Exam of Discrete Event Systems - 31.01.2014

Exercise 1 (both parts/only first part)

A mobile robot without steering wheels receives three types of signals from a remote controller: STOP, FORWARD and BACKWARD. If the robot is still, sending of the signal STOP is deactivated. Similarly, if the robot moves forward or backward, sending of the signals FORWARD and BACKWARD is deactivated, respectively. A timer is associated to each signal. When the corresponding timer expires, the signal is sent. The timer of a signal is reset every time the robot changes state (for instance, the robot is still and starts moving forward, etc.). For the sake of simplicity, it is assumed that the change of direction (for instance, from moving forward to moving backward) is instantaneous.

1. Taking as origin the position of the robot at time zero with robot still, determine the position of the robot relative to the origin after 60 seconds, assuming that the robot moves with speed $v = 1$ m/s in both directions, and the timers associated with the three signals have the following durations expressed in seconds: $V_{\text{STOP}} = \{7, 14, 6, 5, \ldots\}$, $V_{\text{FORWARD}} = \{15, 12, 10, 9, \ldots\}$, $V_{\text{BACKWARD}} = \{20, 10, 8, 12, 15, 7, \ldots\}$.

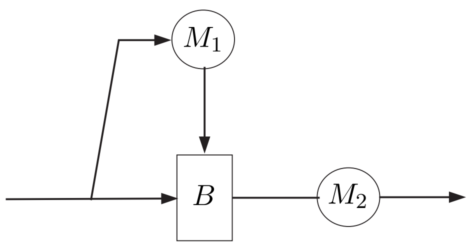
Assume that the timers associated with the signals STOP and BACKWARD follow uniform distributions over the intervals [7,14] and [5,10] seconds, respectively, while the timer associated with the signal FORWARD has constant durations equal to 8.5 seconds.

2. Compute the cumulative distribution function of the state holding time when the robot moves forward.

3. Compute the probability that the robot, initially still, moves backward and then stops again.

Exercise 2 (both parts/only first part)

Consider the production system in the figure, composed by two machines M_1 and M_2, and a unitary buffer B. Raw parts arrive as generated by a Poisson process with rate $\lambda = 0.2$ parts/minute. An arriving part requires preprocessing in M_1 with probability $p = 1/3$. If M_1 is busy, an arriving part requiring preprocessing in M_1 is rejected. Similarly, an arriving part not requiring preprocessing in M_1 is rejected if B is full. After preprocessing in M_1, the part must be processed in M_2, provided that M_2 is available. If this is not the case, the pre-processed part is either stored in B (provided that B is empty) or kept by M_1 (blocking state) until B becomes empty. Tasks in M_1 and M_2 have random durations following exponential distributions with expected value 180 and 150 seconds, respectively.



1. Compute the average state holding time when both machines are working and B is empty.

2. Assuming that both machines are working and B is full, compute the probability that the system is empty before a new arriving part is accepted.

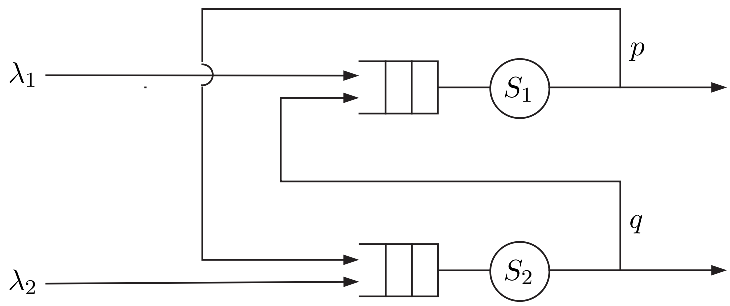
3. Compute the probability that there are five arrivals of parts in 15 minutes and all require preprocessing in M_1.

Only both parts:

4. Verify the condition $\lambda_{eff} = \mu_{eff}$ for the system at steady-state.
5. Compute the average waiting time in B of a generic part at steady state.

