# Endterm Exam - Discrete Event Systems - 16.01.2014

## Exercise 1

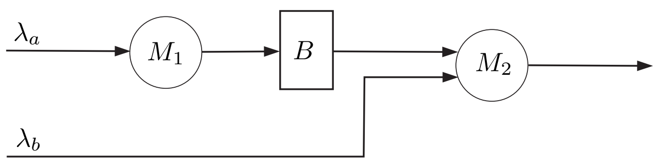
An express courier serves six towns, numbered from 1 to 6. Every day the courier visits only one town. If at day $t$ he visits town $i$, at day $t+1$ he visits town $j$ with probability $p_{i,j}$ given by the transition probability matrix

$$P = [p_{i,j}] = \begin{bmatrix} 1/3 & 1/2 & 0 & 1/6 & 0 & 0 \\ 1/8 & 0 & 1/4 & 1/4 & 3/8 & 0 \\ 1/6 & 0 & 1/3 & 1/4 & 0 & 1/4 \\ 0 & 1/15 & 1/5 & 1/5 & 1/5 & 1/3 \\ 0 & 3/8 & 0 & 1/8 & 0 & 1/2 \\ 0 & 0 & 0 & 1/6 & 2/3 & 1/6 \end{bmatrix}$$

1. Assuming that the courier is initially in town 1, compute the probability that at day 5 he is in town 4.
2. Compute the probability that the courier remains in town 4 for at least three days.
3. Compute the probability that the courier reaches town 6 from town 1 without visiting town 3.
4. Compute the average number of days to reach town 6 from town 1.

## Exercise 2

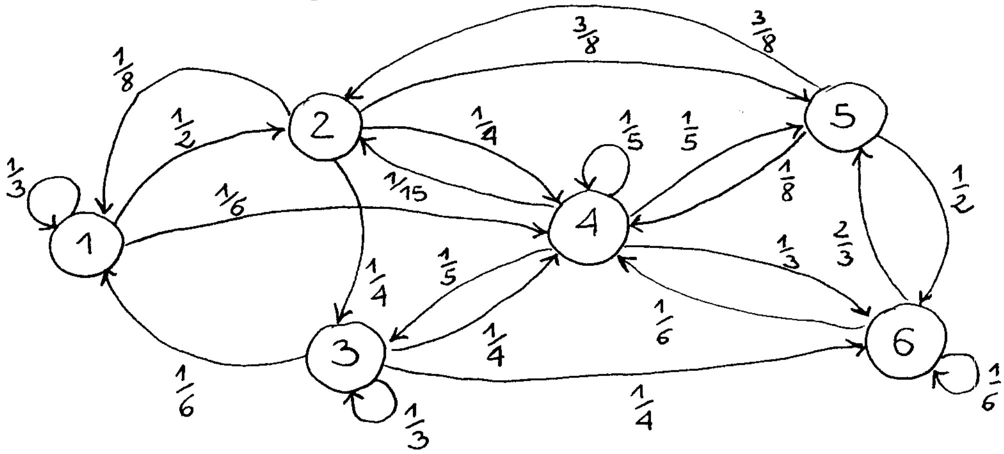
Consider the production system in the figure, composed by two machines $M_1$ and $M_2$, and a unitary buffer $B$. Machine $M_1$ produces one semi-finished product from one part of type $a$. Machine $M_2$ produces one finished product by assembling one semi-finished product and one part of type $b$. Parts of type $a$ and type $b$ arrive as generated by Poisson processes with rates $\lambda_a = 0.2$ and $\lambda_b = 0.8$ parts/minute, respectively. Parts of type $a$ arriving when $M_1$ is busy, are rejected. When $M_1$ terminates the production of a semi-finished product, the semi-finished product is stored in the buffer $B$, provided that the latter is empty. Otherwise, it is kept in $M_1$ (blocking state) until $B$ becomes empty. Parts of type $b$ are accepted only if there is a semi-finished product in $B$ and $M_2$ is idle. When a part of type $b$ is accepted, $M_2$ starts assembling the semi-finished product from $B$ and the part of type $b$. Tasks in $M_1$ and $M_2$ have random durations following exponential distributions with expected value 90 and 120 seconds, respectively.



