Endterm test of Discrete Event Systems - 20.12.2017

Student: ____

Exercise 1

A computer system may execute programs of four different types, numbered from 1 to 4. Execution of a program can be interrupted by the operator. A new program is started immediately after the termination or the interruption of the previous program. For logical reasons, after a type 1 program, the computer system may execute either a type 2 or a type 3 program. After a type 2 program, it may execute either a program of the same type or a type 3 program. After a type 3 program, it may execute either a type 2 or a type 4 program. After a type 4 program, it may execute only a type 1 program.

1. Assume that programs are executed in the order (1, 3, 2, 2, 3, 4, 1, 2, 3, 4), and the scheduled durations of the programs are 2.0 and 2.5 min for type 1, 4.0, 6.0 and 3.5 min for type 2, 5.0, 3.0 and 5.5 min for type 3, and 4.5 and 3.0 min for type 4. Moreover, assume that interruptions occur at times 14.0, 23.0 and 25.0 min from start. What programs are interrupted? What is the total processing time?

Assume that interruptions are disabled, and that, except when the current program is of type 4, the next program is selected randomly, being $q = 3/5$ the probability that the candidate next program with smallest type number is selected. Moreover, assume that durations of type 1 and type 4 programs are uniformly distributed over the intervals $[3.0, 6.0]$ min and $[2.5, 5.0]$ min, respectively, whereas durations of type 2 and type 3 programs are deterministic, and all equal to 3.5 min and 4.0 min, respectively.

2. Model the system through a stochastic timed automaton $(\mathcal{E}, \mathcal{X}, \Gamma, p, x_0, F)$, assuming that the first scheduled program is of type 1.

3. Compute the probability that a type 4 program is executed before a type 2 program.

4. Compute the probability that the program sequence (1, 2, 3, 4) is executed, and it takes no more than 16.5 min.

Exercise 2

A vending machine is replenished with snacks at time intervals following an exponential distribution with expected value equal to 6 hours. Among other snacks, three sandwiches are available in the machine after every supply. Customers show up to buy from the machine according to a Poisson process with average interarrival time equal to 15 minutes. The probability that a customer is willing to buy a sandwich is $q = 1/8$. The machine is initially full.

1. Model the dynamics of the number of sandwiches in the machine using a stochastic timed automaton $(\mathcal{E}, \mathcal{X}, \Gamma, p, x_0, F)$.

2. Assume that two sandwiches are available in the machine. Compute the probability that both of them are bought before the machine is replenished.

3. Compute the average time that only one sandwich is available in the machine.

4. Compute the probability that there is a daily demand of exactly 10 sandwiches.

Exercise 3 (only CAE-R&A)

A project is composed of four sequential stages. Every month, the project team meets to evaluate the progress of the project. The team may decide to carry on the ongoing stage, to move forward to the next stage, or even to stop the project. When the project is in stage 1, progress to stage 2 occurs with probability 0.4, whereas the project is stopped with probability 0.1. When the project is in stage 2, progress to stage 3 occurs with probability 0.3, whereas the project is stopped with probability 0.08. When the project is in stage 3, progress to stage 4 occurs with probability 0.5, whereas the project is stopped with probability 0.05. Finally, when the project is in stage 4, it is completed with probability 0.6. The project is never stopped while in stage 4.

1. Model the progress of the project using a discrete-time homogeneous Markov chain.
2. Compute the probability that the third stage is repeated at least three times.
3. Compute the probability that the project is completed.
4. Compute the probability that the project is completed in exactly ten months.
5. Compute the average duration of the project, no matter it is stopped or completed.

