# A comparison between classical robust stability conditions*

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**Abstract**

The paper is focused on the comparison between some classical robust stability conditions for continuous-time linear time-invariant systems. Such conditions are given as lemmas in the paper, since their statements include some generalizations, needed in order to better compare (and use) them. The analysis is carried out by comparing pairwise the families of systems whose stabilization through a given compensator is guaranteed by each of the considered robust stability conditions. Some properties of the families are formally derived and stated, and some very simple examples are exhibited in order to illustrate the presented properties and comparisons. A theorem, representing an extension of one of the considered conditions, is proposed in order to overcome some difficulties arising in the use of the classical result.

*Keywords:* robust stabilization, linear time-invariant systems, unstructured perturbations, small-gain theorem, strengthened stability.

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# 1 Introduction

Stability robustness in spite of system uncertainties is one of the main purposes in control system design. Since the actual system to be stabilized is not exactly known, a common approach to the problem is to assume it belongs to a set of systems defining the so-called *uncertainty set* (usually expressed in terms of a nominal system and a description of the uncertainty around it), and to design a compensator guaranteeing that stability is preserved for all the systems in this set. To this aim, it can be helpful to resort to several robust stability conditions that are available in the literature (see, e.g., [2]-[12] and the references therein), exploiting different representations of the mismatch between the nominal and the actual system. Notice that, when the covering of the possible systems through the uncertainty set is too large (e.g. because it either neglects the structure of the uncertainty, or takes into account perturbations in the system parameters that rarely or never occur), this approach leads to a conservative design. In the last decades great effort has therefore been made towards less conservative analysis and synthesis methods ($\mu$-analysis and synthesis are a perfect example of this [9, 13]).

In this paper we will focus on the robust stability conditions derived by Doyle and Stein [14] and Chen and Desoer [15], based on a generalized version of the Nyquist criterion, and on some robust stability conditions based on the small-gain theorem [9, 16]. In both these types of conditions (which will be recalled in Section 2), uncertainty affecting the controlled system is represented in the form of matrix perturbations, either additive or multiplicative, relating its actual transfer matrix $P(s)$ to the nominal one, namely $P_0(s)$. Recall that such common representations are quite general, within the hypothesis of linear behaviour of the plant, since they allow to take into account both any kind of inaccuracy that may occur in modelling or identifying the system, and uncertainty in the system parameters.

The purpose of the paper is a comparison between the families of (possibly multivariable) systems whose stabilization is guaranteed by pairs of the mentioned robust stability conditions, for the same choice of the feedback compensator stabilizing the nominal system. After introducing some notations and preliminaries in Section 2, a detailed comparison will be carried out in Section 3 between the families of systems whose stabilization through the given compensator is guaranteed by those of the mentioned conditions referring to multiplicative perturbations of $P_0(s)$. First, a formal proof will be given of the invariance - in locations and algebraic multiplicities - of the unstable eigenvalues of the perturbed systems whose stabilization is guaranteed by the condition based on the small-gain theorem. This seems to be a stronger property than the one stated through informal arguments in References [17, 18], where multiplicative (or additive) perturbations free of unstable poles were considered. On the other hand, for the perturbed systems whose stabilization through the given compensator is guaranteed by the stability condition in References [14, 15] (not requiring the stability of the multiplicative perturbation), a formal proof will be given of the invariance - in locations and algebraic multiplicities - of the only eigenvalues lying on the imaginary axis. It will be also recalled, and shown through an example, that the other unstable eigenvalues can instead move [14]. These formal properties of the two families constitute the first contribution of the paper. They imply in particular that the former family contains only perturbed systems having just the same unstable eigenvalues - in locations and algebraic multiplicities - as the nominal one. This, and other arguments arising from the discussion in Section 3, will suggest that, if the nominal system has unstable eigenvalues or transmission zeros, the former family is less (and possibly much less) significant than the latter. Such an analysis constitutes another contribution of the paper.

For the sake of completeness, a similar comparison will be sketched in Section 4 also for the case of additive perturbations, although some of the related reasonings are straightforward in this case. In the same section, the families of perturbed systems whose stabilization through the given compensator is guaranteed by the two different robust stability conditions in References [14, 15] (the one referred to multiplicative perturbations, the other to additive perturbations) will be compared in some significant cases. This latter comparison, whose results depend on the properties of the (possibly multivariable) nominal controlled system, is a further contribution of the paper, and can be helpful in order to choose which type of perturbations (either additive or multiplicative) is more advantageous to use in each particular situation.

The last contribution of the paper is represented by Theorem 1, which is an extension of the classical robust stability condition recalled in Lemma 3. It is proposed in Section 4 in order to allow also additive perturbations having poles on the imaginary axis, which is a forbidden situation in the classical statement.

The considered robust stability conditions are often referred to the mere asymptotic stability of the closed-loop control system. However, in the following, after introducing proper notations, a possibly stronger requirement will be considered, i.e. it will be required the eigenvalues of the closed-loop control system to lie in the half-plane $\mathrm{Re}(s) < -\alpha$, for a given $\alpha \geq 0$. As it will be discussed in Section 3, a suitable choice of $\alpha > 0$ may be useful not only in order to obtain a faster convergence to zero of the free responses of the closed-loop control system, but also in order to overcome some difficulties arising in the classical use of the robust stability conditions in References [14, 15] with $\alpha=0$. Notice that the aforementioned comparisons (and the related conclusions) still hold in the case $\alpha>0$ by replacing the imaginary axis and the right half-plane with the axis $\mathrm{Re}(s)=-\alpha$ and the half-plane $\mathrm{Re}(s)>-\alpha$, respectively. They will be carried out in the paper for the general case $\alpha\geq 0$.

Very simple examples will be used to illustrate the presented results.

## 2 Notations and preliminaries

In this paper we consider finite dimensional continuous-time linear time-invariant dynamical systems described by equations of the type

$$\begin{aligned} \dot{x}(t) &= A\,x(t) + B\,u(t) \\ y(t) &= C\,x(t) + D\,u(t) \end{aligned} \tag{1}$$

where $t\in\mathbb{R}$ is time, $x(t)\in\mathbb{R}^n$ is the state, $u(t)\in\mathbb{R}^p$ is the input, $y(t)\in\mathbb{R}^q$ is the output, and $A$, $B$, $C$ and $D$ are real constant matrices with suitable dimensions. Let a system $\mathcal{P}$ be described by equations (1). For the sake of brevity, the eigenvalues of the state matrix $A$ of $\mathcal{P}$ will be also called eigenvalues of $\mathcal{P}$. Moreover, $P(s)$ will denote the input/output transfer matrix of $\mathcal{P}$, i.e. $P(s)=C(sI-A)^{-1}B+D$.

In the next sections we will focus on the closed-loop system $\Sigma$ shown in Figure 1, where $\mathcal{P}$ stands for the controlled system, and $\mathcal{K}$ is a given compensator. The state space description of $\mathcal{K}$ is wholly similar to (1) by means of matrices $A_K$, $B_K$, $C_K$ and $D_K$, and its transfer matrix will be

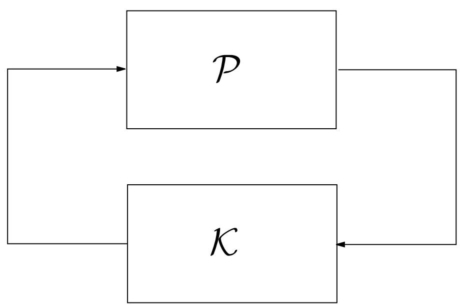


Figure 1: The closed-loop system $\Sigma$.

denoted by $K(s)$. Denoting by $A_\Sigma$ the state matrix of $\Sigma$, it is recalled that, if $\Sigma$ is well-posed, i.e. $\det(I{-}DD_K){\neq}0$, then the following useful relation holds [19]:

$$\det(sI - A_\Sigma) = \frac{\det(sI - A)\det(sI - A_K)\det\big(I - P(s)K(s)\big)}{\det(I - DD_K)} \tag{2}$$

Asymptotic stability is an obvious requirement for $\Sigma$, but possibly too weak, if a prescribed rate of exponential decay of the closed-loop free responses is to be guaranteed. Therefore, for a fixed $\alpha{\geq}0$, define

$$\mathbb{C}_g \triangleq \{s \in \mathbb{C} : \mathrm{Re}(s) < -\alpha\}$$

If all the eigenvalues of $\Sigma$ belong to $\mathbb{C}_g$, $\Sigma$ will be said to be $\mathbb{C}_g$-stable. In addition, system $\mathcal{P}$ will be said to be $\mathbb{C}_g$-stabilizable (or $\mathbb{C}_g$-detectable) if there exists $F{\in}\mathbb{R}^{p\times n}$ (or $V{\in}\mathbb{R}^{n\times q}$) such that all the eigenvalues of $A{+}BF$ (or $A{+}VC$) belong to $\mathbb{C}_g$. It is clear that, if $\alpha{>}0$, $\mathbb{C}_g$-stability is a stronger requirement than the mere asymptotic stability (obtained for $\alpha{=}0$), so it will be referred to also as *strengthened stability*. Further reasons for using $\alpha{>}0$ will be discussed in the subsequent Section 3.

**Remark 1** We recall that the poles of $P(s)$ are the eigenvalues of the reachable and observable part of system $\mathcal{P}$, so that, in general, they are only a subset of the eigenvalues of $\mathcal{P}$.