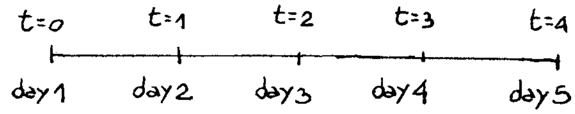

1. Provide an appropriate model of the system.
2. Assuming that the production system is initially empty, compute the probability that a part of type $a$ arriving after exactly 10 minutes is not accepted. Justify the answer.
3. Verify the condition $\lambda_{eff} = \mu_{eff}$ for the system at steady-state.
4. Compute the average time between the termination of the production of a semi-finished product and the start of its assembling with a part of type $b$.
5. Compute the utilization of both machines at steady state.

# Exercise 1

discrete-time homogeneous Markov chain:




1.

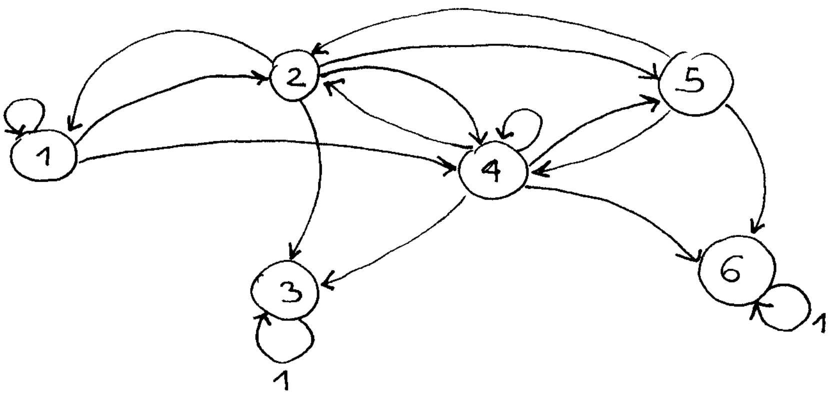



$$\Rightarrow\ p_{1,4}(4) = [1\ 0\ 0\ 0\ 0\ 0]\, P^4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \frac{766}{3949} \simeq 0.1940$$

2. $P(V(4) \geq 3) = 1 - P(V(4)=1) - P(V(4)=2) = 1 - (1-p_{4,4}) - (1-p_{4,4})p_{4,4}$

$$= p_{4,4}^2 = \frac{1}{25} = 0.040$$

3. We modify the model as follows:

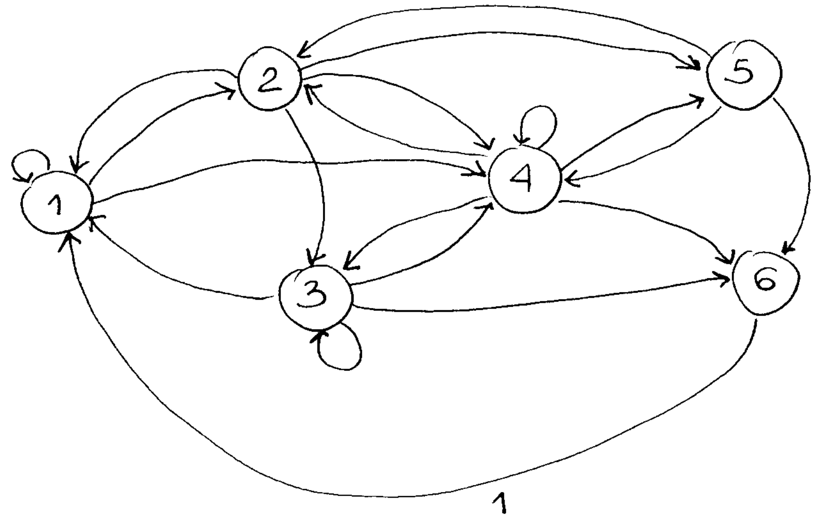



$$\tilde{P} = \begin{bmatrix} \frac{1}{3} & \frac{1}{2} & 0 & \frac{1}{6} & 0 & 0 \\ \frac{1}{8} & 0 & \frac{1}{4} & \frac{1}{4} & \frac{3}{8} & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & \frac{1}{15} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{3} \\ 0 & \frac{3}{8} & 0 & \frac{1}{8} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Using this model, the answer is:

$$\pi_6 = \lim_{t\to\infty} \pi_6(t) = \lim_{t\to\infty} [1\ 0\ 0\ 0\ 0\ 0]\, \tilde{P}^t \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \frac{293}{525} \simeq 0.5581$$

4. We modify the model as follows:




$$\bar{P} = \begin{bmatrix} \frac{1}{3} & \frac{1}{2} & 0 & \frac{1}{6} & 0 & 0 \\ \frac{1}{8} & 0 & \frac{1}{4} & \frac{1}{4} & \frac{3}{8} & 0 \\ \frac{1}{6} & 0 & \frac{1}{3} & \frac{1}{4} & 0 & \frac{1}{4} \\ 0 & \frac{1}{15} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{3} \\ 0 & \frac{3}{8} & 0 & \frac{1}{8} & 0 & \frac{1}{2} \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Using this model, the answer is:

$$E[T_{1,6}] = \underbrace{E[T_{6,6}]}_{\text{recurrence time of state 6}} - 1 \underbrace{=}_{\text{irreducible, aperiodic and finite}} \frac{1}{\pi_6} - 1 = \frac{419}{68} \simeq 6.1618$$

where $\pi_6 = \lim_{t\to\infty} \pi_6(t) = \lim_{t\to\infty} [1\ 0\ 0\ 0\ 0\ 0]\,\bar{P}^t \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \frac{68}{487}$

## Exercise 2

1. stochastic timed automaton

state: $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$
- $x_1 \rightarrow M_1$: idle (0), working (1), blocked (2)
- $x_2 \rightarrow B$: empty (0), full (1)
- $x_3 \rightarrow M_2$: idle (0), working (1)

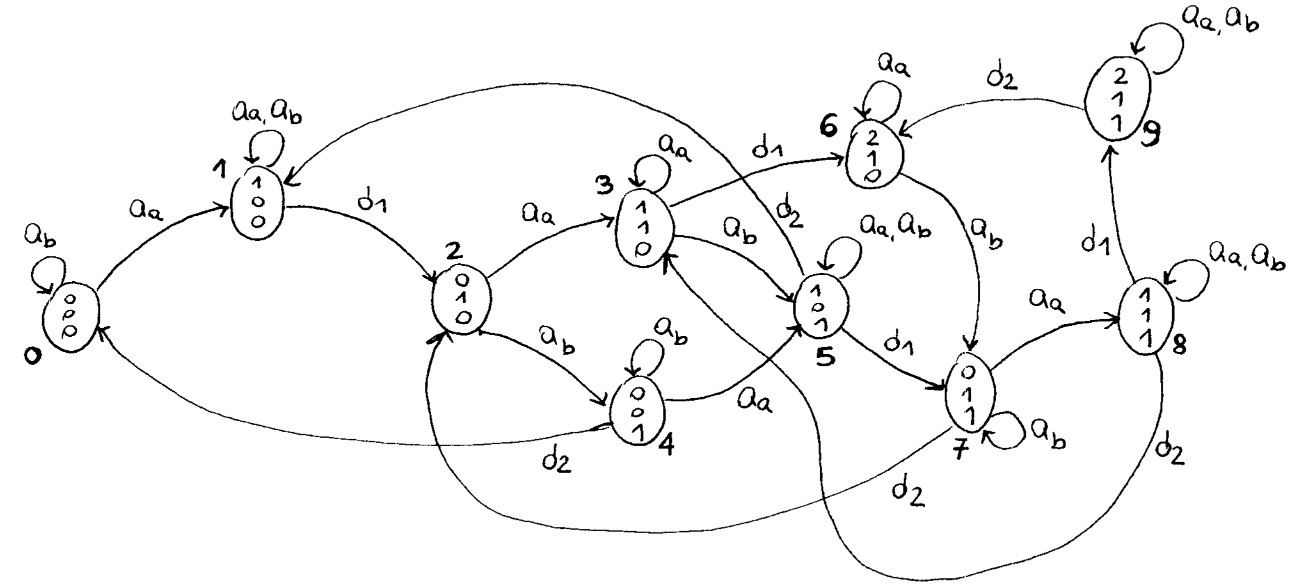
events: $\mathcal{E} = \{a_a, a_b, d_1, d_2\}$
- $a_a$: arrival type a
- $a_b$: arrival type b
- $d_1$: termination in $M_1$
- $d_2$: termination in $M_2$