EXERCISE 1

1. Let x = type of running program $\in \{1,2,3,4\}$ be the state of the system.
Moreover, define the event set $\mathcal{E} = \{d_1, d_2, d_3, d_4, i\}$, where

d_j = termination of type j program, $j = 1,2,3,4$

i = interruption

Clock sequences:

$V_{d_1} = \{\cancel{2.0}, \cancel{2.5}\}$

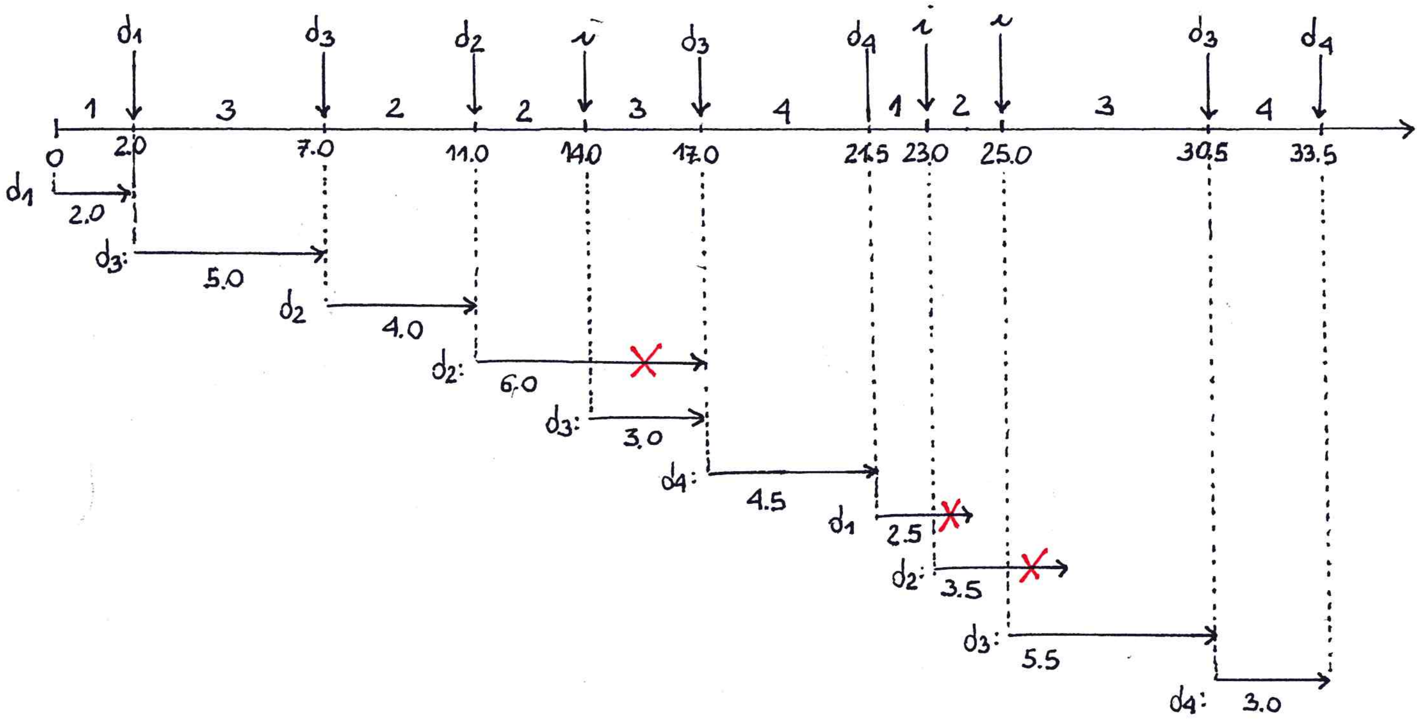
$V_{d_2} = \{\cancel{4.0}, \cancel{6.0}, \cancel{3.5}\}$

$V_{d_3} = \{\cancel{5.0}, \cancel{3.0}, \cancel{5.5}\}$

$V_{d_4} = \{\cancel{4.5}, \cancel{3.0}\}$

Event i occurs at times 14.0, 23.0 and 25.0.

Execution list: $(\cancel{1}, \cancel{3}, \cancel{2}, \cancel{2}, \cancel{3}, \cancel{4}, \cancel{1}, \cancel{2}, \cancel{3}, \cancel{4})$.



