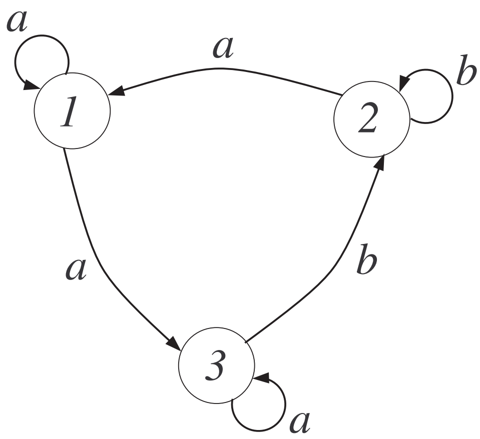
Exam of Discrete Event Systems - 03.02.2015

Exercise 1

Consider the stochastic timed automaton represented in the figure, where $p(1|1,a) = 1/3$, the total lifetimes of event a are uniformly distributed in the interval $[2,4]$ minutes and the total lifetimes of event b are deterministic and equal to 90 seconds. The initial state is $x_0 = 1$.

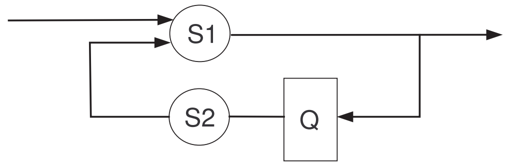


1. Denote by $P(X_k = x)$ the probability that the state after the kth event is x. Compute the state probabilities $P(X_1 = x)$ and $P(X_2 = x)$ for all $x = 1, 2, 3$.

2. Compute the probability to visit all states and return to state 1 in at most 5 minutes.

Exercise 2

Consider the queueing system in the figure, where servers $S1$ and $S2$ represent the sales department and the customer care of a company, respectively. The queue Q has a capacity equal to one. Customers arrive as generated by a Poisson process with rate $\lambda = 3$ arrivals/hour. Customers arriving when $S1$ is busy are rejected. A customer served by $S1$ is satisfied with probability $p = 0.75$. In case the customer is not satisfied, she is routed to $S2$. If $S2$ is busy and Q is empty, the customer waits in Q. If both $S2$ is busy and Q is full, the customer blocks $S1$ until she can move to Q. A customer served by $S2$ is sent back to $S1$, if $S1$ is available. Otherwise, the customer blocks $S2$ until she can move to $S1$. Services in $S1$ and $S2$ have random durations following exponential distributions with expected values 15 and 10 minutes, respectively.



1. Model the queueing system through a stochastic timed automaton $(\mathcal{E}, \mathcal{X}, \Gamma, p, x_0, F)$, assuming the system initially empty.

2. Assume that both $S1$ and $S2$ are busy and Q is empty. Compute the probability that the system is emptied with the minimum number of events.

3. Assume that $S1$ is idle, $S2$ is busy and Q is empty. Show how to compute the probability that the system is emptied with the minimum number of state transitions and within $T = 20$ minutes **(the numerical computation is not required)**.

4. Assume that $S1$ is busy, $S2$ is blocked and Q is empty. Compute the average blocking time of $S2$.

Exercise 3

Consider the queueing system of Exercise 2.

1. Verify the condition $\lambda_{eff} = \mu_{eff}$ for the system at steady-state.
2. Compute the utilizations of $S1$ and $S2$ at steady state.
3. Compute the average time spent by a generic customer in $S1$ at steady state.

Exercise 4

A shop is open from Monday to Saturday. For a given merchandise, the staff checks the stock size on Saturday evening, and four new items are ordered if and only the stock is empty. Ordered items are available at the shop already on Monday morning. Let p_i be the probability that the weekly demand of the considered merchandise is i, and assume $p_0 = 0.5$, $p_1 = 0.3$ and $p_2 = 0.2$. Demand that cannot be satisfied is not backlogged.