$F_{a_a}(t) = 1 - e^{-\lambda_a t}$, $t \geq 0$ where $\lambda_a = 0.2$ parts/min

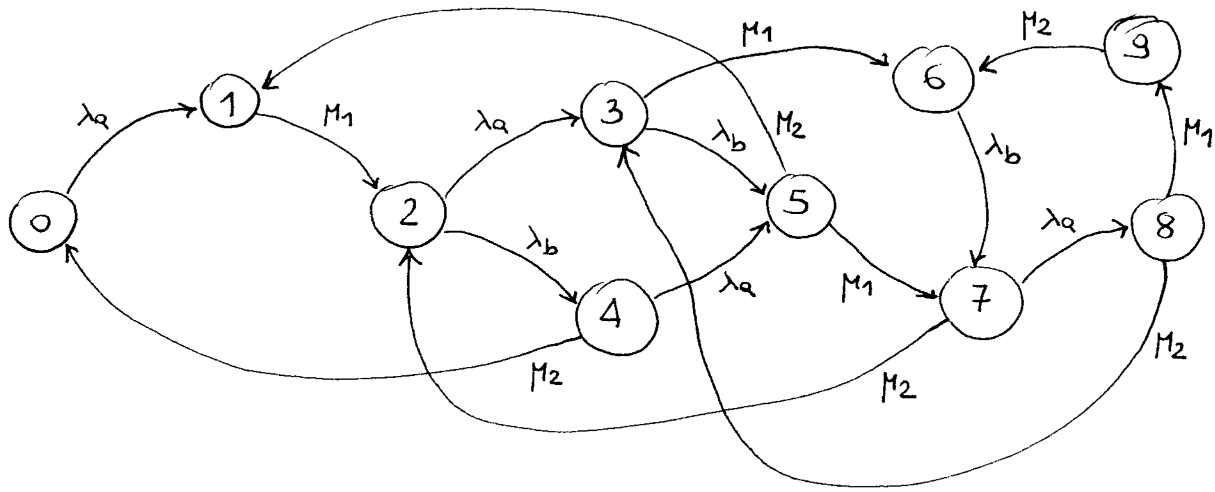
$F_{a_b}(t) = 1 - e^{-\lambda_b t}$, $t \geq 0$ where $\lambda_b = 0.8$ parts/min

$F_{d_1}(t) = 1 - e^{-\mu_1 t}, \quad t \geq 0$ where $\mu_1 = \frac{2}{3}$ parts/min

$F_{d_2}(t) = 1 - e^{-\mu_2 t}, \quad t \geq 0$ where $\mu_2 = \frac{1}{2}$ products/min