Interrupted programs:

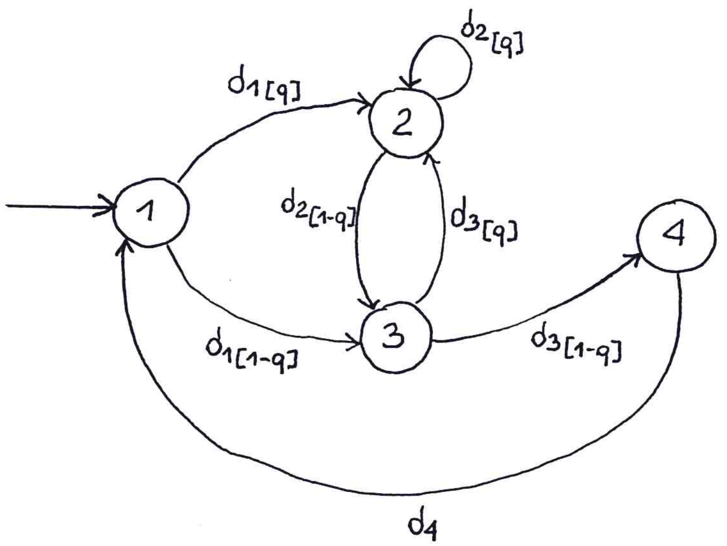
- type 2 at time 14.0
- type 1 at time 23.0
- type 2 at time 25.0

Total time: 33.5 min

2. Interruptions are disabled => event i is discarded.

State x = type of running program $\in \{1,2,3,4\}$

Events $\mathcal{E} = \{d_1, d_2, d_3, d_4\}$



$q = \frac{3}{5}$

$V_{d_1} \sim U(3.0, 6.0)$

$V_{d_2} = 3.5$

$V_{d_3} = 4.0$

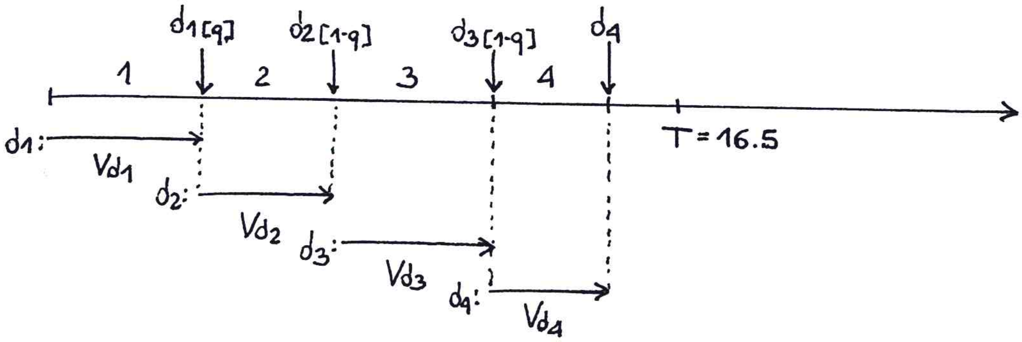
$V_{d_4} \sim U(2.5, 5.0)$

3. The only admissible case is

$$1 \xrightarrow{d_1[1-q]} 3 \xrightarrow{d_3[1-q]} 4$$

$$\Rightarrow P(\dots) = 1 \cdot (1-q) \cdot 1 \cdot (1-q) = (1-q)^2 = \frac{4}{25} = 0.1600$$

4. It corresponds to the sample path:



$$\Rightarrow P(\dots) = q(1-q)^2 \cdot P(V_{d_1} + V_{d_2} + V_{d_3} + V_{d_4} \leq T)$$

$$= q(1-q)^2 \cdot \underbrace{P(V_{d_1} + V_{d_4} \leq 9.0)}_{= 11/15} \simeq 0.0704$$

