_{3,4}+p_{3,6}+p_{3,7} \\ 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$



$$\Rightarrow E[\underbrace{T_{1,4'}}_{\text{duration of the game}}] = E[\underbrace{T_{4',4'}}_{\text{recurrence time of } 4'}] - 1 = \frac{1}{\tilde{\pi}_{4'}} - 1 \simeq 3.3577$$