Continuous time homogeneous Markov chain → $F_{a_a}$, $F_{a_b}$, $F_{d_1}$ and $F_{d_2}$ are all exponential distributions




$$Q = \begin{bmatrix}
-\lambda_a & \lambda_a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\mu_1 & \mu_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -(\lambda_a+\lambda_b) & \lambda_a & \lambda_b & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -(\lambda_b+\mu_1) & 0 & \lambda_b & \mu_1 & 0 & 0 & 0 \\
\mu_2 & 0 & 0 & 0 & -(\lambda_a+\mu_2) & \lambda_a & 0 & 0 & 0 & 0 \\
0 & \mu_2 & 0 & 0 & 0 & -(\mu_1+\mu_2) & 0 & \mu_1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -\lambda_b & \lambda_b & 0 & 0 \\
0 & 0 & \mu_2 & 0 & 0 & 0 & 0 & -(\lambda_a+\mu_2) & \lambda_a & 0 \\
0 & 0 & 0 & \mu_2 & 0 & 0 & 0 & 0 & -(\mu_1+\mu_2) & \mu_1 \\
0 & 0 & 0 & 0 & 0 & 0 & \mu_2 & 0 & 0 & -\mu_2
\end{bmatrix}$$

2. $P(M_1 \text{ is busy at time } t=10 \mid \text{there is an arrival of type } a \text{ at time } t=10) =$

$$\underset{\substack{\downarrow \\ \text{PASTA} \\ \text{property}}}{=} \pi_1(10)+\pi_3(10)+\pi_5(10)+\pi_6(10)+\pi_8(10)+\pi_9(10) \simeq 0.2561$$

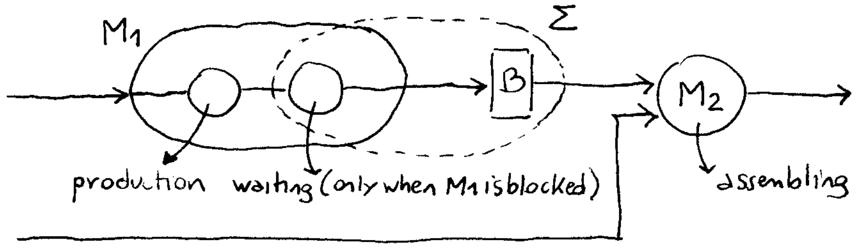
where $\pi(10) = \underbrace{[1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0]}_{\pi(0)} e^{Q\cdot 10}$

3. $\lambda_{eff} = \lambda_a(\pi_0+\pi_2+\pi_4+\pi_7)+\lambda_b(\pi_2+\pi_3+\pi_6) = \dfrac{304}{1035}$

$$\mu_{eff} = \underset{\substack{\downarrow \\ \text{one product} \\ \text{is formed by} \\ \text{two parts}}}{2} \cdot \mu_2(\pi_4+\pi_5+\pi_7+\pi_8+\pi_9) = \frac{304}{1035}$$

ok!

4. We split machine $M_1$ in two "sub-machines":




Using this trick, the answer is:

$$E[S_\Sigma] = \frac{E[X_\Sigma]}{\lambda_\Sigma} = \frac{0\cdot(\pi_0+\pi_1+\pi_4+\pi_5)+1\cdot(\pi_2+\pi_3+\pi_7+\pi_8)+2\cdot(\pi_6+\pi_9)}{\lambda_a(\pi_0+\pi_2+\pi_4+\pi_7)}$$

$$= \frac{308}{137} \simeq 2.2482 \text{ min}$$

5. $U_1 = \pi_1+\pi_3+\pi_5+\pi_8 = \dfrac{76}{345} \simeq 0.2203 \quad (22.03\%)$

$U_2 = \pi_4+\pi_5+\pi_7+\pi_8+\pi_9 = \dfrac{304}{1035} \simeq 0.2937 \quad (29.37\%)$

... where $\pi$, computed from $\begin{cases} \pi Q = 0 \\ \sum \pi_i = 1 \end{cases}$, is

$$\pi = \left[ \frac{530}{1401}\ \ \frac{787}{5450}\ \ \frac{371}{2802}\ \ \frac{61}{2738}\ \ \frac{212}{1401}\ \ \frac{241}{5847}\ \ \frac{113}{4113}\ \ \frac{153}{2200}\ \ \frac{101}{8152}\ \ \frac{101}{6114} \right].$$