If system $\mathcal{P}$ is $\mathbb{C}_g$-stabilizable and $\mathbb{C}_g$-detectable, then any eigenvalue $\lambda$ of $\mathcal{P}$ outside $\mathbb{C}_g$ is also a pole of $P(s)$ - so that, under the mentioned hypotheses, these two terms will be used equivalently - and the algebraic multiplicity of $\lambda$ (i.e. the multiplicity of $\lambda$ in the characteristic polynomial of $A$) equals its multiplicity as a pole of $P(s)$, counted according to the McMillan degree of $P(s)$ (i.e. it equals the multiplicity of $\lambda$ in the least common denominator of all non zero minors of $P(s)$ [20]). □

Since the values of $n$ and of matrices $A$, $B$, $C$ and $D$ are uncertain for the actual controlled system, its available description $\mathcal{P}_0$ of dimension $n_0$, characterized by matrices $A_0$, $B_0$, $C_0$ and $D_0$ in equations wholly similar to (1), will be called the nominal system, whose transfer matrix will be denoted by $P_0(s)$. Then, the nominal closed-loop system, obtained by replacing $\mathcal{P}$ with $\mathcal{P}_0$ in Figure 1, will be called $\Sigma_0$, and the state matrix of $\Sigma_0$ will be denoted by $A_{\Sigma_0}$.

The topics of this paper are well-known sufficient conditions, to be checked on the nominal closed-loop system $\Sigma_0$, guaranteeing that the property of $\mathbb{C}_g$-stability, which is assumed for $\Sigma_0$, is preserved by the closed-loop system

$\Sigma$ for all the perturbed systems $\mathcal{P}$ ranging in different uncertainty sets, but primarily described in terms of matrix perturbations, either additive or multiplicative, relating the transfer matrix $P(s)$ of $\mathcal{P}$ to the nominal one $P_0(s)$. Recall that the representation of the uncertainties in the form of *additive perturbations* [15, 21], i.e.

$$P(s) = P_0(s) + \delta_P(s) \tag{3}$$

is always possible and, within the hypothesis that the actual system behaves linearly, allows to take into account any error in identification and all inaccuracies associated with modelling (including parameter changes and neglected dynamics, thus allowing the order $n$ of $\mathcal{P}$ to be different and even much larger than the order $n_0$ of $\mathcal{P}_0$), whereas the representation of the uncertainties in the form of *output multiplicative perturbations* [14, 15, 21] requires the existence for $P(s)$ of a $q \times q$ rational matrix $\delta^P(s)$ such that[1]

$$P(s) = \big(I + \delta^P(s)\big)P_0(s) \tag{4}$$

or, equivalently, such that

$$\delta_P(s) = \delta^P(s)P_0(s) \tag{5}$$

In this latter case, denoting by $T_0(s) \triangleq P_0(s)K(s)\big(I - P_0(s)K(s)\big)^{-1}$ the so-called output complementary sensitivity matrix of the nominal closed-loop system $\Sigma_0$ (under the assumption that $\Sigma_0$ is well-posed), the following useful identity

$$I - P(s)K(s) = \big(I - \delta^P(s)T_0(s)\big)\big(I - P_0(s)K(s)\big) \tag{6}$$

can be derived from

$$\begin{aligned} I - P(s)K(s) &= I - P_0(s)K(s) - \delta_P(s)K(s) \\ &= (I - \delta_P(s)K(s)\big(I - P_0(s)K(s)\big)^{-1})(I - P_0(s)K(s)) \end{aligned}$$

in view of the hypothesis (5) and the definition of $T_0(s)$. Notice that the existence of $\delta^P(s)$ such that (4) holds, is guaranteed for all $P(s)$ if $P_0(s)$ has full column rank in the rational field.

**Remark 2** It is stressed that equation (4) only relates algebraically $P(s)$ to $P_0(s)$, so that system $\mathcal{P}$, whose transfer matrix is $P(s)$ (and whose stabilization through the given compensator $\mathcal{K}$ is possibly guaranteed by the

[1] It is stressed that the symbol $\delta^P(s)$ has a different meaning from $\delta_P(s)$.

subsequent Lemma 1, or Lemma 2), may actually not be the series connection of $\mathcal{P}_0$ and a realization of $I+\delta^P(s)$, as (4) could instead suggest. This implies that unstable cancellations between $I+\delta^P(s)$ and $P_0(s)$ are allowed in (4), in principle, since they do not necessarily imply any loss of stabilizability or detectability in $\mathcal{P}$ with respect to $\mathcal{P}_0$ (see, e.g., the subsequent Example 1). □

Let $\bar{\sigma}[\cdot]$ and $\underline{\sigma}[\cdot]$ denote the largest and the smallest singular value of the argument matrix, respectively, and let $\mathbb{R}^+$ denote the set of non-negative real numbers. The following lemma gives conditions for the robust $\mathbb{C}_g$-stability of system $\Sigma$ [14, 15], and can be proven by means of a suitable extension of the Nyquist criterion, in which the role usually played by the imaginary axis is taken by the boundary of $\mathbb{C}_g$ (for this reason, in the following, we will call *Nyquist contour* the boundary of $\mathbb{C}_g$).

**Lemma 1** *If, for a fixed* $\alpha\geq 0$ *characterizing* $\mathbb{C}_g$*,*

*(i)* $\Sigma_0$ *is well-posed and* $\mathbb{C}_g$*-stable;*

*(ii)* $\bar{\sigma}\left[T_0(-\alpha+j\omega)\right]\leq\dfrac{1}{l_m(\omega)}$, $\forall\omega\in\mathbb{R}^+$,

*where* $l_m(\omega)$ *is a positive and continuous function of* $\omega\in\mathbb{R}^+$*, then system* $\Sigma$ *is* $\mathbb{C}_g$*-stable for all the perturbed systems* $\mathcal{P}$ *such that:*

*(a) systems* $\mathcal{P}$ *and* $\mathcal{P}_0$ *have the same number of eigenvalues outside* $\mathbb{C}_g$ *(including algebraic multiplicities);*

*(b)* $\Sigma$ *is well-posed;*

*(c)* $P(s)$ *can be expressed in the form (4) (i.e. there exists* $\delta^P(s)$ *such that (4) holds), with* $\delta^P(s)$ *satisfying the inequality* $\bar{\sigma}\left[\delta^P(-\alpha+j\omega)\right]<l_m(\omega)$, $\forall\omega\in\mathbb{R}^+$.

In view of Remark 2, in the following the family of perturbed systems $\mathcal{P}$ satisfying the conditions (a), (b) and (c) of Lemma 1 will be denoted by $\mathcal{M}_{bu}(\alpha, l_m)$. The subscript recalls that the transfer matrices $P(s)$ of all the perturbed systems in this family could be viewed as belonging to a (normalized) 'ball of uncertainty' around the nominal transfer matrix $P_0(s)$ [15].

For a fixed $\sigma\in\mathbb{R}$, let $\mathcal{RL}_{\sigma,\infty}$ be the set of all proper rational matrices with real coefficients and free of poles on the $\sigma$+j$\omega$-axis. Then, for all $G(s)\in\mathcal{RL}_{\sigma,\infty}$, the following quantity is well-defined:

$$\|G(s)\|_{\sigma,\infty} \triangleq \sup_{\omega\in\mathbb{R}} \bar{\sigma}\left[G(\sigma+\mathrm{j}\omega)\right]$$

which is a straightforward extension of the usual $\mathcal{L}_\infty$ norm. The subset of $\mathcal{RL}_{\sigma,\infty}$ consisting of all proper rational matrices with real coefficients and free of poles with real part greater than or equal to $\sigma$ will be denoted by $\mathcal{RH}_{\sigma,\infty}$. For $\sigma$=0 the usual set $\mathcal{RH}_\infty$ is obtained. Also the following lemma gives conditions for the robust $\mathbb{C}_g$-stability of system $\Sigma$. It generalizes well-known results (see, e.g., References [9, 16]), and can be proven by means of a simple extension of the small-gain theorem (see Reference [12] for a suitable version of it in the setting of this paper).

**Lemma 2** *If, for a fixed $\alpha \geq 0$ characterizing $\mathbb{C}_g$,*

*(i)* $\Sigma_0$ *is well-posed and $\mathbb{C}_g$-stable;*

*(ii)* $\|W_2(s)T_0(s)W_1(s)\|_{-\alpha,\infty} \leq \rho^{-1}$,

*where $\rho>0$ and $W_1(s)$, $W_2(s) \in \mathcal{RH}_{-\alpha,\infty}$, then system $\Sigma$ is well-posed and $\mathbb{C}_g$-stable for all the perturbed systems $\mathcal{P}$ such that:*

*(a) system $\mathcal{P}$ is $\mathbb{C}_g$-stabilizable and $\mathbb{C}_g$-detectable;*

*(b) $P(s)$ can be expressed in the form (4) (i.e. there exists $\delta^P(s)$ such that (4) holds), with $\delta^P(s)$ satisfying the relation $\delta^P(s)=W_1(s)\Delta(s)W_2(s)$, where $\Delta(s) \in \mathcal{RH}_{-\alpha,\infty}$ is such that $\|\Delta(s)\|_{-\alpha,\infty} < \rho$.*

In view of Remark 2, in the following the family of perturbed systems $\mathcal{P}$ satisfying the conditions (a) and (b) of Lemma 2, for fixed weighting matrices $W_1(s)$ and $W_2(s)$ characterizing the spatial and frequency structure of the uncertainty, will be denoted by $\mathcal{M}_{sg}(\alpha, \rho)$. The subscript refers to the fact that Lemma 2 can be proven by means of (a suitable extension of) the small-gain theorem, thus recalling that the multiplicative perturbations, relating the transfer matrices $P(s)$ of all the perturbed systems in this family to the nominal one $P_0(s)$, belong to $\mathcal{RH}_{-\alpha,\infty}$.

