First Partial Exam - Discrete Event Systems - 29.11.2012

Exercise 1

The activities of a small consulting firm are organized in projects. Given the small dimensions, the firm may carry on only one project at a time, but luckily its services are much appreciated by the customers, and therefore the firm may start a new project immediately after the end of another. Each project is organized in four tasks T_1, T_2, T_3 and T_4. Task T_1 precedes task T_2. Task T_3 starts simultaneously with task T_1, and may be performed in parallel with tasks T_1 and T_2. Task T_4 starts only when the other three tasks have been completed.

1. Define a logical model $(\mathcal{E}, \mathcal{X}, \Gamma, f, x_0)$ of the activities of the consulting firm, assuming the startup of the project (when tasks T_1 and T_3 are launched) as the initial state.

2. Assume the following durations for the four tasks, expressed in days:

	Project #1	Project #2	Project #3	Project #4
T_1	5	12	3	4
T_2	3	4	6	5
T_3	10	8	7	9
T_4	14	16	16	12

   Construct the time diagram of the system, and compute the average project duration. Verify the constructed time diagram using Matlab.

3. Assume that the durations of the four tasks follow uniform distributions in the intervals $[3, 12]$ days for T_1, $[3, 6]$ days for T_2, $[7, 10]$ days for T_3, and $[12, 16]$ days for T_4. Using Matlab, estimate the probability that T_3 terminates before T_2.

4. Assume that the durations of the four tasks follow exponential distributions with expected values 7.5 days for T_1, 4.5 days for T_2, 8.5 days for T_3, and 14 days for T_4. Compute the probability that T_3 terminates before T_2.

5. Assume exponential distributions as in point 4. Show, also with the help of Matlab, if $\lim_{k\to\infty} P(X_k = x)$ exists for all $x \in \mathcal{X}$, and try to explain the results.

Exercise 2

Active Demand (AD) is a new concept in smart electrical grids. Upon receiving suitable price-volume signals, customers may decide to adjust their consumption patterns in change of a monetary reward. In a very simplified setting, consider a customer with only two controllable appliances. The customer may receive two different AD signals. Each AD signal specifies the maximum number of appliances that can be let operate during the period of validity of the signal: none under signal ad_0, and one under signal ad_1. When AD signal ad_0 arrives, any operating appliance is interrupted. If AD signal ad_1 arrives when only one appliance is operating, the appliance is not interrupted. If AD signal ad_1 arrives when both appliances are operating, one appliance is interrupted. An interrupted appliance is restarted as soon as possible. It is assumed that AD signals may not overlap: no other AD signal may arrive while another AD signal is active.

Assume that arrivals of AD signals ad_0 and ad_1 have exponentially distributed lifetimes with rates $\lambda_0 = 0.4$ and $\lambda_1 = 1.5$ arrivals/day, respectively. Periods of validity of the AD signals are exponentially distributed with expected value 2 hours. Lifetimes of the event of start of an appliance are exponentially distributed with expected value 8 hours. Durations of appliance programs follow an exponential distribution with rate $\mu = 0.5$ programs/hour.

1. Model the AD mechanism through a stochastic state automaton $(\mathcal{E}, \mathcal{X}, \Gamma, p, x_0, F)$, assuming that no appliance is operating and no AD signal is active at time $t_0 = 0$.

2. Assume that both appliances are operating. Compute the probability that both appliances complete operation after at most three events.

3. Assume that both appliances are operating. Compute the probability that, in two hours, no AD signal arrives and both appliances complete operation.

4. Compute the expected value of the sojourn time in a state where one appliance is interrupted and the other is operating.

Instructions about Matlab files

At the end of the exam, please copy into the folder `\\Sunto\Esami\Discrete Event Systems` the following files:

- script `firstname_lastname_ex1_p2.m` relative to point 2 of Exercise 1;
- script `firstname_lastname_ex1_p3.m` relative to point 3 of Exercise 1;
- script `firstname_lastname_ex1_p5.m` relative to point 5 of Exercise 1.

Warning! During the exam, save the files in your local folder, and only at the end copy them in the above specified folder. The system prevents from overwriting.

