Endterm test of Discrete Event Systems - 20.12.2018

Exercise 1

A machine processes raw parts of two types. Type 1 parts need only the processing phase, while type 2 parts need also a pre-processing phase. Arrivals of raw parts at the machine are always possible, and accepted only if the machine is idle. The machine is initially idle.

1. Assume that type 1 parts arrive at times 3.0, 9.0, 18.0, 21.0, 28.0, 40.0, 47.0, 52.0 min, type 2 parts arrive at times 6.0, 21.0, 34.0, 38.0, 50.0, 56.0 min, duration of the pre-processing phase is 5 min, and duration of the processing phase is 8 min. Determine the throughput of finished parts and the utilization of the machine over the interval [0,60] min.

In the following, assume that arrivals of raw parts at the machine occur with interarrival times following a uniform distribution over the interval [5.0,15.0] min. An arriving part is of type 1 with probability $q = 0.6$. Durations of the pre-processing phase are deterministic and all equal to 4.0 min, while durations of the processing phase follow a uniform distribution over the interval [3.0,8.0] min.

2. Model the system through a stochastic timed automaton $(\mathcal{E}, \mathcal{X}, \Gamma, p, x_0, F)$.

3. Compute the probability that the third event is the arrival of a raw part.

4. Compute the probability that exactly two finished parts are released by the machine over the interval [0,15) min.

5. Compute the average total processing time (including pre-processing) of a generic part processed by the machine.

Exercise 2

A production line is composed of a machine preceded by a one-place buffer. The machine processes parts of two types, namely type 1 and type 2. Normally, the system operates according to a first-in first-out scheduling policy. However, if a urgent request for a type 1 finished product is received, pre-emption is possible, i.e., processing of a type 2 part is suspended as soon as a type 1 part becomes available, and processing of the type 1 part is immediately started. The suspended type 2 part waits in the buffer, and its processing will be later continued from where it was suspended. Arrivals of urgent requests may occur only if there are no other pending urgent requests, and the machine is not processing a type 1 part. Moreover, arrivals of type 2 parts are suspended as long as a type 2 part is in the system. Assume that arrivals of type 1 parts are generated by a Poisson process with rate 5 arrivals/h. Lifetimes of the arrivals of type 2 parts and of the urgent requests follow exponential distributions with expected values 10 min and 30 min, respectively. Processing times of type 1 and type 2 parts follow exponential distributions with expected values 20 min and 25 min, respectively. The system is initially empty with no pending urgent request.

1. Model the system through a stochastic timed automaton $(\mathcal{E}, \mathcal{X}, \Gamma, f, x_0, F)$.

2. Compute the average duration of a situation when a type 2 part is being processed and a type 1 part is waiting in the buffer.

3. Assume that a type 2 part is being processed and a type 1 part is waiting in the buffer. Compute the probability that the system is emptied before a urgent request arrives and another type 1 part is accepted.

4. Assume that a type 2 part is being processed and a type 1 part is waiting in the buffer. Compute the probability that no arrivals occur and the system is emptied in the next 15 min.

Exercise 3

A conveyor belt moves with constant velocity. Packages can be placed at the beginning of the conveyor belt only at time instants multiple of the sampling time $T = 30$ s. At every time step, a package is placed on the conveyor belt with probability $p = 3/5$. At the end of the conveyor belt, packages are picked by a robot, called *master*. Normally, the robot is able to accomplish the task within one sampling time, thus being available for a new task in the next sampling time. However, the robot may be unsuccessful in manipulating the package with probability $q = 1/10$. In this case, the task is repeated in the next sampling time, with the same probability of failure. In case of two consecutive failures, the robot is stopped for maintainance. If a package transits in front of the robot and the robot is busy with another task, the package is picked by another robot, called *slave*, which is always able to to accomplish the task within one sampling time.

1. Show that the robot *master* can be modelled by a discrete-time homogeneous Markov chain.

2. Assume that the robot *master* is initially idle. Compute the probability that the robot is stopped for maintainance in the tenth sampling time.

