Exam of Discrete Event Systems - 18.02.2015

Exercise 1

A small manufacturing workstation consists of a machine M preceded by a one-place buffer B. The machine processes raw parts of two types, namely type 1 and type 2. Parts arriving when the system is full are rerouted to another workstation. For technological reasons, arrivals of type 1 parts are disabled after two consecutive arrivals of type 1 parts accepted in the system, and enabled again after the next arrival of a type 2 part accepted in the system. Arrivals of type 2 parts are always possible.

1. Define a logical model $(\mathcal{E}, \mathcal{X}, \Gamma, f, x_0)$ of the system, leaving the initial state x_0 undefined.

2. Assume that the system is initially empty and two consecutive arrivals of type 1 parts can be accepted; arrivals of type 1 parts have total lifetimes equal to 5.0, 3.0 and 8.0 min; arrivals of type 2 parts occur at times 12.0, 18.0 and 27.0 min; processing of type 1 parts requires 9.0, 7.0 and 8.5 min, and processing of type 2 parts requires 4.0, 3.5 and 5.0 min. Construct a timing diagram for the system sample path over the interval [0.0,25.0] minutes, and compute the fraction of time that arrivals of type 1 parts are disabled.

3. Assume that total lifetimes of the arrivals of type 1 parts follow a uniform distribution over the interval [5.0,8.0] min, whereas arrivals of type 2 parts occur every 4.0 min. Moreover, assume that at time $t = 0$ (initialization) the system is empty and arrivals of type 1 parts are disabled. Compute the probability that exactly one arrival of a type 1 part occurs over the interval [0.0,15.0] minutes.

Exercise 2

Consider the manufacturing workstation whose logical model was defined in item 1 of Exercise 1, and assume that the total lifetimes of the arrivals of type 1 parts follow an exponential distribution with rate $\lambda_1 = 0.2$ arrivals/minute; arrivals of type 2 parts are generated by a Poisson process with rate $\lambda_2 = 0.25$ arrivals/minute; processing times of type 1 and type 2 parts are exponentially distributed with expected values 8.0 and 6.0 minutes, respectively.

1. Assume that the system is full with two type 1 parts. Compute the probability that the system reaches with the minimum number of events a state where it is empty and arrivals of type 1 parts are enabled.

Exercise 3

Consider the manufacturing workstation whose logical model was defined in item 1 of Exercise 1 and whose stochastic clock structure is defined in Exercise 2.

2. Verify the condition $\lambda_{eff} = \mu_{eff}$ for the system at steady-state.

3. Compute the average waiting time in B of a generic part at steady state.

4. It is known that at time $t = 20$ minutes the arrival of a type 1 part occurs. Is it possible to apply the PASTA property to compute the probability that the arriving type 1 part is rejected? Why?

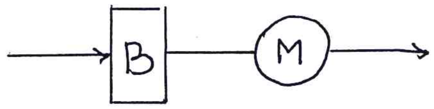
Exercise 4

Consider a sequence of $N = 3$ genes. Each gene can be one of two types: A or a. From one generation to another, each gene can independently mutate to the other type. Let $p = 1/8$ be the probability that gene a mutates to gene A, and $q = 1/5$ the probability that gene A mutates to gene a.

1. Determine the average number of genes of type A in a gene sequence at steady state.
2. Assuming that the initial gene sequence is AAa, show the procedure to compute the probability that the gene sequence aaA is obtained before AAA and aaa.
3. Assuming that the initial gene sequence is AAa, show the procedure to compute the average number of generations to obtain the gene sequence aaA.