1. Model the stock size at the beginning of a week as a discrete-time homogeneous Markov chain, assuming that the stock size is four at the beginning of the first week.
2. Compute the probability that the stock size is four at the beginning of at least three consecutive weeks.
3. Compute the average stock size at the beginning of a generic week at steady state.
4. Assuming that the stock size at the beginning of a week is three, compute the average number of weeks to have again four items in the stock.

EXERCISE 1

1. $p(1|1,a) = 1/3$

$V_a \sim U(2,4) \leftarrow$ minutes

$V_b = 90$ seconds $= 1.5$ minutes

$x_0 = 1 \implies P(X_0=1)=1,\ P(X_0=2)=P(X_0=3)=0$

Only one favorable case for $\{X_1=1\}$:

$$1 \xrightarrow{a} 1$$

$\implies P(X_1=1) = p(1|1,a) = 1/3$

No favorable case for $\{X_1=2\}$

$\implies P(X_1=2)=0$

$\implies P(X_1=3) = 1 - P(X_1=1) - P(X_1=2) = 2/3$

Only one favorable case for $\{X_2=1\}$:

$$1 \xrightarrow{a} 1 \xrightarrow{a} 1$$

$\implies P(X_2=1) = [p(1|1,a)]^2 = 1/9$

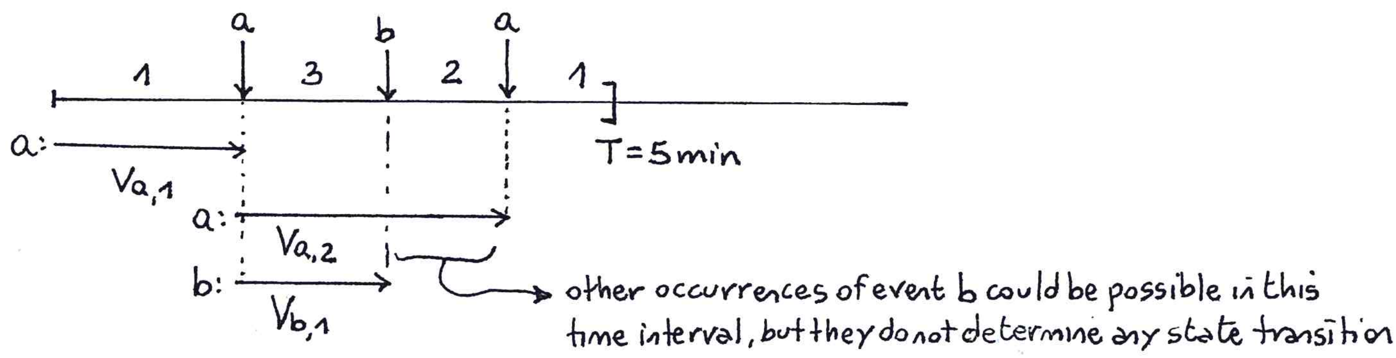
Only one favorable case for $\{X_2=2\}$:

$$1 \xrightarrow{a} 3 \xrightarrow{b} 2$$

$$\implies P(X_2=2) = p(3|1,a) \cdot P(V_{b,1} < V_{a,2}) = \frac{2}{3} \cdot \underbrace{P(V_{a,2} > 1.5)}_{= 1} = 2/3$$

$\implies P(X_2=3) = 1 - P(X_2=1) - P(X_2=2) = 1 - 1/9 - 2/3 = 2/9$

2. Considering the stochastic clock structure of this problem (recall, for instance, that $V_a \sim U(2,4)$ implies that V_a can never take values less than 2 and greater than 4), the only favorable case is the following:



$$\Rightarrow P(\ldots) = P(V_{a,1} < T,\ V_{b,1} < V_{a,2},\ V_{a,1} + V_{b,1} < T,\ V_{a,1} + V_{a,2} < T) \cdot p(3|1,a)$$