3. Assume that the robot *master* is initially idle. Compute the average number of sampling times to stop the robot for maintainance.

4. Assume that the robot *master* is initially idle. Compute the probability that the robot is busy repeating a task at least twice over the first ten sampling times.

EXERCISE 1

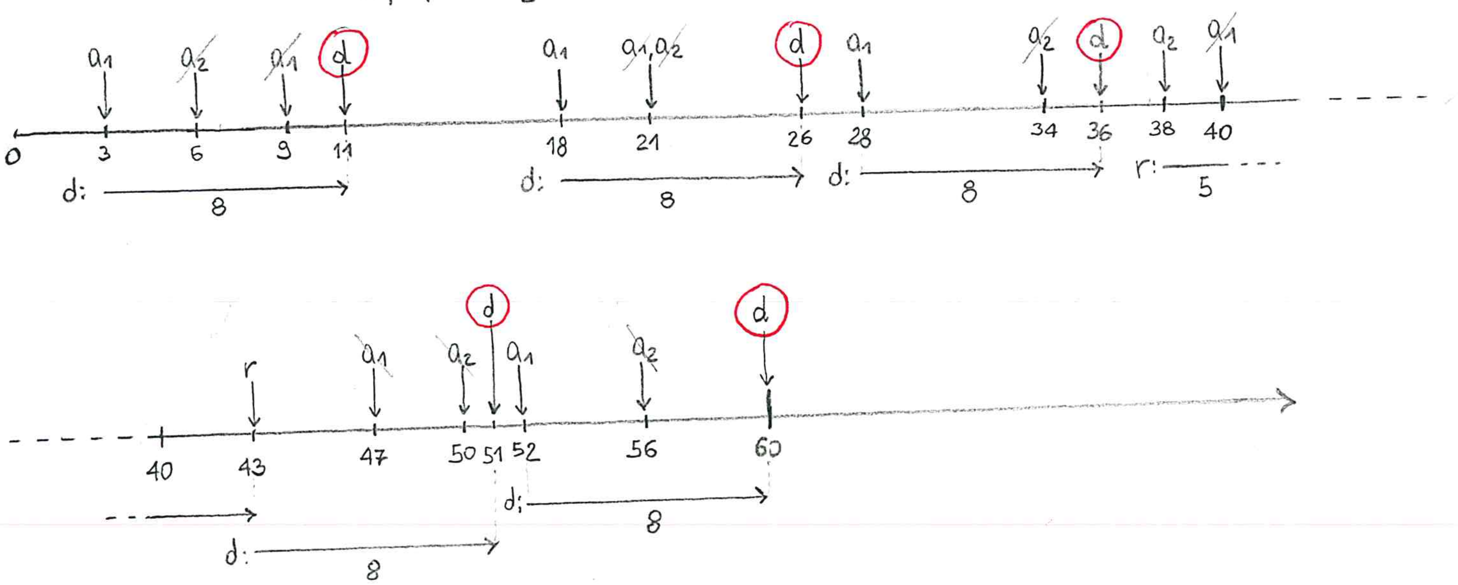
1. $\mathcal{E} = \{ a_1, a_2, r, d \}$

a_1: arrival of type 1 part

a_2: arrival of type 2 part

r: termination of preprocessing

d: termination of processing



$$\text{throughput} = \frac{5\,[\text{parts}]}{1\,[h]} = 5 \text{ parts/h}$$

$$\text{utilization} = \frac{8+8+8+(5+8)+8}{60} = \frac{45}{60} = 75\%$$

2. $\mathcal{E} = \{ a, r, d \}$

a: arrival of a raw part

r: termination of preprocessing

d: termination of processing

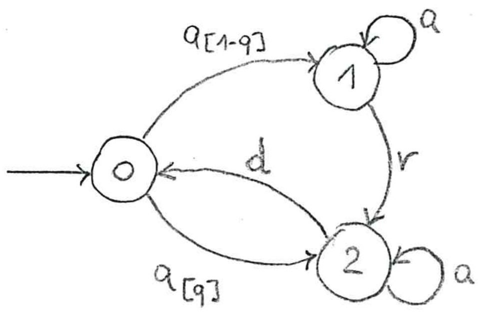
$$\text{state } x = \begin{cases} 0: & \text{idle} \\ 1: & \text{preprocessing} \\ 2: & \text{processing} \end{cases}$$