**Remark 3** The strict inequality in condition (c) of Lemma 1 (or (b) of Lemma2) can be relaxed to $\leq$ if the inequality in the corresponding hypothesis (ii) is strengthened to $<$. Notice that both Lemmas 1 and 2 (as well as subsequent Lemmas 3 and 4) can be effectively used not only to check *a posteriori* the robust $\mathbb{C}_g$-stability of the closed-loop system $\Sigma$ in Figure 1, but also to design a compensator $\mathcal{K}$ satisfying *a priori* robustness requirements (see, e.g., References [9, 10, 14, 16]). □

**Remark 4** When the uncertainty affecting the description of the controlled system is characterized in the form of *input multiplicative perturbation*, i.e. in the form

$$P(s) = P_0(s)\big(I + \delta^P(s)\big) \tag{7}$$

where $\delta^P(s)$ is a $p{\times}p$ rational matrix (whose existence is guaranteed for all $P(s)$ by $P_0(s)$ having full row rank in the rational field), both Lemmas 1 and 2 (as well as the discussion to follow) still hold by replacing (4) with (7), and $T_0(s)$ with the so-called input complementary sensitivity matrix $R_0(s) \triangleq K(s)P_0(s)\big(I-K(s)P_0(s)\big)^{-1}$ of $\Sigma_0$. □

# 3 Families of systems that are robustly stabilized under multiplicative perturbations

In this section, the families $\mathcal{M}_{bu}(\alpha, l_m)$ and $\mathcal{M}_{sg}(\alpha, \rho)$ defined in the previous section will be considered in detail. First, notice that the hypotheses and the conditions of Lemma 1, guaranteeing the $\mathbb{C}_g$-stability of $\Sigma$ in addition to the one of $\Sigma_0$, also guarantee the $\mathbb{C}_g$-stabilizability and the $\mathbb{C}_g$-detectability of systems $\mathcal{P}$ and $\mathcal{P}_0$, as well as the hypotheses and the conditions of Lemma 2 do. Thus, the implication stressed by Remark 1 follows about the use of the terms 'eigenvalue of $\mathcal{P}$ outside $\mathbb{C}_g$' and 'pole of $P(s)$ outside $\mathbb{C}_g$'.

Under the hypotheses and the conditions of Lemma 1, it is obvious that $P(s)$ cannot have poles on the $-\alpha$+j$\omega$-axis different from those of $P_0(s)$, since such poles should be introduced by the factor $I+\delta^P(s)$, in contradiction with condition (c). On the other hand, still under the hypotheses of Lemma 1, and if $P_0(s)$ has poles on the $-\alpha$+j$\omega$-axis, one could ask whether there exist perturbed systems $\mathcal{P}$ in the family $\mathcal{M}_{bu}(\alpha, l_m)$ such that these poles disappear in $P(s)$ or reduce their algebraic multiplicities (this might be compatible with conditions (a) and (c) of Lemma 1 if such poles of $P_0(s)$ are shifted to poles of $P(s)$ in the right-hand side of the $-\alpha$+j$\omega$-axis). The answer is negative, as it is stated by the following proposition.

**Proposition 1** *Under hypotheses (i) and (ii) of Lemma 1, the nominal system $\mathcal{P}_0$ and all the perturbed systems $\mathcal{P}$ in the family $\mathcal{M}_{bu}(\alpha, l_m)$ have the same eigenvalues on the $-\alpha$+j$\omega$-axis, with the same algebraic multiplicities.*

***Proof.*** From (i) it follows that $T_0(s)$ has no poles on the $-\alpha$+j$\omega$-axis, whereas from (c) it follows that $\delta^P(s)$ has no poles on the $-\alpha$+j$\omega$-axis. Hence

$\det(I-\delta^P(s)T_0(s))$ has no poles on the $-\alpha$+j$\omega$-axis, either. Moreover,

$$\begin{aligned}\bar{\sigma}\left[\delta^P(-\alpha+\mathrm{j}\omega)T_0(-\alpha+\mathrm{j}\omega)\right] &\leq \bar{\sigma}\left[\delta^P(-\alpha+\mathrm{j}\omega)\right]\bar{\sigma}\left[T_0(-\alpha+\mathrm{j}\omega)\right] \\ &< l_m(\omega)\frac{1}{l_m(\omega)} = 1, \quad \forall\omega\in\mathbb{R}^+\end{aligned}$$

Hence, $\underline{\sigma}\left[I-\delta^P(-\alpha+\mathrm{j}\omega)T_0(-\alpha+\mathrm{j}\omega)\right]>0,\ \forall\omega\in\mathbb{R}^+$, and therefore the rational function $\det(I-\delta^P(s)T_0(s))$ has not even zeros on the $-\alpha$+j$\omega$-axis. On the other hand, identities (2) and (6) (the former written both for the nominal and the perturbed system) yield

$$\frac{\det(I-DD_K)}{\det(I-D_0D_K)}\frac{\det(sI-A_\Sigma)\det(sI-A_0)}{\det(sI-A)\det(sI-A_{\Sigma_0})} = \det(I-\delta^P(s)T_0(s)) \qquad (8)$$

Since $\Sigma_0$ and $\Sigma$ are $\mathbb{C}_g$-stable, and hence free of eigenvalues on the $-\alpha$+j$\omega$-axis, the proven lack of zeros and poles of $\det(I-\delta^P(s)T_0(s))$ on the $-\alpha$+j$\omega$-axis implies that all the eigenvalues of $A_0$ on the $-\alpha$+j$\omega$-axis are also eigenvalues of $A$ with the same algebraic multiplicities (and *vice versa*). □

Similarly, under the hypotheses and the conditions of Lemma 2, it is obvious that $P(s)$ cannot have poles outside $\mathbb{C}_g$ different from those of $P_0(s)$, since such poles should be introduced by the factor $I+\delta^P(s)$, in contradiction with condition (b), implying $\delta^P(s)\in\mathcal{RH}_{-\alpha,\infty}$. On the other hand, still under the hypotheses of Lemma 2, and if $P_0(s)$ has poles outside $\mathbb{C}_g$, one could ask whether there exist perturbed systems $\mathcal{P}$ in the family $\mathcal{M}_{sg}(\alpha,\rho)$ such that these poles disappear in $P(s)$ or reduce their algebraic multiplicities (this might be compatible with the conditions of Lemma 2 if such poles of $P_0(s)$ are, e.g., shifted to poles of $P(s)$ in the left-hand side of the $-\alpha$+j$\omega$-axis, so as to violate condition (a) of Lemma 1). Also in this case the answer is negative, as it is stated by the following proposition.

**Proposition 2** *Under hypotheses (i) and (ii) of Lemma 2, the nominal system $\mathcal{P}_0$ and all the perturbed systems $\mathcal{P}$ in the family $\mathcal{M}_{sg}(\alpha,\rho)$ have the same eigenvalues outside $\mathbb{C}_g$, with the same algebraic multiplicities.*

***Proof.*** For any $\Delta(s)$ characterizing a system $\mathcal{P}\in\mathcal{M}_{sg}(\alpha,\rho)$ according to condition (b) of Lemma 2, let systems $\mathcal{M}$ and $\mathcal{D}$ be minimal realizations of $M(s)=W_2(s)T_0(s)W_1(s)$ and $\Delta(s)$, respectively, and consider the closed-loop system $\hat{\Sigma}$ obtained by replacing $\mathcal{P}$ and $\mathcal{K}$ with $\mathcal{M}$ and $\mathcal{D}$ in Figure 1. Since $M(s)$ and $\Delta(s)$ belong to $\mathcal{RH}_{-\alpha,\infty}$ (so that $\mathcal{M}$ and $\mathcal{D}$ are $\mathbb{C}_g$-stable), $\|M(s)\|_{-\alpha,\infty}\leq\rho^{-1}$ and $\|\Delta(s)\|_{-\alpha,\infty}<\rho$, then a suitable extension of the small-gain theorem implies that $\hat{\Sigma}$ is well-posed and $\mathbb{C}_g$-stable. Therefore, by rewriting equation (2) with $\Sigma$, $\mathcal{P}$ and $\mathcal{K}$ replaced by $\hat{\Sigma}$, $\mathcal{M}$ and $\mathcal{D}$, respectively, and noticing that all the eigenvalues of $\hat{\Sigma}$, $\mathcal{M}$ and $\mathcal{D}$ belong to $\mathbb{C}_g$ in view of their $\mathbb{C}_g$-stability, it follows that $\det(I-\Delta(s)M(s))$ has neither zeros nor poles outside $\mathbb{C}_g$. On the other hand, identities (2) and (6) (the former written both for the nominal and the perturbed system) yield relation (8), where

$$\begin{aligned} \det(I - \delta^P(s)T_0(s)) &= \det(I - W_1(s)\Delta(s)W_2(s)T_0(s)) \\ &= \det(I - \Delta(s)W_2(s)T_0(s)W_1(s)) \end{aligned}$$

Since $\Sigma$ and $\Sigma_0$ are $\mathbb{C}_g$-stable, the proven lack of zeros and poles of the rational function $\det(I-\Delta(s)M(s))$ outside $\mathbb{C}_g$ implies that all the eigenvalues of $A_0$ outside $\mathbb{C}_g$ are also eigenvalues of $A$ with the same algebraic multiplicities (and *vice versa*). □

## 3.1 Comparison between families of perturbed systems whose robust stabilization is guaranteed by Lemmas 1 and 2

It is now interesting to compare the families $\mathcal{M}_{bu}(\alpha, l_m)$ and $\mathcal{M}_{sg}(\alpha, \rho)$ under the hypotheses of both Lemmas 1 and 2. In order to simplify the exposition, we could choose $l_m(\cdot)$ and $\rho$ providing as large as possible families $\mathcal{M}_{bu}(\alpha, l_m)$ and $\mathcal{M}_{sg}(\alpha, \rho)$. This means taking $\rho$ as the inverse of $\|W_2(s)T_0(s)W_1(s)\|_{-\alpha,\infty}$, and $l_m(\omega)$ as the inverse of $\bar{\sigma}\,[T_0(-\alpha+\mathrm{j}\omega)]$ for all $\omega\in\mathbb{R}^+$, unless this quantity is zero for some $\omega$'s, in which case the above choice is possible outside arbitrarily small neighborhoods of such $\omega$'s.