Exercise 3 (only second part)

Consider the queueing network in the figure, where each node is represented by a M/M/1 queueing system. Server S_1 has a service rate $\mu_1 = 16$ services/minute, while server S_2 has a service rate $\mu_2 = 144$ services/minute. External arrivals are generated by Poisson processes with rates $\lambda_1 = 7$ arrivals/minute and $\lambda_2 = 1/9$ arrivals/minute, respectively. A fraction $p = 1/16$ of parts processed by S_1 is fed back to node 2, while a fraction $q = 1/11$ of parts processed by S_2 is fed back to node 1.



1. Determine the average system time of a generic part at steady state.

Exercise 4 (both parts/only second part)

Four players sit around a table, and are numbered from 1 to 4 clockwise. They play according to the following rules. At each round of the game, the active player tosses two dies. If the sum of the results is an even number different from 12, the next active player will be the one on the right-hand side. If the sum of the results is an odd number, the next active player will be the one on the left-hand side. Otherwise, the next active player will be the one in front of the currently active player.

1. How many rounds of the game does a player wait on average before being active again? Justify the answer.
2. Assuming that player 1 starts the game, compute the probability that player 4 is never active through the first 10 rounds of the game.
3. Assuming that player 1 starts the game, compute the probability that, when player 1 will be active again, the other three players have been active at least once.

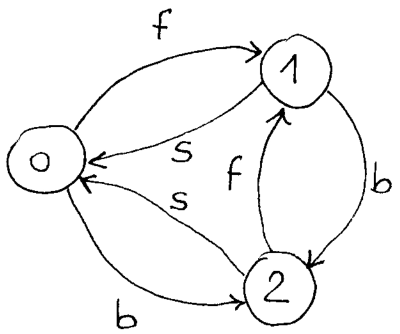
Exercise 1

model

$$\text{state} \quad x = \begin{cases} 0 & \text{if the robot is still} \\ 1 & \text{if the robot moves forward} \\ 2 & \text{if the robot moves backward} \end{cases}$$

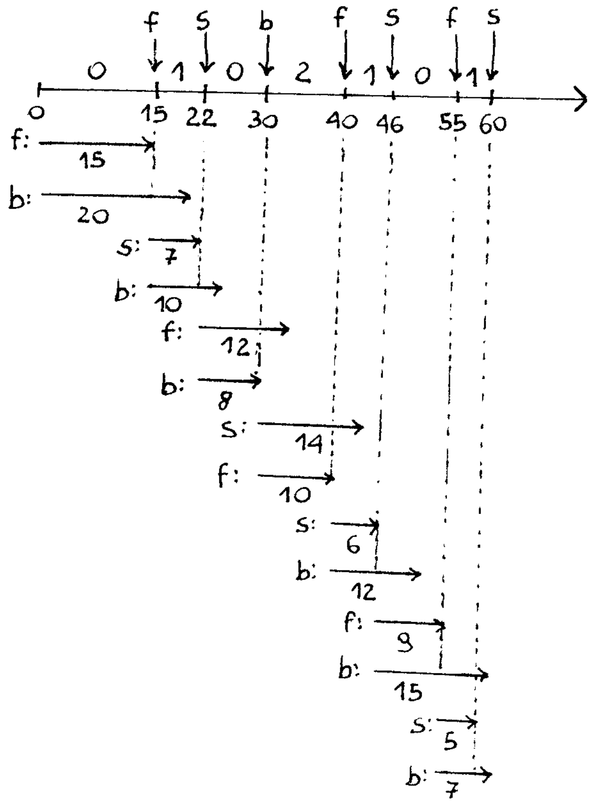
events $\mathcal{E} = \{ s, f, b \}$

- s: arrival of STOP signal
- f: arrival of FORWARD signal
- b: arrival of BACKWARD signal



1. clock structure $V = \{V_s, V_f, V_b\}$, where:

$V_s = \{7, 14, 6, 5, \ldots\}$, $V_f = \{15, 12, 10, 9, \ldots\}$, $V_b = \{20, 10, 8, 12, 15, 7, \ldots\}$



During 60 seconds, the robot moves forward for $7+6+5 = 18$ seconds $\Rightarrow$ 18 m forward and moves backward for 10 seconds $\Rightarrow$ 10 m backward

The position of the robot after 60 seconds is $18 - 10 = 8$ m from the origin.

REMARK: notice that, according to the problem description, a new lifetime is associated to an event even if the event was possible in the previous state, it did not occur and it remains possible in the new state.

