Midterm Exam - Discrete Event Systems - 19.11.2014

Exercise 1

A machine makes finished products from raw parts that are always available. Occasionally, machine operation is interrupted for inspection. After inspection, the machine either resumes production or goes under maintenance. After maintenance, the machine starts a new production, because during maintenance semifinished parts inside the machine are removed.

1. Model the machine through a stochastic automaton $(\mathcal{E}, \mathcal{X}, \Gamma, p, x_0)$, assuming that the machine is initially working and the probability that the machine goes under maintenance after inspection is $q = 1/10$.

Assume that productions take 12, 25, 18, 20, 23, 16, 15, 21 and 26 minutes; inspections are scheduled at time instants 30, 75 and 120 minutes, and take 10 minutes; maintenance is required only after the third inspection and takes 30 minutes.

2. Determine the average machine throughput (number of finished products per hour) over the time interval [0, 210] minutes.

Assume that productions have random durations following a uniform distribution over the interval $[10, 30]$ minutes; inspections are scheduled 45 minutes after the machine starts/restarts operation and take 10 minutes; maintenance has a random duration following an exponential distribution with expected value 30 minutes.

3. Compute the probability that at least two products are finished before the first inspection starts.
4. Compute the mean time to machine restart from when an inspection starts.

Exercise 2

A dual-core processor is composed of two central processing units (CPUs) and a dedicated cache. Programs executed by the processor can be of two types. Type 1 programs are those that require sequential computing, and therefore are executed in one CPU only. Type 2 programs allow for parallel computing and therefore can be executed simultaneously in both CPUs.

Consider a dual-core processor preceded by a cache which may host only one program waiting to be executed. Programs arrive as generated by a Poisson process with rate $\lambda = 0.05$ arrivals$\cdot$ s^{-1}. Programs are of type 1 with probability $p = 2/3$. The number of operations required by a generic program is a random variable L whose cumulative distribution function can be approximated by an exponential distribution with expected value 10^6 operations. The speed of each CPU is $v = 10^5$ operations$\cdot$ s^{-1}. If a program arrives when the cache is full, it is rejected. A type 2 program is executed only when both CPUs are available. For a type 2 program requiring L operations, $L/2$ operations are executed in one CPU and $L/2$ operations in the other (parallel computing).

1. Model the system through a stochastic timed automaton $(\mathcal{E}, \mathcal{X}, \Gamma, p, x_0, F)$, assuming that both CPUs are initially idle.

2. Assume that both CPUs are executing type 1 programs and the cache hosts a type 1 program. Compute the probability that both CPUs are idle when the next program arrives.

3. Assume that both CPUs are executing type 1 programs and the cache hosts a type 2 program. Compute the mean time to the start of the execution of the type 2 program.

4. Assume that both CPUs are executing type 1 programs and the cache hosts a type 2 program. Show the procedure to compute the probability that the two type 1 programs and the type 2 program are all executed within $T = 15$ s.

Exercise 1

1. state

$$x = \begin{cases} 1: \text{ machine is working} \\ 2: \text{ machine is under inspection} \\ 3: \text{ machine is under maintenance} \end{cases}$$