First, we observe that, for perturbed systems $\mathcal{P}$ in the family $\mathcal{M}_{sg}(\alpha, \rho)$, the multiplicative factor $\delta^P(s)$ belongs to $\mathcal{RH}_{-\alpha,\infty}$, i.e. it is proper and with all poles in $\mathbb{C}_g$, whereas for perturbed systems $\mathcal{P}$ in the family $\mathcal{M}_{bu}(\alpha, l_m)$, it must be simply free of poles on the $-\alpha+\mathrm{j}\omega$-axis, and can be also not proper, if $l_m(\omega)$ is unbounded as $\omega$ goes to infinity (what allows to take into account, e.g., the case of unmodelled dynamics [14]). Moreover, under the hypotheses of Lemma 2, systems $\mathcal{P}$ in the family $\mathcal{M}_{sg}(\alpha, \rho)$ have all the eigenvalues outside $\mathbb{C}_g$ of the nominal system $\mathcal{P}_0$, with the same locations and algebraic multiplicities (see Proposition 2), whereas, in view of Proposition 1, under the hypotheses of Lemma 1, systems $\mathcal{P}$ in the family $\mathcal{M}_{bu}(\alpha, l_m)$ have unchanged in locations and algebraic multiplicities only the eigenvalues of the nominal system $\mathcal{P}_0$ on the $-\alpha+\mathrm{j}\omega$-axis. The latter situation might be not so restrictive if such eigenvalues are actually invariant (e.g. the zero pole of a d.c. motor whose output is position). Other eigenvalues on the right-hand side of the $-\alpha+\mathrm{j}\omega$-axis can instead move, being constrained by condition (a) of Lemma 1 only to remain unchanged in number (see Reference [14] and the subsequent Example 1).

From these arguments it follows that, in the case of $P_0(s)$ having poles outside $\mathbb{C}_g$, and under the hypotheses of both Lemmas 1 and 2, the family $\mathcal{M}_{bu}(\alpha, l_m)$ is much more significant than the family $\mathcal{M}_{sg}(\alpha, \rho)$ for any choice of the weighting matrices $W_1(s)$ and $W_2(s)$, since all systems in the latter family are constrained to have those poles in the same locations and with the same multiplicities. Consider also that it is possible to choose $\alpha \geq 0$ such that the nominal system $\mathcal{P}_0$ has no eigenvalues on the $-\alpha + \mathrm{j}\omega$-axis.

In a particular (but significant in practice) case, it can be easily shown that, under the hypotheses of Lemma 2, the family $\mathcal{M}_{bu}(\alpha, l_m)$ contains the family $\mathcal{M}_{sg}(\alpha, \rho)$, so that the former is larger than the latter (see again the subsequent Example 1). For a fixed scalar proper rational function $r(s)$ free of poles and zeros outside $\mathbb{C}_g$, and for $W_1(s) = I$ and $W_2(s) = r(s)I$, suppose to have found a compensator $\mathcal{K}$ such that all the hypotheses of Lemma 2 are satisfied for some $\rho > 0$. Notice that hypothesis (ii) of Lemma 2 is now equivalent to

$$\rho^{-1} \geq \bar{\sigma}\left[W_2(-\alpha + \mathrm{j}\omega)T_0(-\alpha + \mathrm{j}\omega)\right] = |r(-\alpha + \mathrm{j}\omega)|\, \bar{\sigma}\left[T_0(-\alpha + \mathrm{j}\omega)\right]$$

for all $\omega \in \mathbb{R}^+$, which implies the existence of a positive and continuous function $l_m(\cdot)$, defined by

$$l_m(\omega) \triangleq |r(-\alpha + \mathrm{j}\omega)|\, \rho, \quad \forall \omega \in \mathbb{R}^+ \tag{9}$$

such that $\bar{\sigma}\left[T_0(-\alpha + \mathrm{j}\omega)\right] \leq 1/l_m(\omega)$, $\forall \omega \in \mathbb{R}^+$. With such a choice of $l_m(\cdot)$, it is easy to verify that a perturbed system $\mathcal{P}$ satisfying the conditions of Lemma 2 also satisfies the conditions of Lemma 1. In particular, condition (a) of Lemma 1 is guaranteed by Proposition 2, condition (b) is guaranteed by Lemma 2, and condition (c) is satisfied in view of (9) and of the following relations

$$\begin{aligned} \bar{\sigma}\left[\delta^P(-\alpha + \mathrm{j}\omega)\right] &= \bar{\sigma}\left[\Delta(-\alpha + \mathrm{j}\omega)W_2(-\alpha + \mathrm{j}\omega)\right] \\ &\leq |r(-\alpha + \mathrm{j}\omega)|\, \|\Delta(s)\|_{-\alpha,\infty} \\ &< |r(-\alpha + \mathrm{j}\omega)|\, \rho, \quad \forall \omega \in \mathbb{R}^+ \end{aligned}$$

The above choice for $W_1(s)$ and $W_2(s)$ (that is possible for any multivariable system) does not imply any loss of generality in the special case $q=1$, so that, for $q=1$ and under the hypotheses of Lemma 2, there always exists a choice of the bound $l_m(\cdot)$ such that the family $\mathcal{M}_{bu}(\alpha, l_m)$ actually contains the family $\mathcal{M}_{sg}(\alpha, \rho)$. This is what occurs in the following very simple example, which shows that, under the hypotheses of Lemma 1, the family $\mathcal{M}_{bu}(\alpha, l_m)$ actually includes perturbed systems $\mathcal{P}$ having eigenvalues lying in the right-hand side

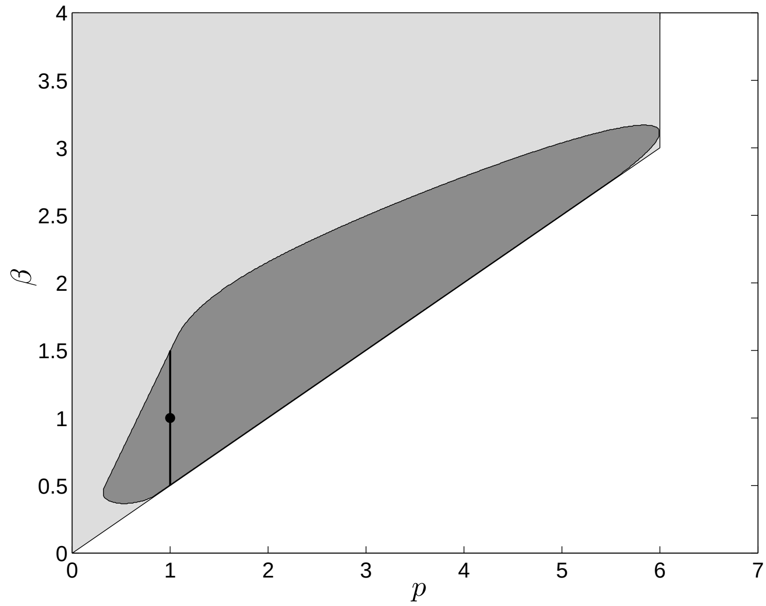


Figure 2: Robust stability regions considered in Example 1.

of the $-\alpha$+j$\omega$-axis that are different from those of the nominal system $\mathcal{P}_0$. It also confirms that the family $\mathcal{M}_{bu}(\alpha, l_m)$ may be much more significant than the family $\mathcal{M}_{sg}(\alpha, \rho)$.

**Example 1** Consider a nominal stabilizable and detectable unstable system $\mathcal{P}_0$ having transfer function $P_0(s)$=$1/(s-1)$ , and a compensator $\mathcal{K}$ with transfer function $K(s)$=$-12/(s+6)$ , such that hypothesis (i) of both Lemmas 1 and 2 is satisfied. Then, let $\Pi$ be the set of all stabilizable and detectable systems $\mathcal{P}$ characterized by transfer functions of the type

$$P(s) = \frac{\beta}{s-p}\,, \quad \beta, p \in \mathbb{R}$$

If $\mathcal{P}$$\in$$\Pi$, it is easy to see that $\delta^P(s)$=$[(\beta-1)s+(p-\beta)]/(s-p)$, and therefore the pole of $P(s)$ is also a pole of the corresponding multiplicative perturbation $\delta^P(s)$.

Fixed $\alpha$=0, it is very simple, by applying the Routh criterion, to compute the set of pairs $(\beta, p)$ characterizing all systems $\mathcal{P}$$\in$$\Pi$ being stabilized by the compensator $\mathcal{K}$. A portion of this set is represented by the whole gray region in Figure 2. The set of pairs $(\beta, p)$ characterizing systems $\mathcal{P}$$\in$$\Pi$ satisfying the conditions of Lemma 1 with $l_m(\omega)$ chosen as the inverse of $\bar{\sigma}\,[T_0(\mathrm{j}\omega)]$, is represented by the dark gray region in Figure 2. The set of pairs $(\beta, p)$ characterizing systems $\mathcal{P}\in\Pi$ satisfying the conditions of Lemma 2 is instead always included in the vertical segment drawn in the same figure, for any choice of the weighting functions $W_1(s)$, $W_2(s)\in\mathcal{RH}_\infty$ (this is not surprising as, in view of Proposition 2, the unstable eigenvalue of $\mathcal{P}_0$ cannot change its location), and reduces to the nominal point, if at least one of such functions is chosen strictly proper.