Exercise 1

1. state space: $\mathcal{X} = \left\{ \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ - \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ - \end{bmatrix}, \begin{bmatrix} - \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \end{bmatrix} \right\}$

$\begin{bmatrix} 1 \\ 3 \end{bmatrix}$: task 1 and task 3 are being performed

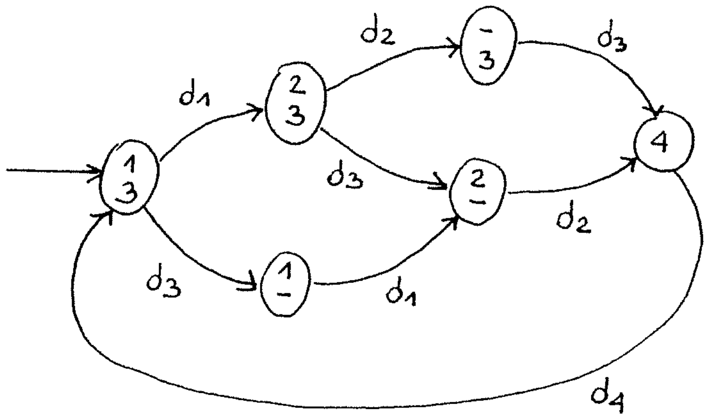
$\begin{bmatrix} 1 \\ - \end{bmatrix}$: task 1 is being performed, task 3 has been terminated

. . .

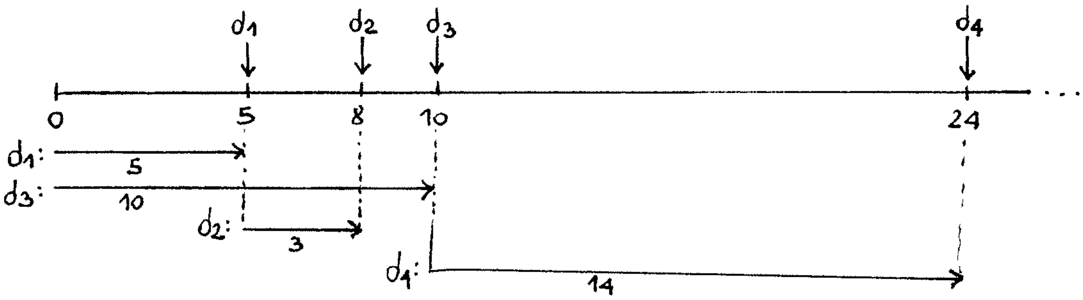
$\begin{bmatrix} - \\ 3 \end{bmatrix}$: task 1 and task 2 have been performed task 3 is being performed

$\begin{bmatrix} 4 \end{bmatrix}$: task 4 is being performed

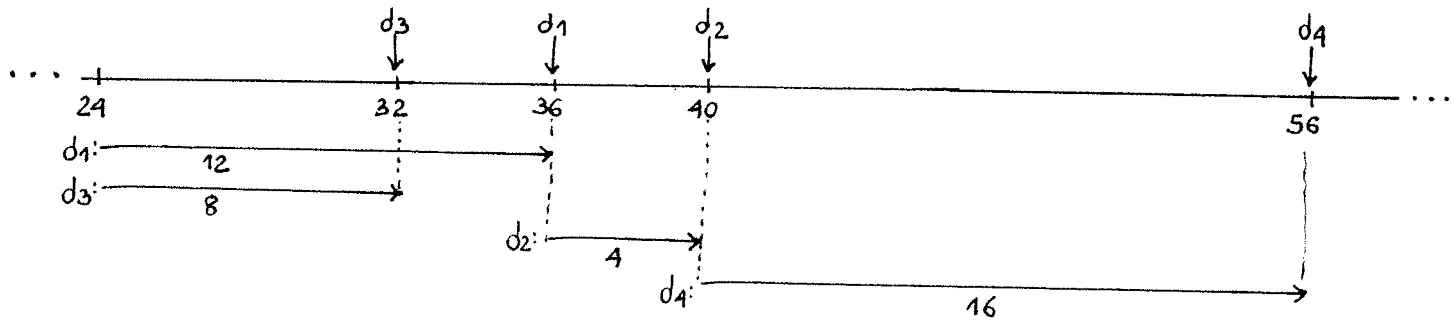
events: $\mathcal{E} = \{ d_1, d_2, d_3, d_4 \}$, where d_i = end of task i, $i = 1,2,3,4$