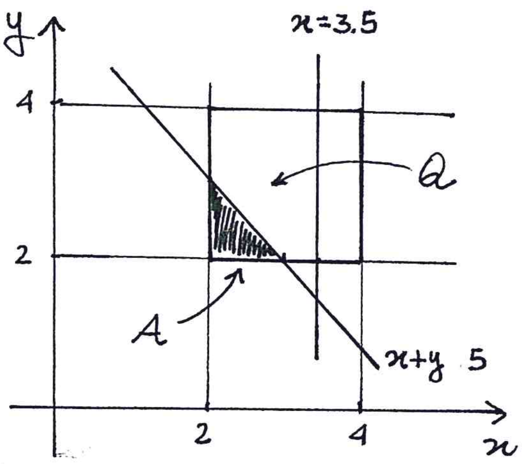
$$= P(V_{a,1} < 5,\ V_{a,2} > 1.5,\ V_{a,1} < 3.5,\ V_{a,1} + V_{a,2} < 5) \cdot p(3|1,a)$$

sempre verificati

$$= P(\underbrace{V_{a,1}}_{x} < 3.5,\ V_{a,1} + \underbrace{V_{a,2}}_{y} < 5) \cdot p(3|1,a)$$

$$= \frac{\text{AREA}(A)}{\text{AREA}(Q)} \cdot p(3|1,a)$$

$$= \frac{\frac{1 \cdot 1}{2}}{2 \cdot 2} \cdot \frac{2}{3} = \frac{1}{8} \cdot \frac{2}{3} = \frac{1}{12}$$



EXERCISE 2

1. events $\mathcal{E} = \{a, d_1, d_2\}$

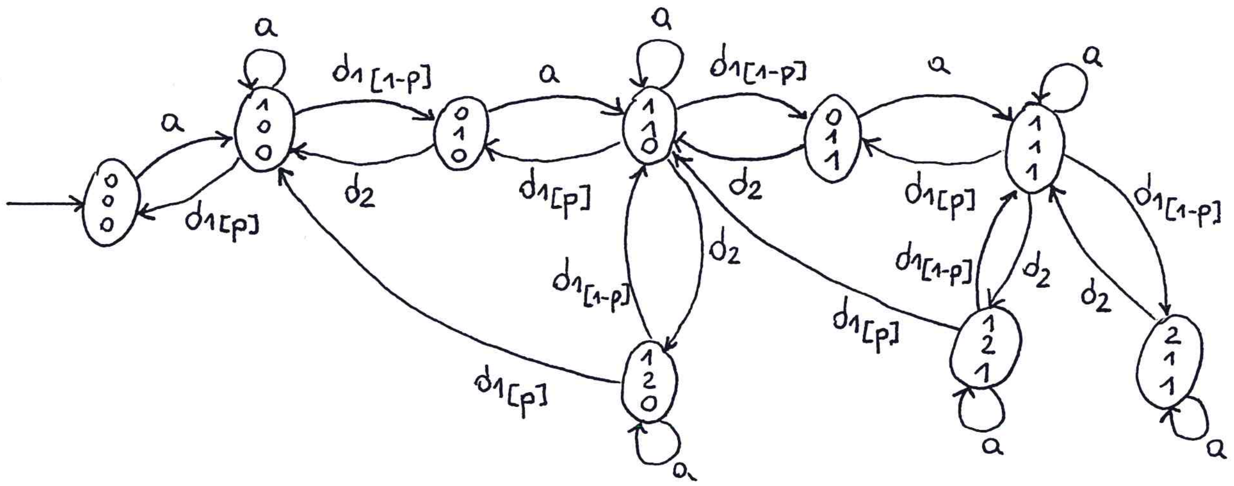
a: arrival of a customer

d_1: termination of a service in S_1

d_2: termination of a service in S_2

state $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$
$x_1 \rightarrow S_1$: 0 (idle), 1 (working), 2 (blocked)
$x_2 \rightarrow S_2$: " " "
$x_3 \rightarrow Q$: 0 (empty), 1 (full)

$p = 3/4$



$F = \{F_a, F_{d_1}, F_{d_2}\}$

$F_a(t) = 1 - e^{-\lambda t}, \; t \geq 0$ with $\lambda = 3$ arrivals/hour

$F_{d_1}(t) = 1 - e^{-\mu_1 t}, \; t \geq 0$ with $\frac{1}{\mu_1} = 15 \text{ min} = \frac{1}{4}$ hours $\Rightarrow \mu_1 = 4$ services/hour