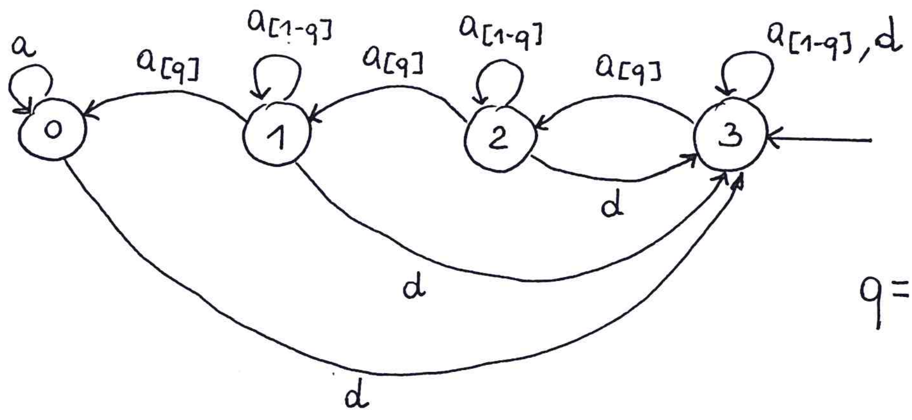
EXERCISE 2

1. state x = # sandwiches in the vending machine $\in \{0,1,2,3\}$

events $\mathcal{E} = \{a, d\}$

a: arrival of a customer

d: replenishment



$q = \frac{1}{8}$

$V_a \sim Exp\left(\frac{1}{\lambda}\right)$ where $\frac{1}{\lambda} = 15$ min $\Rightarrow$ $\lambda = 4$ arrivals/hour

$V_d \sim Exp\left(\frac{1}{\mu}\right)$ where $\frac{1}{\mu} = 6$ hours $\Rightarrow$ $\mu = \frac{1}{6}$ replenishments/hour

2. $X_k = 2$

There is only one possible case:

$$2 \xrightarrow{a[q]} 1 \xrightarrow{a[q]} 0 \quad \text{ignore } a[1-q]$$

$$\Rightarrow P(\ldots) = \frac{\lambda q}{\lambda q + \mu} \cdot \frac{\lambda q}{\lambda q + \mu} \simeq 0.5625$$

3. It corresponds to the average state holding time for state $x=1$.

$$E[V(1)] = \frac{1}{\lambda q + \mu} = 1.5 \text{ hours}$$

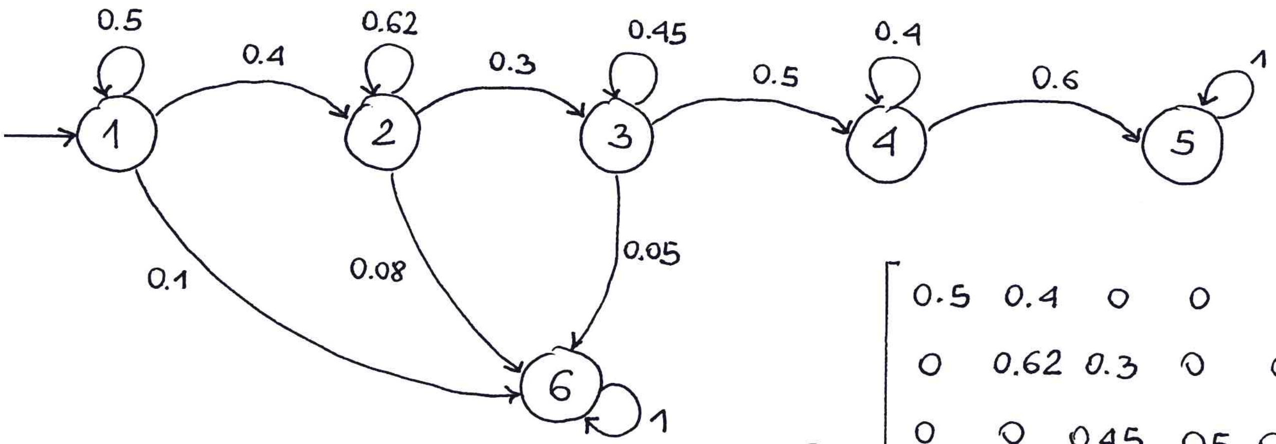
4. The demand of sandwiches occurs with rate λq.