It is stressed that, if the set $\Pi$ includes all stabilizable and detectable systems $\mathcal{P}$ characterized by transfer functions of the type

$$P(s) = \frac{\gamma s + \beta}{s - p}\,, \quad \gamma, \beta, p \in \mathbb{R}$$

(thus allowing an input/output direct link), there is no system $\mathcal{P}\in\Pi$ with $\gamma\neq 0$, satisfying the conditions of Lemma 2, whereas systems $\mathcal{P}\in\Pi$ with $\gamma\neq 0$ actually exist, satisfying the conditions of Lemma 1. $\square$

## 3.2 On the choice of $\alpha$

Although the nonnegative number $\alpha$ seems to be fixed by the desired rate of decay of the free responses of the closed-loop system, in this subsection it will be shown that it can sometimes be advantageous to increase its value, or even to consider different values of $\alpha$ for the same system (see the subsequent Example 3). In particular, although Lemma 1 is usually presented in the literature with reference to $\alpha$=0, its application with $\alpha$>0 may be useful in order to check the corresponding robust strengthened stability for perturbed systems $\mathcal{P}$ that actually may occur in practice, and that may belong to the family $\mathcal{M}_{bu}(\alpha, l_m)$ for some $\alpha$>0, even if they do not belong to the family $\mathcal{M}_{bu}(0, l'_m)$ for any function $l'_m(\cdot)$, since they violate condition (a) for $\alpha$=0. Indeed, if some stable pole of $P_0(s)$ remains stable only for 'small' variations of the parameters of the controlled system, a proper choice of $\alpha$>0 may allow to actually consider perturbed systems $\mathcal{P}$ satisfying the conditions of Lemma 1 for this $\alpha$>0 (if it excludes such a stable pole from $\mathbb{C}_g$), although violating condition (a) of the same lemma for $\alpha$=0 (see the subsequent Example 3). In particular, if the nominal system $\mathcal{P}_0$ is already asymptotically stable, by considering $\alpha$=0 only asymptotically stable perturbed systems $\mathcal{P}$ are allowed in the family $\mathcal{M}_{bu}(0, l'_m)$ for any function $l'_m(\cdot)$, in view of condition (a), so that Lemma 1 would guarantee robust stability for a family of already stable systems only. The choice of a suitable $\alpha > -\mathrm{Re}(\lambda_i)$, where $\lambda_i$ are the stable eigenvalues of $A_0$ possibly moving across the j$\omega$-axis, may allow to overcome this difficulty (see again the subsequent Example 3).

Lastly, a careful choice of $\alpha>0$ might be useful in order to avoid that some pole or transmission zero of $P_0(s)$ lies on the $\mathbb{C}_g$-boundary, since this would restrict severely the family $\mathcal{M}_{bu}(\alpha, l_m)$, in view of Proposition 1 and the subsequent Remark 5, respectively.

Notice that, because of the strong limitations implied by Proposition 2, there are less advantages in the application of Lemma 2 with $\alpha>0$, even if $\mathcal{P}_0$ is asymptotically stable. In addition, in the case of $P_0(s)$ having a transmission zero with nonnegative real part, for any choice of $\alpha\geq 0$ the family $\mathcal{M}_{sg}(\alpha, \rho)$ is severely restricted. In fact, in view of the subsequent Remark 5, perturbed systems $\mathcal{P}$ of actual interest are very likely to be characterized by corresponding $\delta^P(s)$ having such a transmission zero as a pole, so that $\delta^P(s)$ does not belong to $\mathcal{RH}_{-\alpha,\infty}$ for any $\alpha\geq 0$, thus violating condition (b) of Lemma 2.

**Remark 5** Consider the assumption

$$\text{rank}\,[P_0(s)] = \min(q, p) \tag{10}$$

in the rational field. Under this assumption (which is often satisfied in practice), it can be argued that, if $P_0(s)$ has a transmission zero on the $-\alpha+\mathrm{j}\omega$-axis (or outside $\mathbb{C}_g$), then for most perturbed systems $\mathcal{P}$ for which there exists $\delta^P(s)$ satisfying (4), such a transmission zero appears as a pole of $\delta^P(s)$, thus preventing $\mathcal{P}$ to belong to the family $\mathcal{M}_{bu}(\alpha, l_m)$ (or $\mathcal{M}_{sg}(\alpha, \rho)$). Such an assertion can be justified as follows.

If $q<p$, so that $P_0(s)$ has full row rank in the rational field, denote by $P_0^{-R}(s)$ a minimal right inverse of $P_0(s)$. If, for a perturbed system $\mathcal{P}$ and for the corresponding $P(s)$, there exists $\delta^P(s)$ such that (4) holds, from (5) it follows that

$$\delta^P(s) = \delta_P(s) P_0^{-R}(s) \tag{11}$$

If $q\geq p$, so that $P_0(s)$ has full column rank in the rational field, denote by $P_0^{-L}(s)$ a minimal left inverse of $P_0(s)$. Then, for any perturbed system $\mathcal{P}$, and for the corresponding $P(s)$, there exists $\delta^P(s)$ such that (4) holds, and all such $\delta^P(s)$ are expressed by the relation

$$\delta^P(s) = \delta_P(s) P_0^{-L}(s) + \bar{\delta}^P(s) \tag{12}$$

by considering all rational $\bar{\delta}^P(s)$ such that $\bar{\delta}^P(s)P_0(s)=0$.

It is known that all the transmission zeros of $P_0(s)$ are poles of $P_0^{-R}(s)$, or $P_0^{-L}(s)$. Hence, if $P_0(s)$ has a transmission zero on the $-\alpha+\mathrm{j}\omega$-axis (or outside $\mathbb{C}_g$), then in view of (11) or (12), respectively, $\delta^P(s)$ is very likely to have a pole in the same location, thus justifying the assertion. This may occur even if $\delta_P(s) \triangleq P(s) - P_0(s)$ has no poles on the $-\alpha + j\omega$-axis (or outside $\mathbb{C}_g$). Notice that, unlike the SISO case, in the MIMO case it can even occur that $P(s)$ has unchanged transmission zeros with respect to $P_0(s)$, and these nevertheless appear as poles of $\delta^P(s)$. □

The following example, which is physically motivated, illustrates a MIMO case in which the transfer matrix $P_0(s)$ of the nominal system $\mathcal{P}_0$ has an unstable transmission zero. Calculation of the poles of the multiplicative perturbations relating the transfer matrices $P(s)$ of perturbed systems $\mathcal{P}$ of interest to $P_0(s)$, confirms the assertion made in Remark 5.

**Example 2** Consider a finite dimensional linear model of a clamped flexible beam vibrating in a plane. For simplicity, only the three lower order vibration modes are taken into account. It is assumed that two piezoelectric sensors and two piezoelectric actuators are used for vibration damping, so that the two measured outputs are (approximately) the curvatures at the sensor locations, and the two control inputs are (approximately) the bending torques applied by the actuators. The model of the beam, derived according to Euler-Bernoulli theory (valid for small deflections), is a two-input two-output sixth order linear system. Let the beam be made of steel, and be half a meter long, five centimeters wide and one millimeter thick. Let one sensor and one actuator be placed in the location indicated as A in Figure 3 (very close to the clamped end, where all the curvatures of the vibration modes of the beam have a high value). Then, let the second actuator and the second sensor be placed in the locations indicated in Figure 3 as B and C, respectively. Location C is very close to a local maximum in the curvature of the second mode. Assume that each of the vibration modes of the beam is naturally damped by some (small and positive) uncertain viscous coefficient $\zeta_i$, $i{=}1,2,3$, and let the nominal model $\mathcal{P}_0$ of the beam be obtained by setting $\zeta_i{=}0.02$, $i{=}1,2,3$. Notice that all the physical parameters of this example are taken from a real experimental setup, built in the Robotics and Control Laboratory of the DISP at the University of Rome 'Tor Vergata'.

The transfer matrix of the nominal system $\mathcal{P}_0$ is $P_0(s){=}N_0(s)/d_0(s)$, where $N_0(s)$ is a $2{\times}2$ polynomial matrix, and

$$d_0(s) = (s^2 + 0.817\,s + 417.295)(s^2 + 5.121\,s + 16388.8)(s^2 + 14.338\,s + 128491)$$

The McMillan-Smith form of $P_0(s)$ is

$$M_0(s) = \begin{bmatrix} \dfrac{1}{d_0(s)} & 0 \\ 0 & \varepsilon_0(s) \end{bmatrix}$$

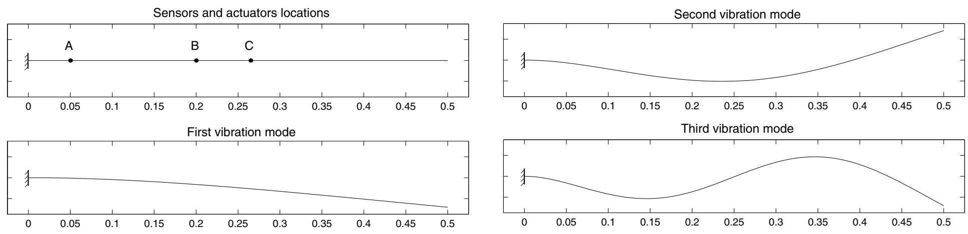


Figure 3: The beam considered in Example 2: locations of sensors and actuators, and shapes of the considered vibration modes.

where $\varepsilon_0(s)=s^2+0.104\,s-7346.7$. The roots of $\varepsilon_0(s)$ are the transmission zeros of $P_0(s)$. Such roots are $-85.7647$ and $85.6607$, i.e. the transfer matrix of the nominal system has one unstable transmission zero. The perturbed systems $\mathcal{P}$ of interest are characterized by transfer matrices of the type $P(s)=N(s)/d(s)$, where $N(s)$ is a $2\times 2$ polynomial matrix whose entries depend on $\zeta_1$, $\zeta_2$ and $\zeta_3$, and

$$d(s) = (s^2+40.8556\,\zeta_1\,s+417.295)(s^2+256.038\,\zeta_2\,s+16388.8)\times \times(s^2+716.913\,\zeta_3\,s+128491)$$

By computing $\delta^P(s)=(P(s)-P_0(s))\,P_0^{-1}(s)$, it can be seen that $\delta^P(s)$ has fourteen poles: the six poles of $P_0(s)$, the six poles of $P(s)$, and the roots of $\varepsilon_0(s)$. Hence, even though all the poles of $P_0(s)$ and $P(s)$ have negative real part, $\delta^P(s)$ has one unstable pole, namely the positive root of $\varepsilon_0(s)$. The fact that this root has been shown to be also a transmission zero of $P_0(s)$, confirms the assertion made in Remark 5. □

The following very simple example shows that the use of Lemma 1 with a proper choice of $\alpha>0$ (thus requiring the corresponding robust strengthened stability) allows to take into account perturbed systems $\mathcal{P}$ violating condition (a) of Lemma 1 for $\alpha=0$ in consequence of possibly small variations of the parameters of the system from their nominal values (such a situation might be quite frequent in practice).