EXERCISE 1

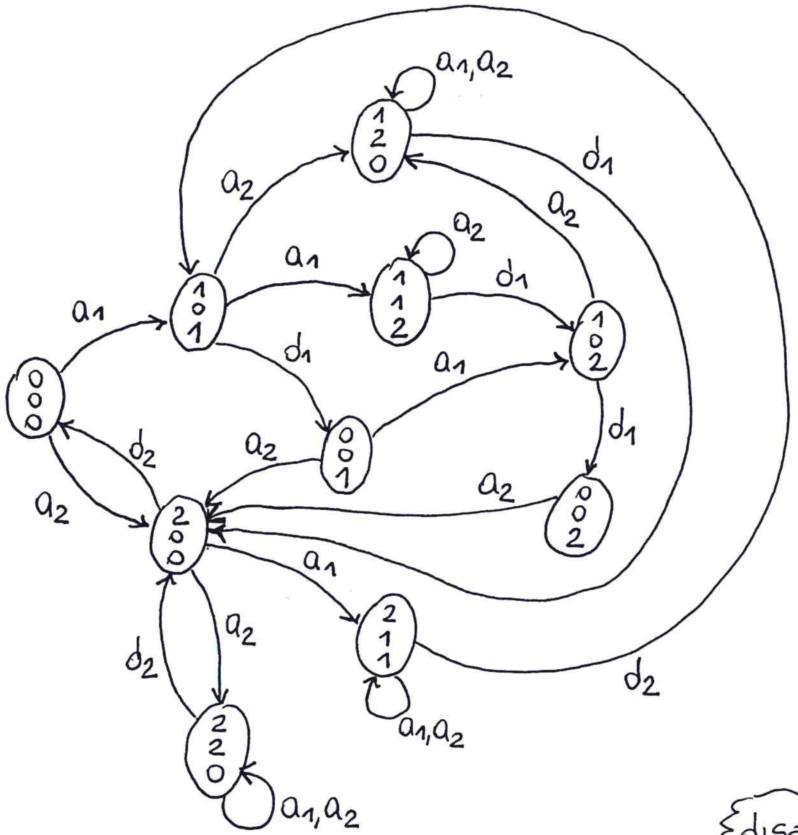
1. The system can be represented as



events $\mathcal{E} = \{ a_1, a_2, d_1, d_2 \}$

- a_1: arrival type 1
- a_2: arrival type 2
- d_1: termination type 1
- d_2: termination type 2

$$\text{state} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \begin{array}{l} \rightarrow M: 0 \text{ (idle)}, 1 \text{ (type 1)}, 2 \text{ (type 2)} \\ \rightarrow B: 0 \text{ (empty)}, 1 \text{ (type 1)}, 2 \text{ (type 2)} \\ \rightarrow \#\text{consecutive accepted type 1 arrivals} \in \{0,1,2\} \end{array}$$



disabled

REMARK: Notice that event a_1 is not possible in states $\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$ because two type 1 arrivals were consecutively accepted.

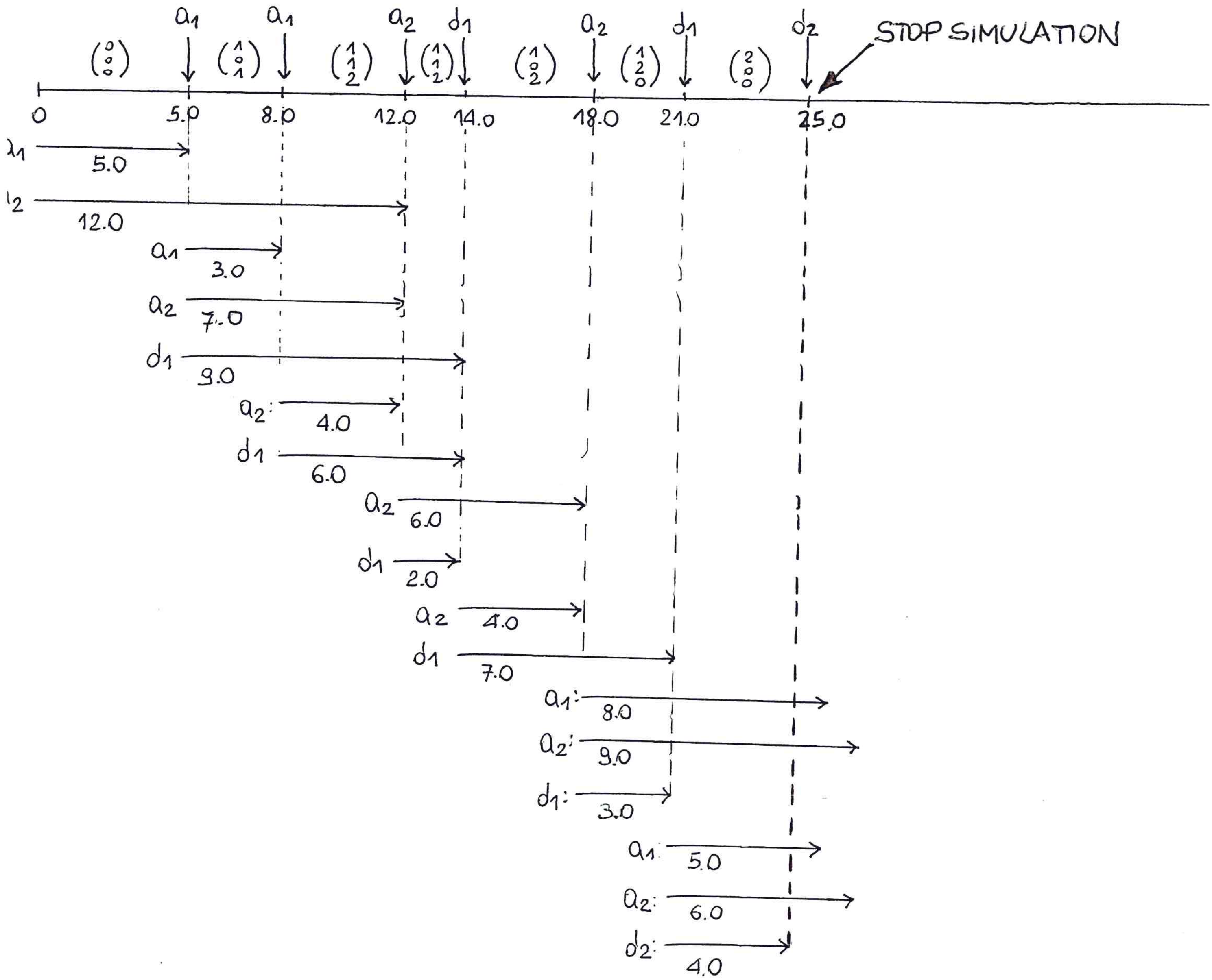
2. $x_0 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$V_{a_1} = \{\cancel{5.0}, \cancel{3.0}, \cancel{8.0}\}$