Stochastic clock structure $F = \{F_s, V_f, F_b\}$ where:

$$F_s(t) = \begin{cases} 0 & \text{if } t<7 \\ \frac{t-7}{14-7} = \frac{t-7}{7} & \text{if } 7 \le t \le 14 \\ 1 & \text{if } t>14 \end{cases}$$

$V_f = \{8.5, 8.5, 8.5, \ldots\}$ => deterministic clock sequence

$$F_b(t) = \begin{cases} 0 & \text{if } t<5 \\ \frac{t-5}{10-5} = \frac{t-5}{5} & \text{if } 5 \le t \le 10 \\ 1 & \text{if } t>10 \end{cases}$$

2. $P(V(1) \le t) = P(\min\{V_s, V_b\} \le t) = 1 - P(\min\{V_s, V_b\} > t)$

holding time in state 1 (the robot moves forward)

lifetime of event s

lifetime of event b

$= 1 - P(V_s > t, V_b > t)$

$= 1 - P(V_s > t)P(V_b > t)$

V_s and V_b are independent by assumption

Notice that:

$$P(V_s > t) = \begin{cases} 1 & \text{if } t<7 \\ \frac{14-t}{7} & \text{if } 7 \le t \le 14 \\ 0 & \text{if } t>14 \end{cases}, \quad P(V_b > t) = \begin{cases} 1 & \text{if } t<5 \\ \frac{10-t}{5} & \text{if } 5 \le t \le 10 \\ 0 & \text{if } t>10 \end{cases}$$

Therefore:

$$P(V(1) \le t) = \begin{cases} 0 & \text{if } t<5 \\ \frac{t-5}{5} & \text{if } 5 \le t < 7 \\ \frac{-t^2+24t-105}{35} & \text{if } 7 \le t \le 10 \\ 1 & \text{if } t>10 \end{cases}$$

3. It is the probability of the following path:

$$0 \xrightarrow{b} 2 \xrightarrow{s} 0$$

$P(\ldots) = P(X_2=0, E_2=s, X_1=2, E_1=b \mid X_0=0) = \ldots = P(E_2=s \mid X_1=2)\, P(E_1=b \mid X_0=0)$

$= P(V_s < V_f)\, P(V_b < V_f) = P(V_s < 8.5)\, P(V_b < 8.5) = \frac{8.5-7}{7} \cdot \frac{8.5-5}{5} = \frac{3}{20} = 0.15$

Exercise 2

model

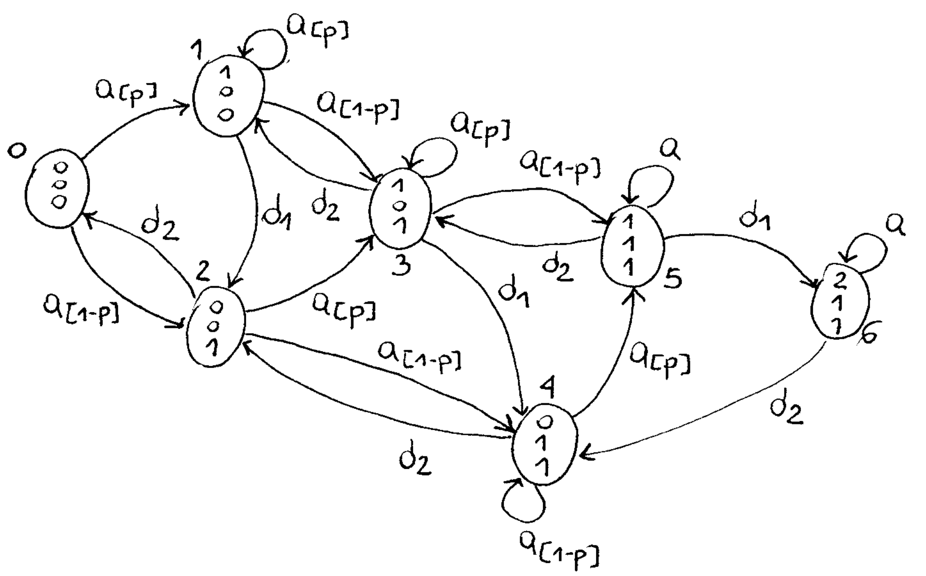
state $x = \begin{bmatrix} x_1 \to M_1: 0\ (\text{idle}),\ 1\ (\text{working}),\ 2\ (\text{blocked}) \\ x_2 \to B: 0\ (\text{empty}),\ 1\ (\text{full}) \\ x_3 \to M_2: 0\ (\text{idle}), 1\ (\text{working}) \end{bmatrix}$