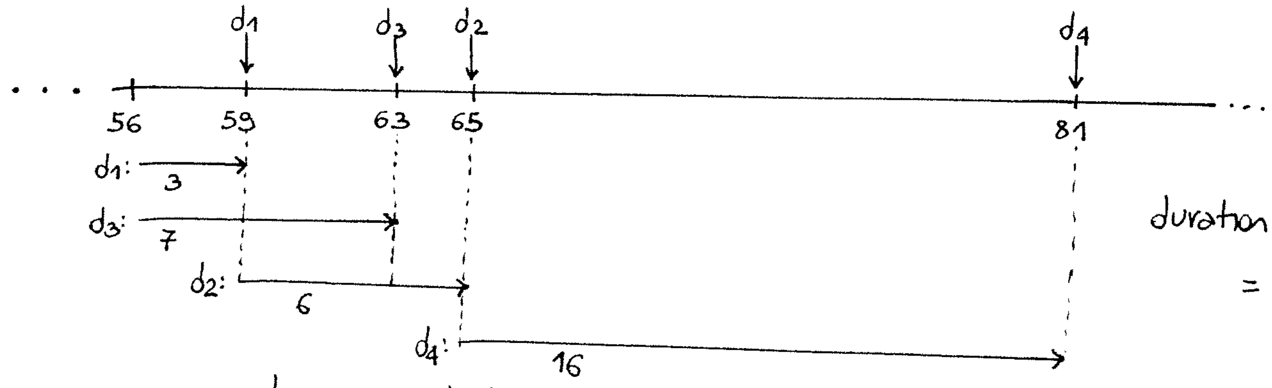
2. We construct the sample path:



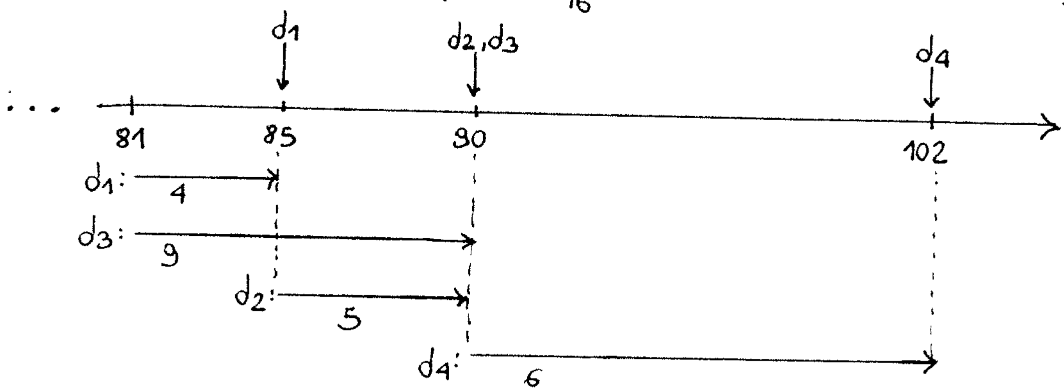
duration of project #1 = 24



duration of project #2 = 32



duration of project #.
= 25



duration of project #4
= 21

$\Rightarrow$ average project duration $= \dfrac{24+32+25+21}{4} = 25.5$ days

3. It corresponds to the probability $P\left(X_2=\left[\begin{smallmatrix}2\\-\end{smallmatrix}\right] \,\middle|\, X_0=\left[\begin{smallmatrix}1\\3\end{smallmatrix}\right]\right)$.
Using Matlab (10000 simulations): $\hat{P}(\ldots) \simeq 0.87200$.

4. $\mu_1 = \frac{1}{7.5}$ tasks/day, $\mu_2 = \frac{1}{4.5}$ tasks/day, $\mu_3 = \frac{1}{8.5}$ tasks/day, $\mu_4 = \frac{1}{14}$ tasks/day

$$P\left(X_2=\left[\begin{smallmatrix}2\\-\end{smallmatrix}\right] \,\middle|\, X_0=\left[\begin{smallmatrix}1\\3\end{smallmatrix}\right]\right) = \frac{\mu_1}{\mu_1+\mu_3}\cdot\frac{\mu_3}{\mu_2+\mu_3} + \frac{\mu_3}{\mu_1+\mu_3}\cdot 1 = \frac{543}{832} \simeq 0.6526$$

5. $\lim_{k\to\infty} P(X_k=x)$ does not exist for any $x \in \mathcal{X}$, since $P(X_k=x)$ is periodic with period 4 (every project is "composed" by 4 events).