$V_a \sim U(5,15)$

$V_r = 4$

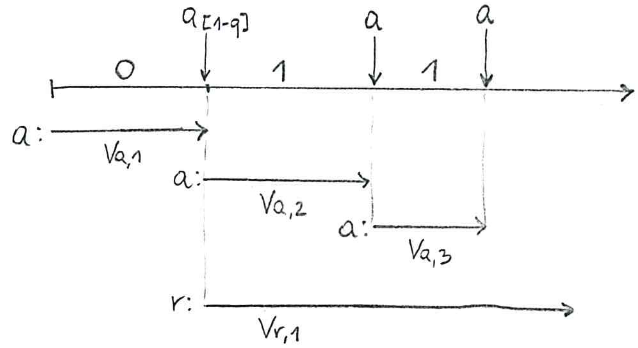
$V_d \sim U(3,8)$

$q = 0.6$ [prob. type 1]



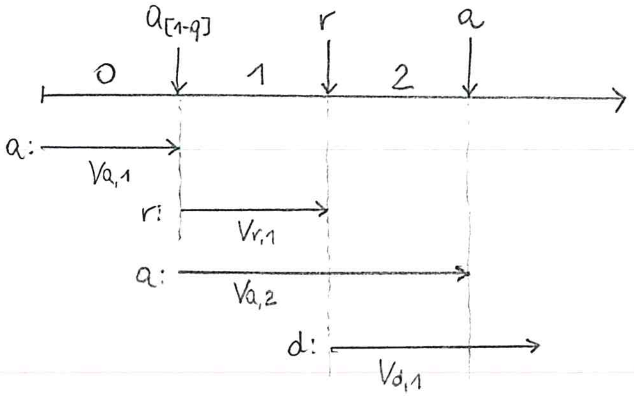
3. There are four possible cases.

$\boxed{1}$



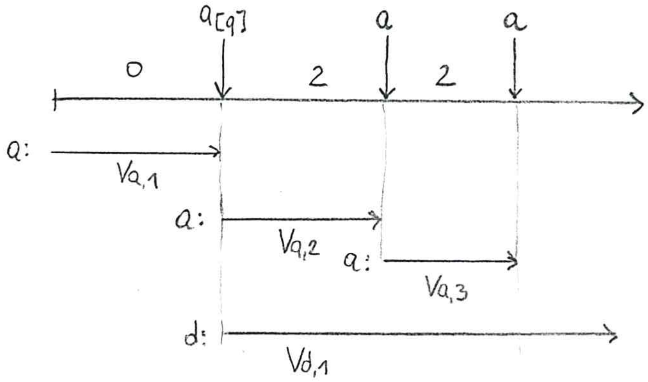
$$\Rightarrow P(\boxed{1}) = (1-q)\,P\left(V_{a,2}+V_{a,3}<V_{r,1}\right)$$
$$= (1-q)\underbrace{P\left(V_{a,2}+V_{a,3}<4\right)}_{=0} = 0$$

$\boxed{2}$



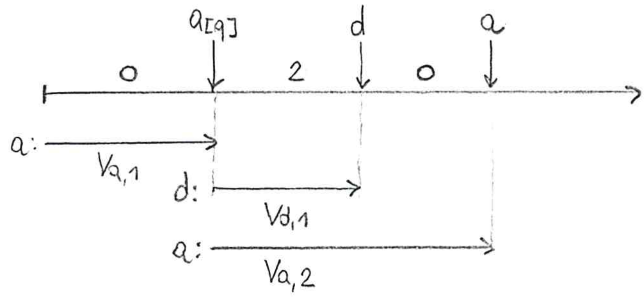
$$\Rightarrow P(\boxed{2}) = (1-q)\,P\left(V_{r,1}<V_{a,2}<V_{r,1}+V_{d,1}\right)$$
$$= (1-q)\,P\left(\underbrace{4<V_{a,2}}_{\text{always true}}<4+V_{d,1}\right)$$
$$= (1-q)\underbrace{P\left(V_{a,2}<V_{d,1}+4\right)}_{=\frac{9}{20}} = 0.18$$