**Example 3** Consider a nominal reachable and observable asymptotically stable system $\mathcal{P}_0$ having transfer function $P_0(s)=1/(s^2+0.2s+1)$. The poles of $P_0(s)$ are very close to the j$\omega$-axis (their real part is $-0.1$). Consider also a compensator $\mathcal{K}$ with transfer function $K(s)=-90(s+2)/(s+16)$, such that hypothesis (i) of Lemma 1 is satisfied. Then, let $\Pi$ be the set of all reachable and observable systems $\mathcal{P}$ characterized by transfer functions of the type

$$P(s) = \frac{\beta}{s^2 + 2\zeta s + 1}\,, \quad \beta, \zeta \in \mathbb{R}$$

If $\mathcal{P}{\in}\Pi$, it is easy to see that $\delta^P(s){=}[(\beta-1)s^2+(0.2\beta-2\zeta)s+(\beta-1)]/(s^2+2\zeta s+1)$, and therefore the poles of $P(s)$ (that in this case are all the eigenvalues of $\mathcal{P}$) are also poles of the corresponding multiplicative perturbation $\delta^P(s)$.

The set of pairs $(\beta, \zeta)$ characterizing systems $\mathcal{P}{\in}\Pi$ satisfying the conditions of Lemma 1 with $l_m(\omega)$ chosen as the inverse of $\bar{\sigma}\,[T_0(-\alpha{+}\mathrm{j}\omega)]$, is represented in Figure 4 for $\alpha{=}0$ and $\alpha{=}1$ (the latter choice is allowed, since the real part of the nominal closed-loop eigenvalues is less than $-1$) by the dark gray region and by the light gray region, respectively. The analysis of these regions confirms that the most convenient choice of $\alpha{\geq}0$ depends on the expected range of values for the parameter $\zeta$: e.g. for $\zeta$ ranging from 0.2 to $-1$ the choice of $\alpha{=}1$ seems to be appropriate, whereas for $\zeta$ ranging from 0 to 1.5 (i.e. if it is known that the eigenvalues of $\mathcal{P}$ remain stable) the choice of $\alpha{=}0$ is still the more suitable.

In this example, the union of the two regions depicted in Figure 4 can be used as an estimate of the set of pairs $(\beta, \zeta)$ characterizing all systems $\mathcal{P}{\in}\Pi$ being (asymptotically) stabilized by the given compensator $\mathcal{K}$. □

# 4 The case of additive perturbations and further comparisons

In this section, well-known sufficient conditions guaranteeing the robust $\mathbb{C}_g$-stability of control system $\Sigma$ under additive perturbations (defined by (3)) affecting the nominal transfer matrix $P_0(s)$ of the controlled system, will be analyzed and a bit extended (see the subsequent Theorem 1) in order to take into account also additive perturbations $\delta_P(s)$ having poles on the $\mathbb{C}_g$-boundary.

Let $V_0(s){\triangleq}K(s)\big(I{-}P_0(s)K(s)\big)^{-1}$ denote the so-called control sensitivity matrix of the nominal closed-loop system $\Sigma_0$ (under the assumption that $\Sigma_0$ is well-posed). Then, a first condition for the robust $\mathbb{C}_g$-stability of system $\Sigma$ is given by the following lemma [15, 21], which can be proven by means of a suitable extension of the Nyquist criterion.

**Lemma 3** *If, for a fixed $\alpha{\geq}0$ characterizing $\mathbb{C}_g$,*

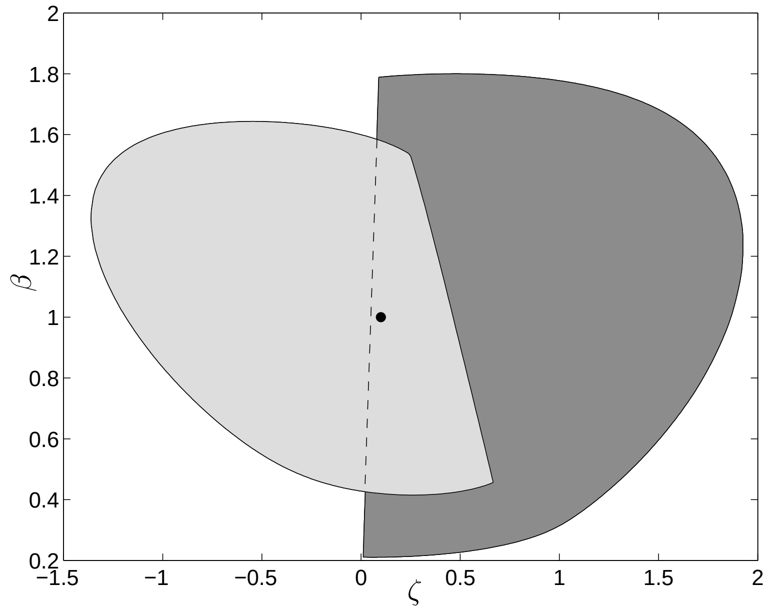


Figure 4: Robust stability regions considered in Example 3.

*(i)* $\Sigma_0$ *is well-posed and* $\mathbb{C}_g$*-stable;*

*(ii)* $\bar{\sigma}\left[V_0(-\alpha+j\omega)\right] \leq \dfrac{1}{l_a(\omega)}\ ,\ \forall \omega \in \mathbb{R}^+,$

*where* $l_a(\omega)$ *is a positive and continuous function of* $\omega \in \mathbb{R}^+$*, then system* $\Sigma$ *is* $\mathbb{C}_g$*-stable for all the perturbed systems* $\mathcal{P}$ *such that:*

*(a) systems* $\mathcal{P}$ *and* $\mathcal{P}_0$ *have the same number of eigenvalues outside* $\mathbb{C}_g$ *(including algebraic multiplicities);*

*(b)* $\Sigma$ *is well-posed;*

*(c)* $\delta_P(s)$ *in (3) satisfies the inequality* $\bar{\sigma}\left[\delta_P(-\alpha+j\omega)\right] < l_a(\omega)\ ,\ \forall \omega \in \mathbb{R}^+.$

In the following, the family of perturbed systems $\mathcal{P}$ satisfying the conditions (a), (b) and (c) of Lemma 3 will be denoted by $\mathcal{A}_{bu}(\alpha, l_a)$. The subscript recalls that the transfer matrices $P(s)$ of all the perturbed systems in this family could be viewed as belonging to a 'ball of uncertainty' around the nominal transfer matrix $P_0(s)$ [15].

The following lemma gives a second type of conditions for the robust $\mathbb{C}_g$-stability of $\Sigma$. It generalizes well-known results (see, e.g., References [9, 16]) and can be proven by means of a suitable extension of the small-gain theorem.

**Lemma 4** *If, for a fixed* $\alpha \geq 0$ *characterizing* $\mathbb{C}_g$*,*

*(i)* $\Sigma_0$ *is well-posed and* $\mathbb{C}_g$*-stable;*

*(ii)* $\|W_2(s)V_0(s)W_1(s)\|_{-\alpha,\infty} \leq \rho^{-1}$*,*

*where* $\rho>0$ *and* $W_1(s)$*,* $W_2(s) \in \mathcal{RH}_{-\alpha,\infty}$*, then system* $\Sigma$ *is well-posed and* $\mathbb{C}_g$*-stable for all the perturbed systems* $\mathcal{P}$ *such that:*

*(a) system* $\mathcal{P}$ *is* $\mathbb{C}_g$*-stabilizable and* $\mathbb{C}_g$*-detectable;*

*(b)* $\delta_P(s)$ *in (3) satisfies the relation* $\delta_P(s) = W_1(s)\Delta(s)W_2(s)$*, where* $\Delta(s)$ *belongs to* $\mathcal{RH}_{-\alpha,\infty}$*, and is such that* $\|\Delta(s)\|_{-\alpha,\infty} < \rho$*.*

In the following, the family of perturbed systems $\mathcal{P}$ satisfying the conditions (a) and (b) of Lemma 4, for fixed weighting matrices $W_1(s)$ and $W_2(s)$, will be denoted by $\mathcal{A}_{sg}(\alpha,\rho)$. The subscript refers to the fact that Lemma 4 can be proven by means of (a suitable extension of) the small-gain theorem, thus recalling that the additive perturbations relating the transfer matrices $P(s)$ of all the perturbed systems in this family to the nominal one $P_0(s)$, belong to $\mathcal{RH}_{-\alpha,\infty}$.