events $\mathcal{E} = \{ a, d_1, d_2 \}$

a: arrival of a raw part

d_1: termination in M_1

d_2: termination in M_2



$p = \frac{1}{3}$

stochastic clock structure $F = \{ F_a, F_{d_1}, F_{d_2} \}$

$F_a(t) = 1 - e^{-\lambda t},\ t \geq 0$ where $\lambda = \frac{1}{5}$ parts/minute

$F_{d_1}(t) = 1 - e^{-\mu_1 t},\ t \geq 0$ where $\mu_1 = \frac{1}{3}$ services/minute

$F_{d_2}(t) = 1 - e^{-\mu_2 t},\ t \geq 0$ where $\mu_2 = \frac{2}{5}$ services/minute

1. $E\left[V\left(\begin{bmatrix}1\\0\\1\end{bmatrix}\right)\right] = \dfrac{1}{\lambda(1-p) + \mu_1 + \mu_2} = \dfrac{15}{13} \simeq 1.1538$ minutes

holding time in state $\begin{bmatrix}1\\0\\1\end{bmatrix}$

2. $X_k = \begin{bmatrix}1\\1\\1\end{bmatrix}$ current state

$P(\ldots) = \dfrac{\mu_2}{\mu_1+\mu_2} \cdot \dfrac{\mu_2}{\lambda(1-p)+\mu_1+\mu_2} \cdot \dfrac{\mu_1}{\lambda(1-p)+\mu_1} \cdot \dfrac{\mu_2}{\lambda+\mu_2} +$

there are three possible paths from $\begin{bmatrix}1\\1\\1\end{bmatrix}$ to $\begin{bmatrix}0\\0\\0\end{bmatrix}$ without accepting new arrivals ...

$$+\frac{\mu_2}{\mu_1+\mu_2}\cdot\frac{\mu_1}{\lambda(1-p)+\mu_1+\mu_2}\cdot\frac{\mu_2}{\lambda p+\mu_2}\cdot\frac{\mu_2}{\lambda+\mu_2}+$$

$$+\frac{\mu_1}{\mu_1+\mu_2}\cdot 1\cdot\frac{\mu_2}{\lambda p+\mu_2}\cdot\frac{\mu_2}{\lambda+\mu_2}\simeq 0.4995$$

3. $P(\ldots)=P\left(N_{a_{[p]}}(T)=5\right)P\left(N_{a_{[1-p]}}(T)=0\right)$

$$=\frac{(\lambda p\cdot T)^5}{5!}e^{-\lambda p\cdot T}\cdot e^{-\lambda(1-p)\cdot T}=p^5\cdot\underbrace{\frac{(\lambda T)^5}{5!}e^{-\lambda T}}_{P(N_a(T)=5)}\underset{T=15\text{min}}{\simeq}0.0004$$

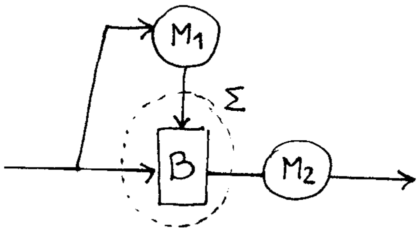
$$Q=\begin{bmatrix}
-\lambda & \lambda p & \lambda(1-p) & 0 & 0 & 0 & 0\\
0 & -[\lambda(1-p)+\mu_1] & \mu_1 & \lambda(1-p) & 0 & 0 & 0\\
\mu_2 & 0 & -(\lambda+\mu_2) & \lambda p & \lambda(1-p) & 0 & 0\\
0 & \mu_2 & 0 & -[\lambda(1-p)+\mu_1+\mu_2] & \mu_1 & \lambda(1-p) & 0\\
0 & 0 & \mu_2 & 0 & -(\lambda p+\mu_2) & \lambda p & 0\\
0 & 0 & 0 & \mu_2 & 0 & -(\mu_1+\mu_2) & \mu_1\\
0 & 0 & 0 & 0 & \mu_2 & 0 & -\mu_2
\end{bmatrix}$$

irreducible and finite continuous-time homogeneous Markov chain

$$\begin{cases}\pi Q=0\\ \sum \pi_i=1\end{cases}\Rightarrow \pi\simeq\left[0.4736\ \ 0.1040\ \ 0.2368\ \ 0.0424\ \ 0.1106\ \ 0.0178\ \ 0.0148\right]$$