$$\Rightarrow P(\underbrace{N_{a[q]}(T)}_{T = 24 \text{ hours}} = 10) = \frac{(\lambda q T)^{10}}{10!} e^{-\lambda q \cdot T} \simeq 0.1048$$

EXERCISE 3

1.

$$\text{state } x = \begin{cases} 1,2,3,4 : \text{ current project stage} \\ 5 : \text{ project completed} \\ 6 : \text{ project stopped} \end{cases}$$



$$P = \begin{bmatrix} 0.5 & 0.4 & 0 & 0 & 0 & 0.1 \\ 0 & 0.62 & 0.3 & 0 & 0 & 0.08 \\ 0 & 0 & 0.45 & 0.5 & 0 & 0.05 \\ 0 & 0 & 0 & 0.4 & 0.6 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\pi(0) = [1 \ 0 \ 0 \ 0 \ 0 \ 0]$$

2. $P(V(3) \geq 4) = 1 - P(V(3)=1) - P(V(3)=2) - P(V(3)=3)$

$V(3)$: state holding time

$$= 1 - (1-p_{3,3}) - p_{3,3}(1-p_{3,3}) - p_{3,3}^2(1-p_{3,3}) = p_{3,3}^3 = (0.45)^3 \simeq 0.0911$$

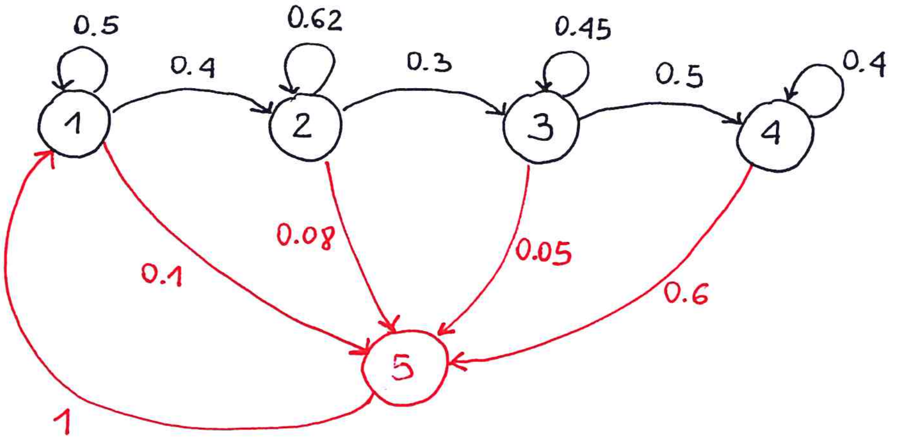
3. $P(\dots) = \lim_{t\to\infty} P(X(t)=5) = \lim_{t\to\infty} \pi(0) P^t \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \simeq 0.5742$

4. $P(\dots) = P(X(10)=5, X(9)\neq 5)$

$= P(X(10)=5, X(9)=4) = P(X(10)=5 \mid X(9)=4)\, P(X(9)=4)$

$= p_{4,5} \cdot \pi(0) P^9 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \simeq 0.0469$

5. We modify the model as follows:



$$\tilde{P} = \begin{bmatrix} 0.5 & 0.4 & 0 & 0 & 0.1 \\ 0 & 0.62 & 0.3 & 0 & 0.08 \\ 0 & 0 & 0.45 & 0.5 & 0.05 \\ 0 & 0 & 0 & 0.4 & 0.6 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \tilde{\pi} \simeq \begin{bmatrix} \underset{\tilde{\pi}_1}{0.2774} & \underset{\tilde{\pi}_2}{0.2920} & \underset{\tilde{\pi}_3}{0.1593} & \underset{\tilde{\pi}_4}{0.1327} & \underset{\tilde{\pi}_5}{0.1387} \end{bmatrix}$$

$$E[T_{1,5}] = E[T_{5,5}] - 1 = \frac{1}{\tilde{\pi}_5} - 1 \simeq 6.2105 \text{ months}$$