Exercise 2

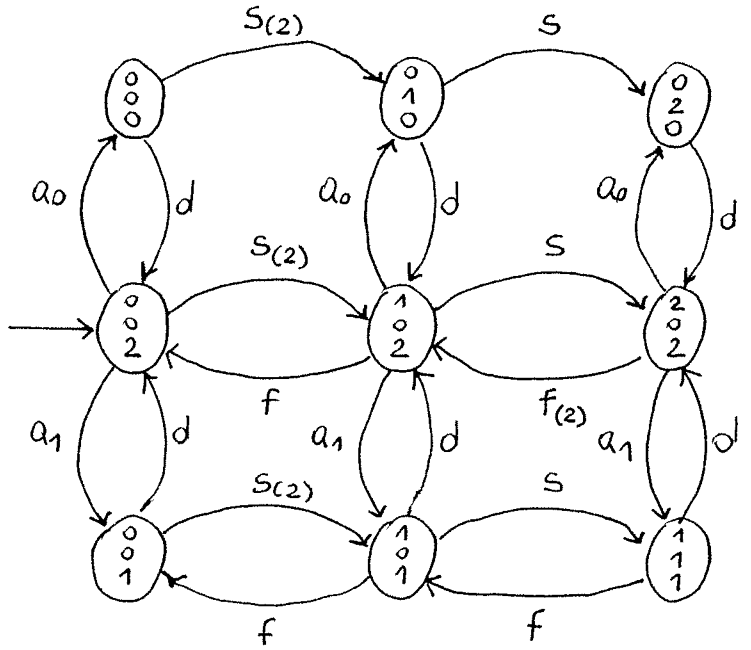
1. Definition of state:

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \begin{array}{l} \to \#\text{active appliances} \in \{0,1,2\} \\ \to \#\text{interrupted appliances} \in \{0,1,2\} \\ \to \#\text{max appliances} \in \{0,1,2\} \end{array}$$

Events:

$\mathcal{E} = \{a_0, a_1, d, s, f\}$

- a_0: arrival of ad_0
- a_1: arrival of ad_1
- d: termination of ad
- s: start of an appliance
- f: end of an appliance program



$a_0 \Rightarrow \lambda_0 = 0.4$ arrivals/day

$a_1 \Rightarrow \lambda_1 = 1.5$ arrivals/day

$d \Rightarrow \delta = 12$ services/day

$s \Rightarrow \gamma = 3$ starts/day

$f \Rightarrow \mu = 12$ programs/day

2. $X_k = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}$

$$P(\ldots) = \frac{2\mu}{\lambda_0+\lambda_1+2\mu}\,\frac{\mu}{\lambda_0+\lambda_1+\gamma+\mu} + \frac{2\mu}{\lambda_0+\lambda_1+2\mu}\,\frac{\lambda_1}{\lambda_0+\lambda_1+\gamma+\mu}\,\frac{\mu}{\delta+\gamma+\mu} +$$

$$+ \frac{2\mu}{\lambda_0+\lambda_1+2\mu}\,\frac{\gamma}{\lambda_0+\lambda_1+\gamma+\mu}\,\frac{\mu}{\lambda_0+\lambda_1+2\mu} + \frac{\lambda_1}{\lambda_0+\lambda_1+2\mu}\,\frac{\mu}{\delta+\mu}\,\frac{\mu}{\delta+\mu+\gamma}$$

$$= \frac{239}{305} \simeq 0.7836$$

3. $X_k = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}$

$$P(\ldots) = P(Y_{a_0} > T)\, P(Y_{a_1} > T) \left[P(Y_f \leq T) \right]^2 \quad \text{where } T = 2 \text{ hours} = \frac{1}{12} \text{ days}$$

$$= e^{-\lambda_0 T} \cdot e^{-\lambda_1 T} \cdot \left(1 - e^{-\mu T}\right)^2 \simeq 0.3411$$

Remark: here Y_{a_0} represents the residual lifetime of event a_0,
Y_{a_1} represents the residual lifetime of event a_1,
and Y_f represents the generic residual lifetime of an event f.

4. $X_k = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$$E\left[V(X_k)\right] = \frac{1}{\delta + \mu} = \frac{1}{24} \text{ days} = 1 \text{ hour}$$