$F_{d_2}(t) = 1 - e^{-\mu_2 t}, \; t \geq 0$ with $\frac{1}{\mu_2} = 10 \text{ min} = \frac{1}{6}$ hours $\Rightarrow \mu_2 = 6$ services/hour

2. The current state is $X_k = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$. The minimum number of events to get from $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ to $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ is three. There are two favorable cases:

- $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \xrightarrow{d_1[p]} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \xrightarrow{d_2} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \xrightarrow{d_1[p]} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

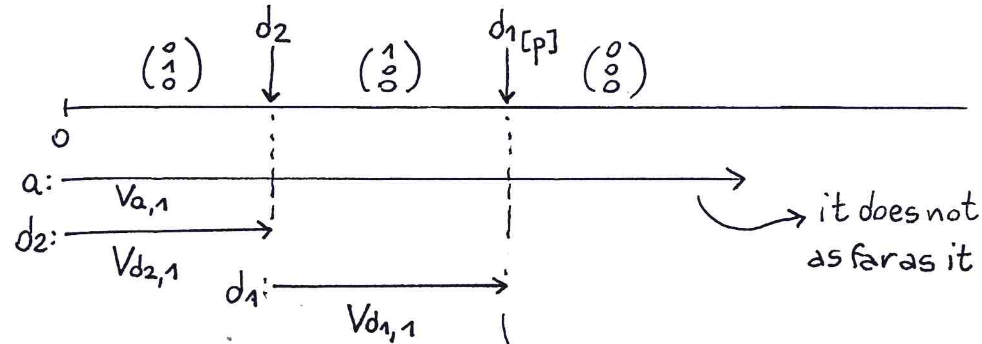
with probability $\dfrac{\mu_1 p}{\lambda + \mu_1 + \mu_2} \cdot \dfrac{\mu_2}{\lambda + \mu_2} \cdot \dfrac{\mu_1 p}{\lambda + \mu_1}$

- $\begin{pmatrix}1\\1\\0\end{pmatrix} \xrightarrow{d_2} \begin{pmatrix}1\\2\\0\end{pmatrix} \xrightarrow{d_{1[p]}} \begin{pmatrix}1\\0\\0\end{pmatrix} \xrightarrow{d_{1[p]}} \begin{pmatrix}0\\0\\0\end{pmatrix}$

  with probability $\dfrac{\mu_2}{\lambda+\mu_1+\mu_2} \cdot \dfrac{\mu_1 p}{\lambda+\mu_1} \cdot \dfrac{\mu_1 p}{\lambda+\mu_1}$

$$\Rightarrow P(\ldots) = \frac{\mu_1 p}{\lambda+\mu_1+\mu_2} \cdot \frac{\mu_2}{\lambda+\mu_2} \cdot \frac{\mu_1 p}{\lambda+\mu_1} + \frac{\mu_2}{\lambda+\mu_1+\mu_2} \left(\frac{\mu_1 p}{\lambda+\mu_1}\right)^2 \simeq 0.1507$$

3. The current state is $X_k = \begin{pmatrix}0\\1\\0\end{pmatrix}$. There is only one favorable case:



it does not matter when event a occurs, as far as it occurs after d_2 and $d_{1[p]}$.

this occurrence must be before T

$$\Rightarrow P(\ldots) = P\big(\underbrace{V_{d_2,1} < T,\ V_{d_2,1} + V_{d_1,1} < T,\ V_{a,1} > V_{d_2,1} + V_{d_1,1}}_{\text{integral over a region of } \mathbb{R}^3}\big) \cdot p \simeq 0.2858$$

estimated numerically

4. The current state is $X_k = \begin{pmatrix}1\\2\\0\end{pmatrix}$.

$$E\left[V\begin{pmatrix}1\\2\\0\end{pmatrix}\right] = \frac{1}{\mu_1} = 15 \text{ min.}$$

state holding time