**Remark 6** The strict inequality in condition (c) of Lemma 3 (or (b) of Lemma 4) can be relaxed to $\leq$ if the inequality in the corresponding hypothesis (ii) is strengthened to $<$. □

An analysis of the families $\mathcal{A}_{bu}(\alpha, l_a)$ and $\mathcal{A}_{sg}(\alpha,\rho)$ can be carried out in a wholly similar way to what has been made in the previous section for the families $\mathcal{M}_{bu}(\alpha, l_m)$ and $\mathcal{M}_{sg}(\alpha,\rho)$. In particular, the following propositions:

**Proposition 3** *Under hypotheses (i) and (ii) of Lemma 3, the nominal system* $\mathcal{P}_0$ *and all the perturbed systems* $\mathcal{P}$ *in the family* $\mathcal{A}_{bu}(\alpha, l_a)$ *have the same eigenvalues on the* $-\alpha+j\omega$*-axis, with the same algebraic multiplicities.*

**Proposition 4** *Under hypotheses (i) and (ii) of Lemma 4, the nominal system* $\mathcal{P}_0$ *and all the perturbed systems* $\mathcal{P}$ *in the family* $\mathcal{A}_{sg}(\alpha,\rho)$ *have the same eigenvalues outside* $\mathbb{C}_g$*, with the same algebraic multiplicities.*

(that are nearly obvious, and can be proven similarly to Propositions 1 and 2, respectively, or by means of appropriate tools of realization theory) suggest that, under the hypotheses of both Lemmas 3 and 4, the family $\mathcal{A}_{bu}(\alpha, l_a)$ is more significant than the family $\mathcal{A}_{sg}(\alpha,\rho)$ in the case of $P_0(s)$ having poles outside $\mathbb{C}_g$, since, according to Proposition 4, all systems in the latter family are constrained to have those poles in the same locations and with the same multiplicities. This can be stressed by stating that, for simple but reasonable choices of $W_1(s)$ and $W_2(s)$, and consequently of $l_a(\omega)$, under the hypotheses of Lemma 4 the family $\mathcal{A}_{bu}(\alpha, l_a)$ contains the family $\mathcal{A}_{sg}(\alpha, \rho)$ (what can be derived in the same way as the similar statement about families $\mathcal{M}_{bu}(\alpha, l_m)$ and $\mathcal{M}_{sg}(\alpha, \rho)$ has been).

Notice that also the choice of $\alpha \geq 0$ can be discussed through arguments similar to the ones adduced in the previous section, when multiplicative perturbations were considered, with the exception that a remark similar to Remark 5 does not hold, obviously, for additive perturbations.

## 4.1 Comparison between families of perturbed systems whose robust stabilization is guaranteed by Lemmas 1 and 3

It seems then useful to carry out the further comparison between families $\mathcal{M}_{bu}(\alpha, l_m)$ and $\mathcal{A}_{bu}(\alpha, l_a)$, under the hypotheses of both Lemmas 1 and 3. A compensator $\mathcal{K}$ such that the nominal closed-loop system $\Sigma_0$ satisfies hypothesis (i) of both lemmas will be considered. For simplicity and compactness of the formulas, we shall refer to the case $\alpha=0$, i.e. to the mere asymptotic stability, but the subsequent conclusions hold for any choice of $\alpha \geq 0$.

First, suppose $P_0(s)$ is free of poles on the Nyquist contour (see a subsequent comment for the significance of this hypothesis), and notice that, in order to obtain a family $\mathcal{A}_{bu}(0, l_a)$ as large as possible, we could choose $l_a(\omega)$ as the inverse of $\bar{\sigma}\left[V_0(\mathrm{j}\omega)\right]$ for all $\omega \in \mathbb{R}^+$, unless this quantity is zero for some $\omega$'s, in which case the above choice is possible outside arbitrarily small neighborhoods of such $\omega$'s. A sufficient condition guaranteeing that the bound $l_m(\cdot)$ satisfies the hypothesis (ii) of Lemma 1 is the following:

$$l_m(\omega)\bar{\sigma}\left[P_0(\mathrm{j}\omega)\right] \leq l_a(\omega), \quad \forall \omega \in \mathbb{R}^+ \tag{13}$$

(which, in particular, allows to define $l_m(\cdot)$ by putting the equality in (13)), since, in this case,

$$\bar{\sigma}\left[T_0(\mathrm{j}\omega)\right] \leq \bar{\sigma}\left[P_0(\mathrm{j}\omega)\right]\bar{\sigma}\left[V_0(\mathrm{j}\omega)\right] \leq \frac{\bar{\sigma}\left[P_0(\mathrm{j}\omega)\right]}{l_a(\omega)} \leq \frac{1}{l_m(\omega)}\,, \quad \forall \omega \in \mathbb{R}^+$$

Then, if the bounds $l_a(\cdot)$ and $l_m(\cdot)$ are chosen so as to satisfy (13), the perturbed systems $\mathcal{P}$ in the family $\mathcal{M}_{bu}(0, l_m)$ also belong to the family $\mathcal{A}_{bu}(0, l_a)$. In particular, they satisfy condition (c) of Lemma 3, since

$$\bar{\sigma}\left[\delta_P(\mathrm{j}\omega)\right] \leq \bar{\sigma}\left[\delta^P(\mathrm{j}\omega)\right]\bar{\sigma}\left[P_0(\mathrm{j}\omega)\right] < l_m(\omega)\bar{\sigma}\left[P_0(\mathrm{j}\omega)\right] \leq l_a(\omega), \quad \forall \omega \in \mathbb{R}^+$$

Therefore, the family $\mathcal{M}_{bu}(0, l_m)$ is contained in the family $\mathcal{A}_{bu}(0, l_a)$.

Second, suppose $P_0(s)$ is free of transmissions zeros on the Nyquist contour (see a subsequent comment for the significance of this hypothesis) and has full column rank in the rational field, so that a left inverse $P_0^{-L}(s)$ of $P_0(s)$ exists, and for all additive perturbations $\delta_P(s)$ defined by (3) there exists $\delta^P(s)$ such that $\delta_P(s){=}\delta^P(s)P_0(s)$, so as to satisfy (4) (in particular, $\delta^P(s)$ can be chosen as $\delta^P(s){=}\delta_P(s)P_0^{-L}(s)$). If $P_0^{-L}(s)$ is chosen to be minimal, its poles are all the transmission zeros of $P_0(s)$, so that $P_0^{-L}(s)$ has no poles on the Nyquist contour. Now, consider a bound $l_m(\cdot)$ such that hypothesis (ii) of Lemma 1 is satisfied (possibly the bound $l_m(\cdot)$ specified at the beginning of Subsection 3.1, so as to enlarge the family $\mathcal{M}_{bu}(0, l_m)$ as much as possible). A sufficient condition guaranteeing that the bound $l_a(\cdot)$ satisfies the hypothesis (ii) of Lemma 3 is the following:

$$l_a(\omega)\bar{\sigma}\left[P_0^{-L}(\mathrm{j}\omega)\right] \leq l_m(\omega), \quad \forall\omega \in \mathbb{R}^+ \tag{14}$$

(which, in particular, allows to define $l_a(\cdot)$ by putting the equality in (14)), since, in this case,

$$\bar{\sigma}\left[V_0(\mathrm{j}\omega)\right] \leq \bar{\sigma}\left[P_0^{-L}(\mathrm{j}\omega)\right]\bar{\sigma}\left[T_0(\mathrm{j}\omega)\right] \leq \frac{\bar{\sigma}\left[P_0^{-L}(\mathrm{j}\omega)\right]}{l_m(\omega)} \leq \frac{1}{l_a(\omega)}, \quad \forall\omega \in \mathbb{R}^+$$

As a consequence, if the bounds $l_m(\cdot)$ and $l_a(\cdot)$ are chosen so as to satisfy (14), the perturbed systems $\mathcal{P}$ in the family $\mathcal{A}_{bu}(0, l_a)$ also belong to the family $\mathcal{M}_{bu}(0, l_m)$. In particular, they satisfy condition (c) of Lemma 1, since

$$\bar{\sigma}\left[\delta^P(\mathrm{j}\omega)\right] \leq \bar{\sigma}\left[\delta_P(\mathrm{j}\omega)\right]\bar{\sigma}\left[P_0^{-L}(\mathrm{j}\omega)\right] < l_a(\omega)\bar{\sigma}\left[P_0^{-L}(\mathrm{j}\omega)\right] \leq l_m(\omega), \quad \forall\omega \in \mathbb{R}^+$$

Therefore, the family $\mathcal{A}_{bu}(0, l_a)$ is contained in the family $\mathcal{M}_{bu}(0, l_m)$.

If $P_0(s)$ is free both of poles and transmission zeros on the Nyquist contour and has full column rank in the rational field, then both previous discussions can be applied, and one could argue the identity of the families $\mathcal{M}_{bu}(0, l_m)$ and $\mathcal{A}_{bu}(0, l_a)$, if both (13) and (14) hold. However, although this is trivially true for SISO systems by choosing $l_m(\omega)|P_0(\mathrm{j}\omega)|{=}l_a(\omega)$, $\forall\omega{\in}\mathbb{R}^+$, a similar choice for MIMO systems is unlikely to be available, since it is readily seen that there exists a pair of positive and continuous functions $l_a(\cdot)$ and $l_m(\cdot)$ satisfying both (13) and (14) if and only if

$$\bar{\sigma}\left[P_0(\mathrm{j}\omega)\right]\bar{\sigma}\left[P_0^{-L}(\mathrm{j}\omega)\right] \leq 1, \quad \forall\omega \in \mathbb{R}^+ \tag{15}$$

Taking into account that $P_0^{-L}(\mathrm{j}\omega)P_0(\mathrm{j}\omega) = I$, (15) is equivalent to

$$\bar{\sigma}\left[P_0^{-L}(\mathrm{j}\omega)\right]\bar{\sigma}\left[P_0(\mathrm{j}\omega)\right] = 1, \quad \forall\omega \in \mathbb{R}^+$$

and this, for $p{=}q$ (i.e. for square nonsingular $P_0(s)$), means that the largest and the smallest singular values of $P_0(\mathrm{j}\omega)$ coincide for all $\omega{\in}\mathbb{R}^+$.