4. $$\left.\begin{aligned}\lambda_{eff}&=\lambda p\cdot(\pi_0+\pi_2+\pi_4)+\lambda(1-p)\cdot(\pi_0+\pi_1+\pi_2+\pi_3)\simeq 0.1630\\ \mu_{eff}&=\mu_2\cdot(\pi_2+\pi_3+\pi_4+\pi_5+\pi_6)\simeq 0.1630\end{aligned}\right\}\text{ ok!}$$

5.



$$E[S_\Sigma]=\frac{E[X_\Sigma]}{\lambda_\Sigma}=\frac{0\cdot(\pi_0+\pi_1+\pi_2+\pi_3)+1\cdot(\pi_4+\pi_5+\pi_6)}{\lambda_{eff}}$$

why?

$\simeq 0.8476$ minutes

Exercise 3

1. $$\begin{cases} \lambda_{1,eff} = \lambda_1 + q\mu_{2,eff} \\ \lambda_{2,eff} = \lambda_2 + p\mu_{1,eff} \end{cases}$$

Since $\lambda_{1,eff} = \mu_{1,eff}$ and $\lambda_{2,eff} = \mu_{2,eff}$ at steady state, we have:

$$\begin{cases} \lambda_{1,eff} = \lambda_1 + q\lambda_{2,eff} \\ \lambda_{2,eff} = \lambda_2 + p\lambda_{1,eff} \end{cases}$$

Solving with respect to $\lambda_{1,eff}$ and $\lambda_{2,eff}$, we have:

$$\lambda_{1,eff} = \lambda_1 + q\lambda_2 + qp\lambda_{1,eff} \quad \Rightarrow \quad \lambda_{1,eff} = \frac{\lambda_1 + q\lambda_2}{1-qp} \simeq 7.0502$$

$$\lambda_{2,eff} = \lambda_2 + p\lambda_1 + qp\lambda_{2,eff} \quad \Rightarrow \quad \lambda_{2,eff} = \frac{\lambda_2 + p\lambda_1}{1-qp} \simeq 0.5517$$

$$\Rightarrow \rho_1 = \frac{\lambda_{1,eff}}{\mu_1} \simeq 0.4406 < 1 \ \underline{ok}, \quad \rho_2 = \frac{\lambda_{2,eff}}{\mu_2} \simeq 0.0038 < 1 \ \underline{ok}$$

$$\Rightarrow E[X_1] = \frac{\rho_1}{1-\rho_1} \simeq 0.7877, \quad E[X_2] = \frac{\rho_2}{1-\rho_2} \simeq 0.0038$$

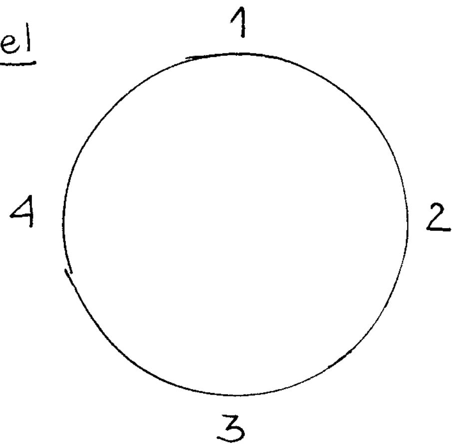
$E[X_1]$: number of parts in node 1

$$\Rightarrow E[S] = \frac{E[X_1] + E[X_2]}{\lambda_1 + \lambda_2} \simeq 0.1113 \text{ minutes}$$