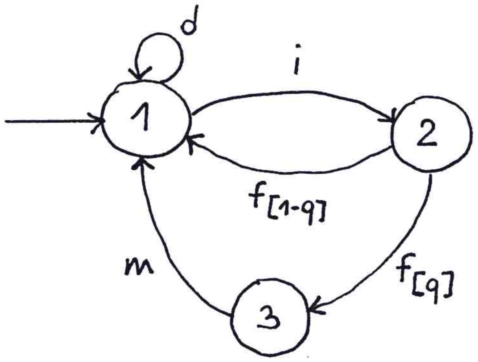
events $\mathcal{E} = \{d, i, f, m\}$

d: termination of a job

i: start of inspection

f: termination of inspection

m: termination of maintenance

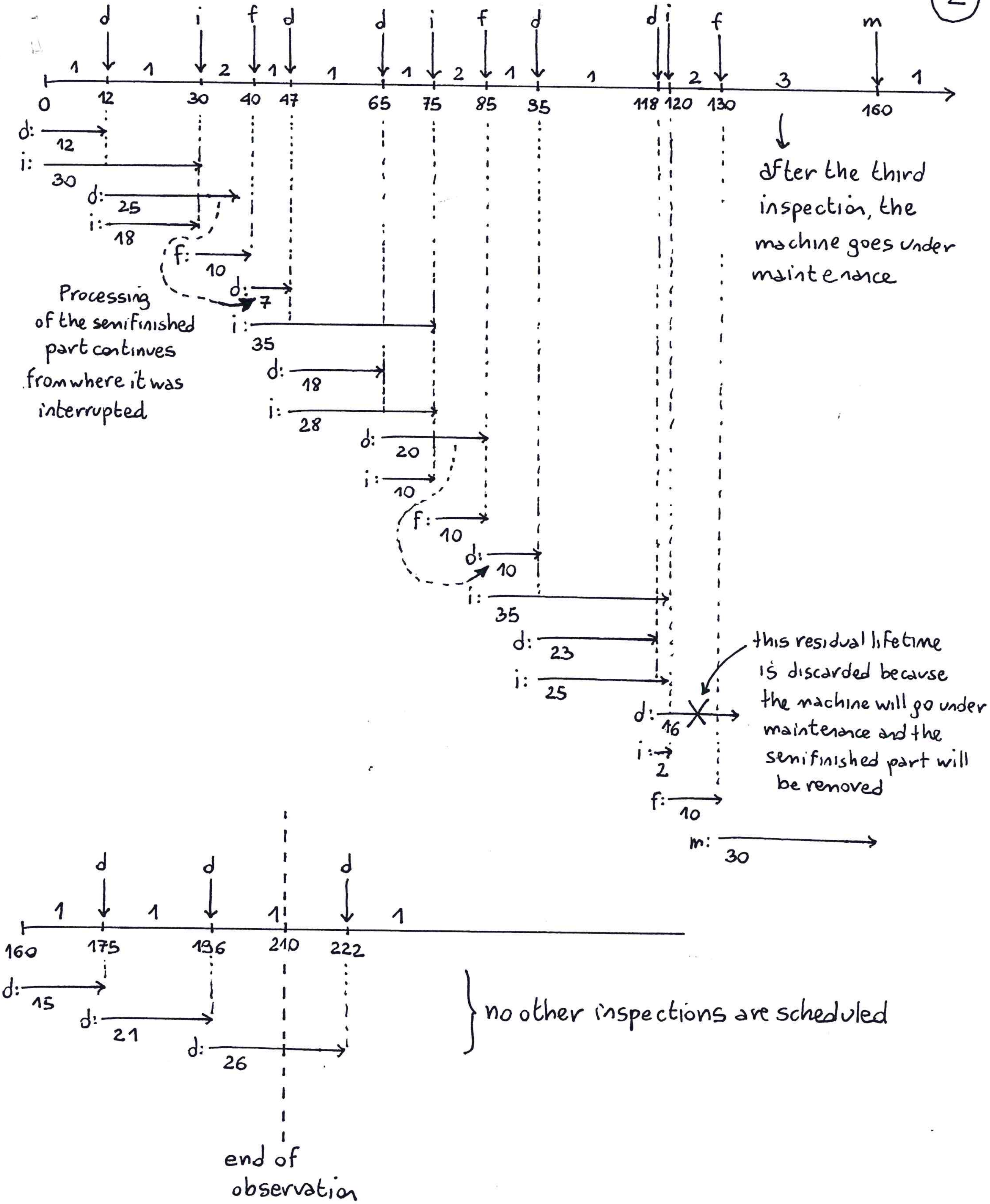


The initial state is $x_0 = 1$. Transition probability $p(3|2,f) = q = 1/10$.

2. We construct the sample path according to the following clock sequences:

- $V_d = \{12, 25, 18, 20, 23, 16, 15, 21, 26\}$
- event i occurs at times 30, 75 and 120
- $V_f = \{10, 10, 10\}$
- $V_m = \{30\}$

[see next page]



We have 7 events d within the time interval [0, 210] minutes, corresponding to 3.5 hours

$\Rightarrow$ average throughput $= \dfrac{7}{3.5} = 2$ products/hour

3. • V_d are uniformly distributed over $[10,30]$ minutes.

total lifetimes of event d

$$\Rightarrow \text{pdf} \quad f_d(t) = \begin{cases} \frac{1}{20} & \text{if } 10 \leq t \leq 30 \\ 0 & \text{otherwise} \end{cases}$$