$\boxed{3}$



$$\Rightarrow P(\boxed{3}) = q\cdot\underbrace{P\left(V_{a,2}+V_{a,3}<V_{d,1}\right)}_{=0} = 0$$

$\boxed{4}$



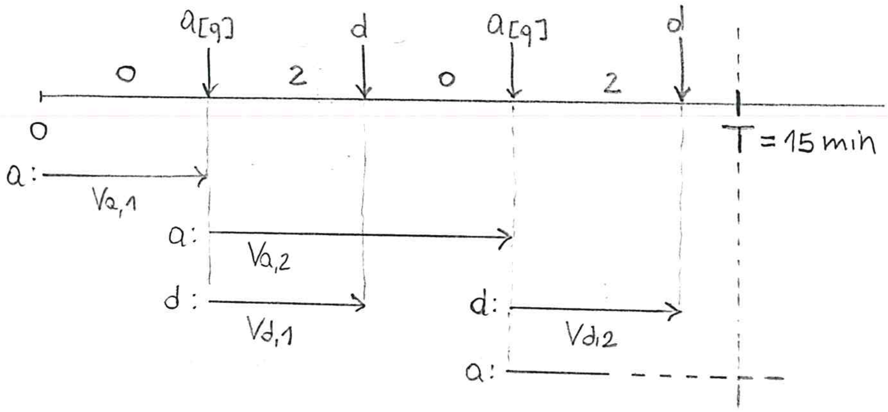
$$\Rightarrow P(\boxed{4}) = q\cdot\underbrace{P\left(V_{d,1}<V_{a,2}\right)}_{\frac{91}{100}} \simeq 0.5460$$

$$\Rightarrow P(\ldots) = P(\boxed{1}) + P(\boxed{2}) + P(\boxed{3}) + P(\boxed{4}) \simeq 0.7260$$

4. With the considered distributions of the lifetimes of the events, the following hold over the interval $[0,15)$ min:

- a maximum of two arrivals of raw parts may occur;
- in order to release two finished parts, the second part has to arrive after the termination of the processing of the first part;
- the first part cannot arrive before time 5 min;
- the second part cannot arrive before time 10 min;
- the duration of the processing cannot be less than 3 min;
- in case of preprocessing, the total duration of pre-processing and processing cannot be less than 7 min.

It turns out that there is only one possible case:



$$\Rightarrow P(\ldots) = q^2\, P\left(V_{d,1} < V_{a,2}\,,\ V_{a,1} + V_{a,2} + V_{d,2} < T\right) \simeq 0.00048$$

estimated with Matlab

5. $E[S] = q \cdot E[V_d] + (1-q)\,E[V_r + V_d] = \frac{3}{5}\cdot\frac{11}{2} + \frac{2}{5}\left(4 + \frac{11}{2}\right) = \frac{71}{10} = 7.1$ min