a_2 occurs at times 12.0, 18.0 and 27.0 $\Rightarrow$ $V_{a_2} = \{\cancel{12.0}, \cancel{6.0}, \cancel{9.0}\}$

$V_{d_1} = \{\cancel{9.0}, \cancel{7.0}, 8.5\}$

$V_{d_2} = \{\cancel{4.0}, 3.5, 5.0\}$



The interval is $T = 25.0$ min.

Arrivals of type 1 are disabled from time 8.0 to time 18.0 $\Rightarrow$ $T_{disabled} = 10.0$ min

$$\Rightarrow f_{disabled} = \frac{T_{disabled}}{T} = \frac{10}{25} = 0.4 = 40\%$$

3. $V_{a_1} \sim U(5.0, 8.0)$

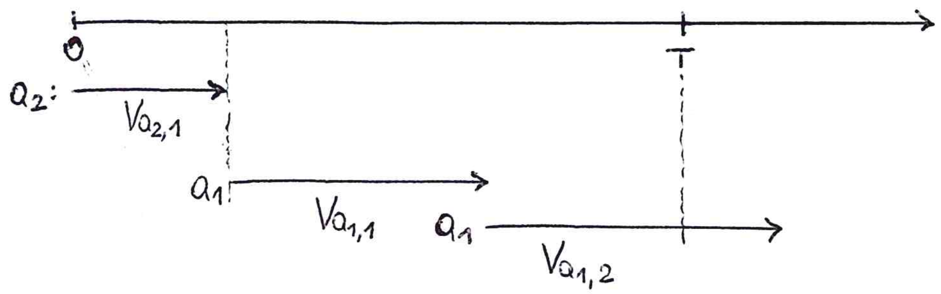
$\downarrow$ generic lifetime of type 1 arrivals

$V_{a_2} = 4.0$ (deterministic)