• $V_i = 45$ minutes

total lifetimes of event i

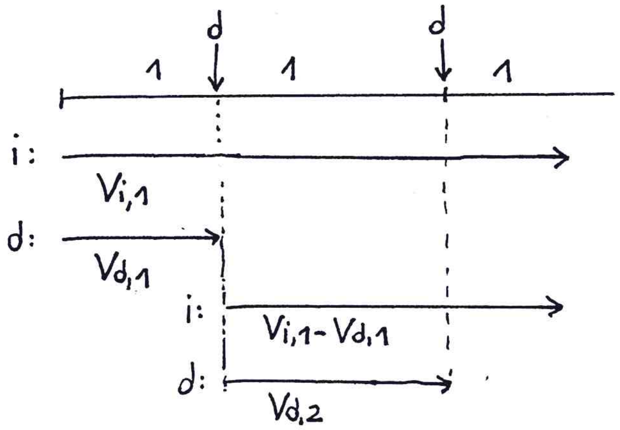
• $V_f = 10$ minutes

total lifetimes of event f

• V_m are exponentially distributed with rate $\lambda = \frac{1}{30}$ minutes^{-1}.

total lifetimes of event m

The probability we need to compute, corresponds to the following sample path:



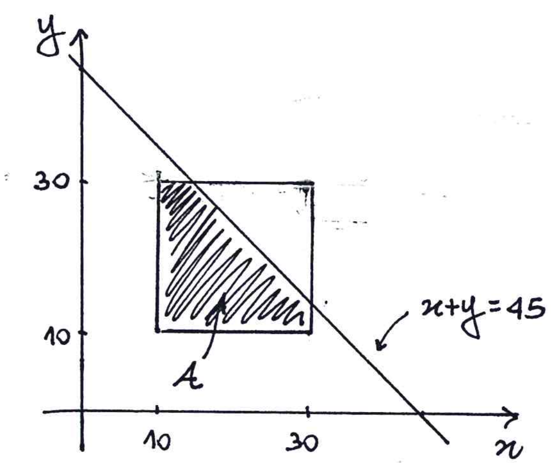
$$\Rightarrow P(\dots) = P(V_{d,1} + V_{d,2} < \underset{45}{\underset{\|}{V_{i,1}}})$$

$$= P(V_{d,1} + V_{d,2} < 45)$$

$$P(V_{d,1} + V_{d,2} < 5) = \frac{\text{area}(A)}{\text{area(square)}} = \frac{20^2 - \frac{15^2}{2}}{20^2}$$

$$= \frac{400 - 112.5}{400} \simeq 0.7188$$

We exploit the fact that $V_{d,1}$ and $V_{d,2}$ are uniformly distributed



4. Let V_f be the duration of inspection and V_m be the duration of maintenance. Moreover, let T_R be the time to machine restart.

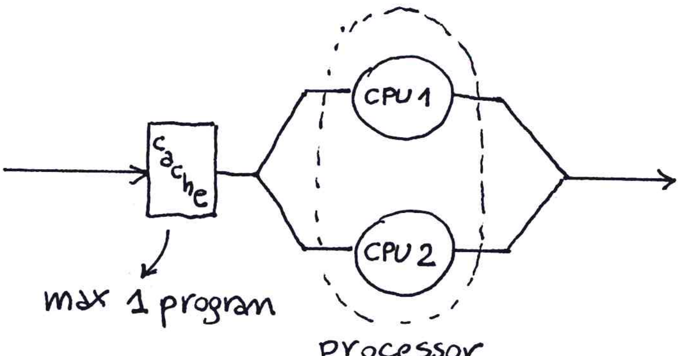
- If no maintenance is required after inspection (this occurs with probability $1-q$), we have $T_R = V_f$.
- If maintenance is required after inspection (this occurs with probability q), we have $T_R = V_f + V_m$.

Therefore (see the Appendix of these notes):

$$E[T_R] = \underbrace{(1-q)}_{\frac{9}{10}} \underbrace{E[V_f]}_{10} + \underbrace{q}_{\frac{1}{10}} \left(\underbrace{E[V_f]}_{10} + \underbrace{E[V_m]}_{30} \right) = 10 + \frac{1}{10} \cdot 30 = 13 \text{ minutes}$$

Exercise 2

The system can be seen as follows:



- type 1 programs: sequential computing (only one CPU)
- type 2 programs: parallel computing (both CPUs simultaneously)

$P(L \leq \ell) = 1 - e^{-\gamma \ell}$, $\ell \geq 0$ with $\gamma = 10^{-6}$ operations^{-1}

L: number of operations required by a program

- execution time in case of sequential computing:

$$Z_s = L/v \Rightarrow P(Z_s \leq t) = P(L/v \leq t) = P(L \leq vt) = 1 - e^{-\gamma v t}$$

If we define $\mu_s = \gamma v = 10^{-6} \cdot 10^{5} = 0.1$ seconds^{-1}, we conclude that Z_s is exponentially distributed with rate μ_s.