total processing time

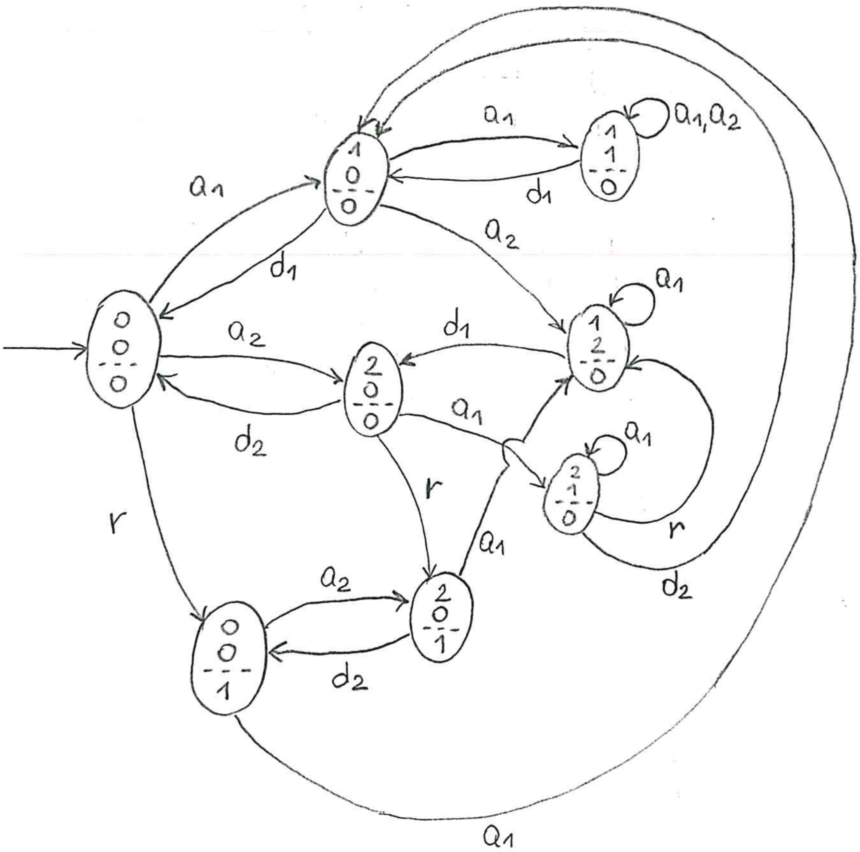
EXERCISE 2

1. events $\mathcal{E} = \{a_1, a_2, r, d_1, d_2\}$

a_1: arrival of a type 1 part
a_2: arrival of a type 2 part
r: arrival of a urgent request
d_1: termination of processing of a type 1 part
d_2: termination of processing of a type 2 part

state $x = \begin{bmatrix} x_1 \\ x_2 \\ \hdashline x_3 \end{bmatrix}$

$x_1 \rightarrow$ machine: 0 (idle), 1 (processing type 1), 2 (processing type 2)
$x_2 \rightarrow$ buffer: 0 (empty), 1 (type 1), 2 (type 2)
$x_3 \rightarrow$ pending request: 0 (no pending request), 1 (pending request)



- $V_{a_1} \sim Exp\left(\frac{1}{\lambda_1}\right)$

  $\lambda_1 = 5$ arrivals/h

- $V_{a_2} \sim Exp\left(\frac{1}{\lambda_2}\right)$

  $\frac{1}{\lambda_2} = 10\,min = \frac{1}{6}h$

  $\Rightarrow \lambda_2 = 6$ arrivals/h

- $V_r \sim Exp\left(\frac{1}{\gamma}\right)$

  $\frac{1}{\gamma} = 30\,min = \frac{1}{2}h$

  $\Rightarrow \gamma = 2$ requests/h

- $V_{d_1} \sim Exp\left(\frac{1}{\mu_1}\right)$

  $\frac{1}{\mu_1} = 20\,min = \frac{1}{3}h$

  $\Rightarrow \mu_1 = 3$ services/h

- $V_{d_2} \sim Exp\left(\frac{1}{\mu_2}\right)$

  $\frac{1}{\mu_2} = 25\,min = \frac{5}{12}h$

  $\Rightarrow \mu_2 = \frac{12}{5}$ services/h

2. $X_k = \begin{pmatrix} 2 \\ 1 \\ \hdashline 0 \end{pmatrix}$

$$E[V(X_k)] = \frac{1}{\gamma + \mu_2} = \frac{5}{22} \simeq 0.2273\,h$$

3. $X_k = \begin{pmatrix}2\\1\\0\end{pmatrix}$

There are two possible cases:

$\boxed{1}$ $\begin{pmatrix}2\\1\\0\end{pmatrix} \xrightarrow[\text{(ignore } a_1)]{d_2} \begin{pmatrix}1\\0\\0\end{pmatrix} \xrightarrow{d_1} \begin{pmatrix}0\\0\\0\end{pmatrix}$

$$P(\boxed{1}) = \frac{\mu_2}{\gamma+\mu_2} \cdot \frac{\mu_1}{\lambda_1+\lambda_2+\mu_1}$$