$\downarrow$ generic lifetime of type 2 arrivals

$$x_0 = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$$

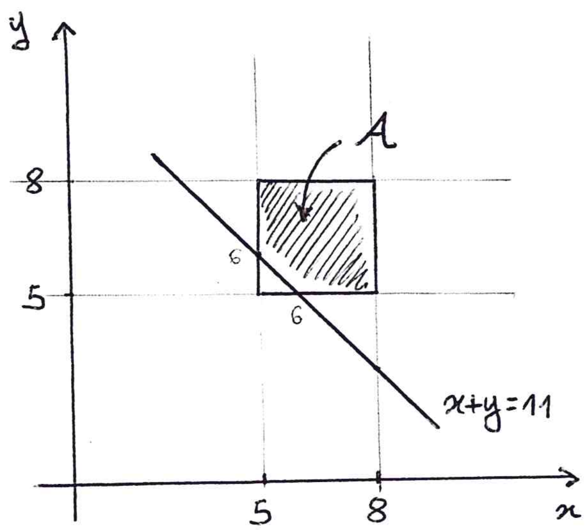
First, we need event a_2 to enable event a_1. Then, we require that the first occurrence of event a_1 is before $T = 15.0$, and the second after T.



$$\Rightarrow P(\underbrace{V_{a_{2,1}}}_{4.0} + V_{a_{1,1}} \leq \underbrace{T}_{15.0},\ \underbrace{V_{a_{2,1}}}_{4.0} + V_{a_{1,1}} + V_{a_{1,2}} > \underbrace{T}_{15.0})$$

$$= P(\underbrace{V_{a_{1,1}} \leq 11}_{\text{always true because } V_{a_1} \sim U(5,8)},\ V_{a_{1,1}} + V_{a_{1,2}} > 11) = P(\underbrace{V_{a_{1,1}}}_{x} + \underbrace{V_{a_{1,2}}}_{y} > 11)$$

$$= \frac{\text{area}(A)}{\underbrace{\text{area}(Q)}_{\text{square}}} = \frac{3^2 - \frac{1^2}{2}}{3^2} = \frac{17}{18} \simeq 0.9444$$



EXERCISE 2

1. stochastic timed automaton $(\underbrace{\mathcal{E}, \mathcal{X}, \Gamma, f, x_0}_{\substack{\text{the logical part}\\ \text{is the same as in}\\ \text{Exercise 1}}}, F)$

$F = \{F_{a_1}, F_{a_2}, F_{d_1}, F_{d_2}\}$

$F_{a_1}(t) = 1 - e^{-\lambda_1 t}, \; t \geq 0$ with $\lambda_1 = 0.2$ arrivals/min

$F_{a_2}(t) = 1 - e^{-\lambda_2 t}, \; t \geq 0$ with $\lambda_2 = 0.25$ arrivals/min

$F_{d_1}(t) = 1 - e^{-\mu_1 t}, \; t \geq 0$ with $\frac{1}{\mu_1} = 8.0$ min $\Rightarrow$ $\mu_1 = 0.125$ services/min

$F_{d_2}(t) = 1 - e^{-\mu_2 t}, \; t \geq 0$ with $\frac{1}{\mu_2} = 6.0$ min $\Rightarrow$ $\mu_2 \simeq 0.1667$ services/min

The current state is $X_k = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$. We have to reach state $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ with the minimum number of events. It is not possible with one, two or three events. It can be done with four events. There are two favorable cases:

- $\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \xrightarrow{d_1} \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \xrightarrow{d_1} \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} \xrightarrow{a_2} \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} \xrightarrow{d_2} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

  with probability $\frac{\mu_1}{\lambda_2 + \mu_1} \cdot \frac{\mu_1}{\lambda_2 + \mu_1} \cdot 1 \cdot \frac{\mu_2}{\lambda_1 + \lambda_2 + \mu_2}$

- $\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \xrightarrow{d_1} \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \xrightarrow{a_2} \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \xrightarrow{d_1} \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} \xrightarrow{d_2} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

  with probability $\frac{\mu_1}{\lambda_2 + \mu_1} \cdot \frac{\lambda_2}{\lambda_2 + \mu_1} \cdot \frac{\mu_1}{\lambda_1 + \lambda_2 + \mu_1} \cdot \frac{\mu_2}{\lambda_1 + \lambda_2 + \mu_2}$

$$\Rightarrow P(\ldots) = \left(\frac{\mu_1}{\lambda_2 + \mu_1}\right)^2 \frac{\mu_2}{\lambda_1 + \lambda_2 + \mu_2} + \frac{\mu_1}{\lambda_2 + \mu_1} \frac{\lambda_2}{\lambda_2 + \mu_1} \frac{\mu_1}{\lambda_1 + \lambda_2 + \mu_1} \frac{\mu_2}{\lambda_1 + \lambda_2 + \mu_2} \simeq 0.0431$$