- execution time in case of <u>parallel</u> computing

$$Z_p = \frac{1}{2} \cdot \frac{L}{v} \Rightarrow P(Z_p \le t) = P\left(\frac{L}{2v} \le t\right) = $$
$$= P(L \le 2vt) = 1 - e^{-2\gamma v t}$$

because half operations are executed on one CPU, and half on the other, in parallel.

If we define $\mu_p = 2\gamma v = 2 \cdot 10^{-6} \cdot 10^{5} = 0.2$, we conclude that Z_p is exponentially distributed with rate μ_p.

1. state

$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ x_1 → processor, x_2 → cache

processor:
- 0: both CPUs are idle
- 1: one CPU is idle, the other is executing a type 1 program
- 2: both CPUs are executing type 1 programs
- 3: a type 2 program is being executed (both CPUs are busy)

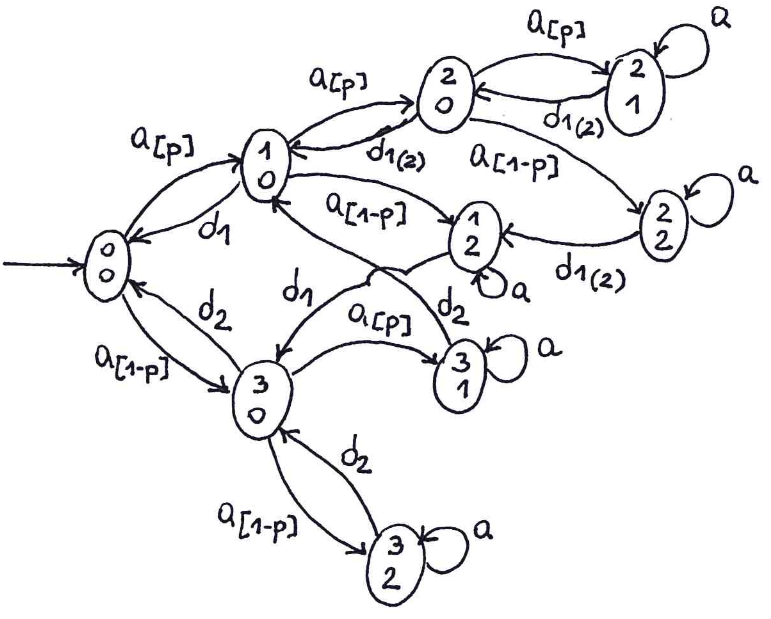
cache:
- 0: empty
- 1: one type 1 program
- 2: one type 2 program

events $\mathcal{E} = \{a, d_1, d_2\}$

a: arrival of a new program

d_1: termination of a type 1 program

d_2: termination of a type 2 program



Initial state is $x_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

$p = \frac{2}{3}$ is the probability of a type 1 program

$\Rightarrow 1-p = \frac{1}{3}$ is the probability of a type 2 program

Stochastic clock structure $F = \{F_a, F_{d_1}, F_{d_2}\}$:

- $F_a(t) = 1 - e^{-\lambda t}$, $t \geq 0$ with $\lambda = 0.05$ arrivals/second
- $F_{d_1}(t) = 1 - e^{-\mu_1 t}$, $t \geq 0$ with $\mu_1 = \mu_S = 0.1$ programs/second
- $F_{d_2}(t) = 1 - e^{-\mu_2 t}$, $t \geq 0$ with $\mu_2 = \mu_P = 0.2$ programs/second

2. The current state is $X_k = \binom{2}{1}$.

We need to compute the probability of the sample path:

$$\binom{2}{1} \xrightarrow{d_1} \binom{2}{0} \xrightarrow{d_1} \binom{1}{0} \xrightarrow{d_1} \binom{0}{0}$$

$$\Rightarrow P(\ldots) = \frac{2\mu_1}{\lambda + 2\mu_1} \cdot \frac{2\mu_1}{\lambda + 2\mu_1} \cdot \frac{\mu_1}{\lambda + \mu_1} \simeq 0.4267$$

3. The current state is $X_k = \binom{2}{2}$.

The time to the start of the execution of the type 2 program can be seen as the sum of the state holding times in $\binom{2}{2}$ and $\binom{1}{2}$.