$\boxed{2}$ $\begin{pmatrix}2\\1\\0\end{pmatrix} \xrightarrow[\text{(ignore } a_1)]{d_2} \begin{pmatrix}1\\0\\0\end{pmatrix} \xrightarrow{a_2} \begin{pmatrix}1\\2\\0\end{pmatrix} \xrightarrow[\text{(ignore } a_1)]{d_1} \begin{pmatrix}2\\0\\0\end{pmatrix} \xrightarrow{d_2} \begin{pmatrix}0\\0\\0\end{pmatrix}$

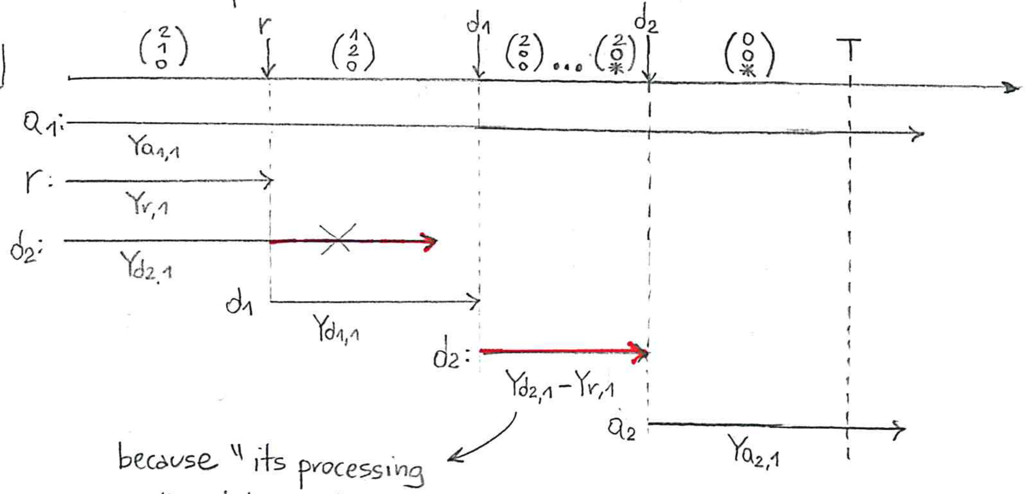
$$P(\boxed{2}) = \frac{\mu_2}{\gamma+\mu_2} \cdot \frac{\lambda_2}{\lambda_1+\lambda_2+\mu_1} \cdot 1 \cdot \frac{\mu_2}{\lambda_1+\gamma+\mu_2}$$

$$\Rightarrow P(\ldots) = P(\boxed{1}) + P(\boxed{2}) \simeq 0.1766$$

4. $X_k = \begin{pmatrix}2\\1\\0\end{pmatrix}$, $T = 15\text{ min} = \frac{1}{4}\text{h}$

There are two possible cases:

$\boxed{1}$

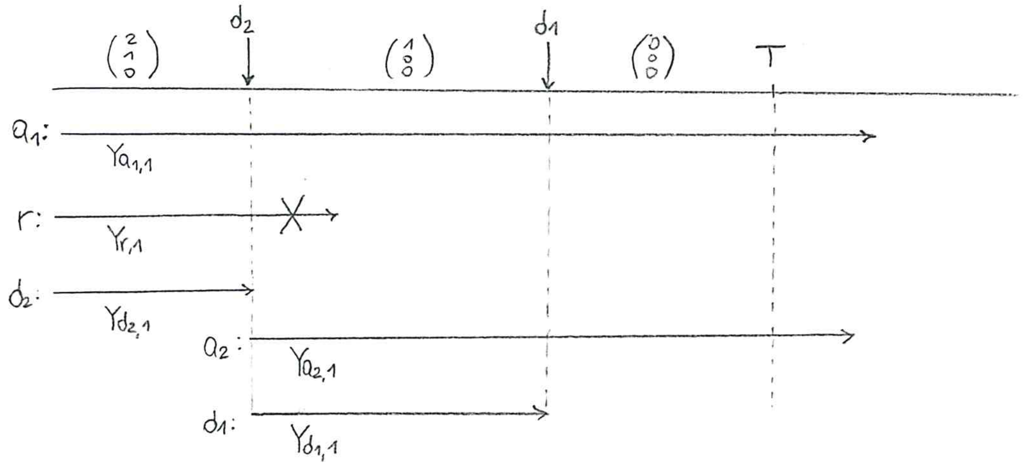


because "its processing will be later continued from where it was suspended"

$$P(\boxed{1}) = P\Big(Y_{a_1,1} > T,\ Y_{r,1} < Y_{d_2,1},\ \overbrace{Y_{r,1} + Y_{d_1,1} + (Y_{d_2,1} - Y_{r,1})}^{Y_{d_1,1}+Y_{d_2,1}} < T,$$

$$\underbrace{Y_{r,1} + Y_{d_1,1} + (Y_{d_2,1} - Y_{r,1})}_{Y_{d_1,1}+Y_{d_2,1}} + Y_{a_2,1} > T\Big)$$

$\boxed{2}$



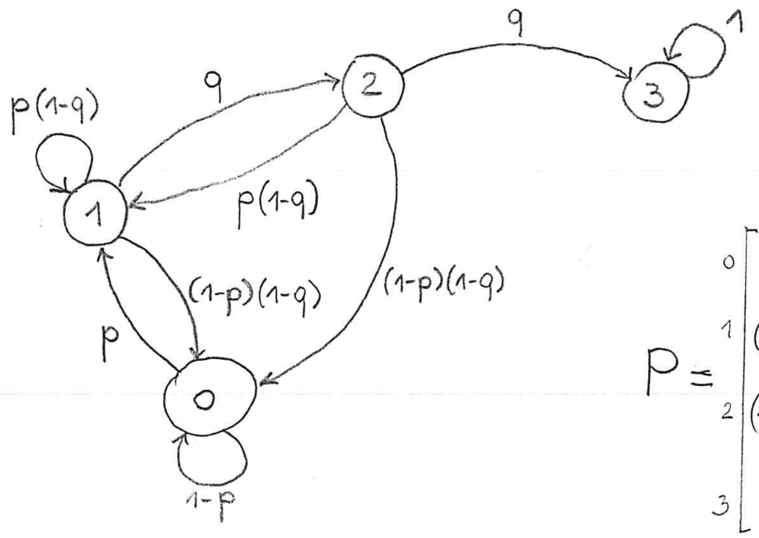
$$P(\boxed{2}) = P\left(Y_{a_1,1} > T,\ Y_{d_2,1} < Y_{r,1},\ Y_{d_2,1} + Y_{a_2,1} > T,\ Y_{d_2,1} + Y_{d_1,1} < T\right)$$

$$\Rightarrow P(\ldots) = P(\boxed{1}) + P(\boxed{2}) \simeq 0.0172 \quad \text{(estimated using Matlab)}$$

EXERCISE 3

1.

$$\text{state} \quad x = \begin{cases} 0: & \text{idle} \\ 1: & \text{picking a package } (1^{st} \text{ time}) \\ 2: & \quad '' \quad '' \quad '' \quad (2^{nd} \text{ time}) \\ 3: & \text{under maintainance} \end{cases}$$



$p = 3/5$

$q = 1/10$

$$P = \begin{array}{c c} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 1-p & p & 0 & 0 \\ (1-p)(1-q) & p(1-q) & q & 0 \\ (1-p)(1-q) & p(1-q) & 0 & q \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

2. $X(0)=0 \quad \Rightarrow \quad \pi_0 = [1 \quad 0 \quad 0 \quad 0]$

$$P(\ldots) = P(X(10)=3, X(9)\neq 3) = P(X(10)=3, X(9)=2)$$

$$= P(X(10)=3 \mid X(9)=2)\, P(X(9)=2) = p_{2,3} \cdot \pi_2(9) \simeq 0.0055$$

$p_{2,3} \to q$

$\pi_2(9) \to \pi(9) = [\pi_0(9)\ \pi_1(9)\ \pi_2(9)\ \pi_3(9)] = \pi_0 \cdot P^9$