$E[S]$: system time of a generic part

Exercise 4

model



$$p_{RIGHT} = \frac{17}{36}$$
$$p_{LEFT} = \frac{1}{2}$$
$$p_{FRONT} = \frac{1}{36}$$

state x = current active player $\in \{1, 2, 3, 4\}$

$$P = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{36} & \frac{17}{36} \\ \frac{17}{36} & 0 & \frac{1}{2} & \frac{1}{36} \\ \frac{1}{36} & \frac{17}{36} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{36} & \frac{17}{36} & 0 \end{bmatrix}$$

1. The discrete-time homogeneous Markov chain is irreducible, aperiodic and finite. The vector of stationary probabilities is

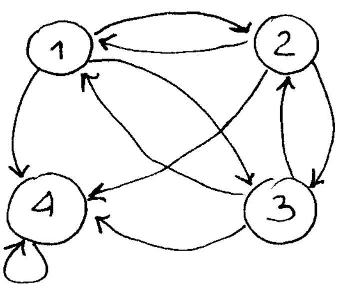
$$\Pi = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{bmatrix} \qquad (\ldots \text{why?})$$

$$\Rightarrow E[T_{i,i}] = \frac{1}{\Pi_i} = 4$$

recurrence time of state i

on average, a player waits four rounds of the game before being active again

2. We modify the model as follows:



$$\tilde{P} = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{36} & \frac{17}{36} \\ \frac{17}{36} & 0 & \frac{1}{2} & \frac{1}{36} \\ \frac{1}{36} & \frac{17}{36} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\Rightarrow$ 4 is an absorbing state

Using this model, the answer is:

$$P(\tilde{X}(9) \neq 4 \mid \tilde{X}(0)=1) = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \tilde{P}^9 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \simeq 0.0323$$

3. To answer this question, we need a definition of state which has memory of the players who have been active at least once and of the current player.

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \begin{matrix} x_1 \to \text{visited states} \\ x_2 \to \text{current player} \end{matrix}$$

$$\mathcal{X} = \left\{ \begin{array}{cccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \begin{bmatrix} \{1\} \\ 1 \end{bmatrix}, & \begin{bmatrix} \{1,2\} \\ 2 \end{bmatrix}, & \begin{bmatrix} \{1,3\} \\ 3 \end{bmatrix}, & \begin{bmatrix} \{1,4\} \\ 4 \end{bmatrix}, & \begin{bmatrix} \{1,2,3\} \\ 2 \end{bmatrix}, & \begin{bmatrix} \{1,2,3\} \\ 3 \end{bmatrix}, & \begin{bmatrix} \{1,2,4\} \\ 2 \end{bmatrix}, & \begin{bmatrix} \{1,2,4\} \\ 4 \end{bmatrix}, \end{array} \right.$$

$$\left. \begin{array}{cccc} 8 & 9 & 10 & 11 \\ \begin{bmatrix} \{1,3,4\} \\ 3 \end{bmatrix}, & \begin{bmatrix} \{1,3,4\} \\ 4 \end{bmatrix}, & \begin{bmatrix} \{1,2,3,4\} \\ - \end{bmatrix}, & \begin{bmatrix} - \\ - \end{bmatrix} \end{array} \right\}$$

initial state

absorbing state which tells us that all players have been active at least once

absorbing state which tells us that player 1 has been active again before all other players at least once

$$\bar{P} = \begin{bmatrix}
0 & \frac{1}{2} & \frac{1}{36} & \frac{17}{36} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{36} & 0 & 0 & 0 & \frac{17}{36} \\
0 & 0 & 0 & 0 & \frac{17}{36} & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{36} \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{36} & 0 & \frac{17}{36} & 0 & 0 & \frac{1}{2} \\
0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & \frac{1}{36} & \frac{17}{36} \\
0 & 0 & 0 & 0 & \frac{17}{36} & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{36} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{36} & 0 & 0 & \frac{1}{2} & \frac{17}{36} \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{36} & 0 & 0 & 0 & \frac{17}{36} & \frac{1}{2} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{17}{36} & \frac{1}{36} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{17}{36} & 0 & \frac{1}{36} & \frac{1}{2} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}$$

Using this model, the answer is:

$$\lim_{t\to\infty} P(\bar{X}(t)=10 \mid \bar{X}(0)=1) =$$

$$= \lim_{t\to\infty} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \bar{P}^{t} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \simeq 0.3327$$