Therefore:

$$\underbrace{E[W_2]}_{\text{waiting time of type 2 program}} = E[V(\tbinom{2}{2})] + E[V(\tbinom{1}{2})] = \frac{1}{2\mu_1} + \frac{1}{\mu_1} = \frac{3}{2\mu_1} = 15 \text{ seconds}$$

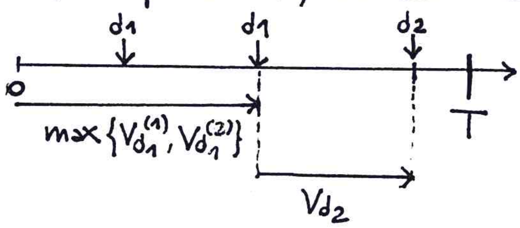
Notice that W_2 can be expressed as

$$W_2 = \max\{V_{d_1}^{(1)}, V_{d_1}^{(2)}\}$$

where $V_{d_1}^{(1)}$ and $V_{d_1}^{(2)}$ are the execution times of the two type 1 programs in the CPUs.

4. Following the previous remark, we have that the probability we need to compute is

$$P(\max\{V_{d_1}^{(1)}, V_{d_1}^{(2)}\} + V_{d_2} < T)$$



where $T = 15$ s.

If one goes on with the computation (<u>not required by the exercise</u>), one has:

$$P\left(\max\left\{V_{d_1}^{(1)}, V_{d_1}^{(2)}\right\} + V_{d_2} < T\right)$$

$$= P\Big(\underbrace{V_{d_1}^{(1)}}_{x} + \underbrace{V_{d_2}}_{z} < T,\ \underbrace{V_{d_1}^{(2)}}_{y} + \underbrace{V_{d_2}}_{z} < T\Big)$$

$$= \int_0^T \mu_2 e^{-\mu_2 z} \left(\int_0^{T-z} \mu_1 e^{-\mu_1 x} dx\right)\left(\int_0^{T-z} \mu_1 e^{-\mu_1 y} dy\right) dz$$

$$= \int_0^T \mu_2 e^{-\mu_2 z} \left[1 - e^{-\mu_1 (T-z)}\right]\left[1 - e^{-\mu_1 (T-z)}\right] dz$$

$$= \int_0^T \left(\mu_2 e^{-\mu_2 z} - 2\mu_2 e^{-\mu_1 T} \cdot e^{(\mu_1-\mu_2) z} + \mu_2 e^{-2\mu_1 T} \cdot \underbrace{e^{\overbrace{(2\mu_1-\mu_2)}^{0} z}}_{1}\right) dz$$

$$= \left[-e^{-\mu_2 z} + \frac{2\mu_2 e^{-\mu_1 T}}{\mu_2 - \mu_1} \cdot e^{-(\mu_2-\mu_1) z} + \mu_2 e^{-2\mu_1 T} \cdot z\right]_0^T$$

$$= 1 - e^{-\mu_2 T} - \frac{2\mu_2 e^{-\mu_1 T}}{\mu_2 - \mu_1}\left[1 - e^{-(\mu_2-\mu_1)T}\right] + \mu_2 e^{-2\mu_1 T} \cdot T \simeq 0.4062$$

Appendix

Let X and Y be random variables with cdf's $F_X(\cdot)$ and $F_Y(\cdot)$, and pdf's $f_X(\cdot)$ and $f_Y(\cdot)$, respectively.

Let $q \in [0,1]$ and define the random variable

$$Z = \begin{cases} X & \text{with probability } (1-q) \\ Y & \text{with probability } q \end{cases}$$

Then,

$$P(Z \leq t) = P(Z \leq t \mid Z = X)\overbrace{P(Z = X)}^{1-q} + P(Z \leq t \mid Z = Y)\overbrace{P(Z = Y)}^{q}$$

total probability rule

$$= (1-q)P(X \leq t) + q\,P(Y \leq t) = (1-q)F_X(t) + qF_Y(t)$$

By deriving with respect to t, we obtain the pdf of Z:

$$f_Z(t) = \frac{dP(Z \leq t)}{dt} = (1-q)\frac{dF_X(t)}{dt} + q\frac{dF_Y(t)}{dt} = (1-q)f_X(t) + q f_Y(t)$$

Finally,

$$E[Z] = \int_{-\infty}^{+\infty} t f_Z(t)\,dt = (1-q)\int_{-\infty}^{+\infty} t f_X(t)\,dt + q\int_{-\infty}^{+\infty} t f_Y(t)\,dt$$

$$= (1-q)E[X] + qE[Y].$$

expected value of Z