3. We modify the model by adding a deterministic transition from 3 to 0:

$$\tilde{P} = \begin{bmatrix} 1-p & p & 0 & 0 \\ (1-p)(1-q) & p(1-q) & q & 0 \\ (1-p)(1-q) & p(1-q) & 0 & q \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

⤳ the new model is irreducible, aperiodic and finite.

In the new model:

$$T_{3,3} = 1 + T_{0,3} \quad \Rightarrow \quad E[T_{0,3}] = E[T_{3,3}] - 1$$

recurrence time of state 3

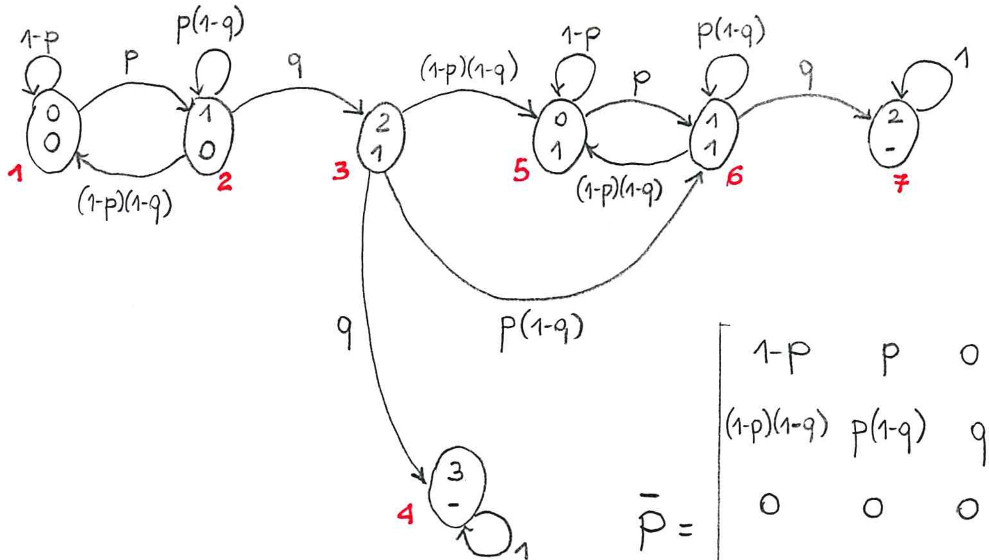
$$= \frac{1}{\tilde{\pi}_3} - 1 \simeq 177.67$$

$\tilde{\pi} = [\tilde{\pi}_0 \ \tilde{\pi}_1 \ \tilde{\pi}_2 \ \tilde{\pi}_3]$ solution of $\begin{cases} \tilde{\pi}\tilde{P} = \tilde{\pi} \\ \sum \tilde{\pi}_i = 1 \end{cases}$

$$\Rightarrow \tilde{\pi} = [\,0.3787 \quad 0.5597 \quad 0.0560 \quad 0.0056\,]$$

4. We modify the definition of the state as follows:

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \begin{matrix} \rightarrow \text{as before} \\ \rightarrow \text{time spent in state 2 (cumulative)} \end{matrix}$$



$$\bar{P} = \begin{bmatrix} 1-p & p & 0 & 0 & 0 & 0 & 0 \\ (1-p)(1-q) & p(1-q) & q & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & q & (1-p)(1-q) & p(1-q) & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1-p & p & 0 \\ 0 & 0 & 0 & 0 & (1-p)(1-q) & p(1-q) & q \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\bar{\pi}_0 = [1\ 0\ 0\ 0\ 0\ 0\ 0]$$

$$P(\ldots) = \bar{\pi}_0 \bar{P}^{10} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \simeq 0.0712$$

state 7 encodes the information that the robot was busy repeating a task at least twice