It is stressed that, in the above analysis, if relation (13), or (14), fails to hold, then the corresponding inclusion between the two families may fail to hold, too. However, even when the bounds $l_a(\cdot)$ and $l_m(\cdot)$ are chosen independently, some further comments may be helpful.

It was shown in Remark 5 that, if (10) holds, $P_0(s)$ has transmission zeros on the Nyquist contour and, for a possible $P(s)$, there exists $\delta^P(s)$ such that (4) holds, then $\delta^P(s)$ is likely to have poles on the Nyquist contour, even if the corresponding $\delta_P(s)$ has not. Therefore, if $P_0(s)$ has transmission zeros, but is free of poles, on the Nyquist contour, and (10) holds, perturbed systems $\mathcal{P}$ might exist that belong to $\mathcal{A}_{bu}(0, l_a)$, but not to $\mathcal{M}_{bu}(0, l_m)$. This does not mean, in general, that, under the mentioned hypotheses, the family $\mathcal{A}_{bu}(0, l_a)$ includes the family $\mathcal{M}_{bu}(0, l_m)$, but it can be argued that in this case the former family may be more significant than the latter.

By converse, if $P_0(s)$ has poles on the Nyquist contour, then from (3) it follows that the additive perturbation $\delta_P(s)$ corresponding to a possible perturbed system $\mathcal{P}$ is likely to have poles on the Nyquist contour, whereas the multiplicative perturbation $\delta^P(s)$ satisfying (4) for the same $P(s)$ (if it exists) might not have such poles, since they might be introduced in the right-hand side of (5) by $P_0(s)$ itself. Therefore, if $P_0(s)$ has poles, but is free of transmission zeros, on the Nyquist contour, perturbed systems $\mathcal{P}$ might exist that belong to $\mathcal{M}_{bu}(0, l_m)$, but not to $\mathcal{A}_{bu}(0, l_a)$. Also in this case, this does not mean, in general, that, under the mentioned hypotheses, the family $\mathcal{M}_{bu}(0, l_m)$ includes the family $\mathcal{A}_{bu}(0, l_a)$, but it can be argued that the former family may be more significant than the latter, for instance when $\delta^P(s)$ satisfying (4) exists for all the perturbed systems $\mathcal{P}$ of interest, as it is shown by the following very simple example.

**Example 4** Consider a nominal stabilizable and detectable unstable system $\mathcal{P}_0$ having transfer function $P_0(s){=}1/[s(s-1)]$, with a pole on the Nyquist contour for $\alpha{=}0$, and a compensator $\mathcal{K}$ with transfer function $K(s){=}{-}(140s{+}60)/(s^2+12s+57)$, such that hypothesis (i) of both Lemmas 1 and 3 is satisfied. Then, let $\Pi$ be the set of all stabilizable and detectable systems $\mathcal{P}$ characterized by transfer functions of the type

$$P(s) = \frac{\beta}{s(s-p)}\,, \quad \beta, p \in \mathbb{R}$$

If $\mathcal{P}{\in}\Pi$, it is easy to see that $\delta_P(s){=}[(\beta-1)s+(p-\beta)]/[s(s-1)(s-p)]$ and $\delta^P(s){=}[(\beta-1)s+(p-\beta)]/(s-p)$, and therefore the pole $s{=}p$ of $P(s)$ is also

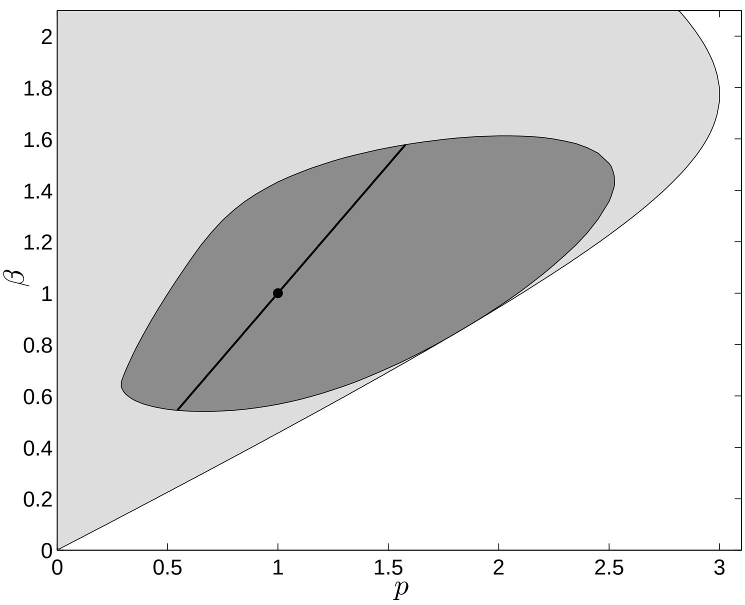


Figure 5: Robust stability regions considered in Example 4.

a pole of the corresponding multiplicative perturbation $\delta^P(s)$, whereas the pole $s=0$ of $P(s)$ is not.

Fixed $\alpha=0$, it is very simple, by applying the Routh criterion, to compute the set of pairs $(\beta, p)$ characterizing all systems $\mathcal{P}\in\Pi$ being stabilized by the compensator $\mathcal{K}$. A portion of this set is represented by the whole gray region in Figure 5. The set of pairs $(\beta, p)$ characterizing systems $\mathcal{P}\in\Pi$ satisfying the conditions of Lemma 1 with $l_m(\omega)$ chosen as the inverse of $\bar{\sigma}\left[T_0(\mathrm{j}\omega)\right]$, is represented by the dark gray region in Figure 5. The set of pairs $(\beta, p)$ characterizing systems $\mathcal{P}\in\Pi$ satisfying the conditions of Lemma 3 is instead always included in the oblique segment drawn in the same figure, for any choice of the positive bound $l_a(\cdot)$ satisfying hypothesis (ii) of Lemma 3. Notice that such a segment belongs to the line of equation $\beta=p$, which implies that the zero pole has the same residual in $P(s)$ as in $P_0(s)$, so that $\delta_P(s)$ has no zero pole. $\square$

In Example 4, the perturbed systems $\mathcal{P}$ of interest have a zero eigenvalue as the nominal system $\mathcal{P}_0$, so that both Propositions 1 and 3 should be not really restrictive in that case. Nevertheless, it is stressed that the robust stability region guaranteed by Lemma 1 is much more significant than the one guaranteed by Lemma 3, as the analysis of Figure 5 attests. This is actually due to the fact that, in order the additive perturbation $\delta_P(s)$ not to have a zero pole (so as to satisfy condition (c) of Lemma 3), the zero pole must have the same residual in $P(s)$ as in $P_0(s)$, which is a further restriction besides Proposition 3.

## 4.2 An extension of Lemma 3

The above discussion suggests to investigate the possibility of extending the results stated in both Lemmas 1 and 3, so as to allow also additive and multiplicative perturbations having poles on the $\mathbb{C}_g$-boundary. Only an extension of Lemma 3 will be here presented, since it may imply a significant enlargement of family $\mathcal{A}_{bu}(\alpha, l_a)$ (see the subsequent Remark 8), whereas this is not true, in general, for the similar extension of Lemma 1.

**Theorem 1** *If, for a fixed $\alpha \geq 0$ characterizing $\mathbb{C}_g$,*

*(i) $\Sigma_0$ is well-posed and $\mathbb{C}_g$-stable;*

*(ii) $\bar{\sigma}\left[V_0(-\alpha+j\omega)\right] \leq \dfrac{1}{l_a(\omega)}$ , $\forall \omega \in \mathbb{R}^+\text{-}\{\omega_1, \ldots, \omega_r\}$,*

*where $l_a(\omega)$ is a positive and continuous function of $\omega \in \mathbb{R}^+\text{-}\{\omega_1, \ldots, \omega_r\}$, for some $r \in \mathbb{Z}$, $r>0$, and some $\omega_1, \ldots, \omega_r \in \mathbb{R}^+$, and is such that*

$$\lim_{\omega \to \omega_i} l_a(\omega) = +\infty, \quad \forall\, i = 1, \ldots, r \tag{16}$$

*then system $\Sigma$ is well-posed and $\mathbb{C}_g$-stable for all the perturbed systems $\mathcal{P}$ such that:*

*(a) systems $\mathcal{P}$ and $\mathcal{P}_0$ have the same number of eigenvalues outside $\mathbb{C}_g$ (including algebraic multiplicities);*

*(b) there exists $\varepsilon \in (0, 1)$ such that $\delta_P(s)$ in (3) satisfies the inequality*

$$\bar{\sigma}\left[\delta_P(-\alpha + j\omega)\right] \leq \varepsilon\, l_a(\omega)\,, \quad \forall \omega \in \mathbb{R}^+\text{-}\{\omega_1, \ldots, \omega_r\}$$

***Proof.*** Under hypotheses (i) and (ii), consider any perturbed system $\mathcal{P}$ satisfying conditions (a) and (b), and define $\Gamma(s) \triangleq \delta_P(s) V_0(s)$. The following relations hold for all $\omega \in \mathbb{R}^+\text{-}\{\omega_1, \ldots, \omega_r\}$:

$$\bar{\sigma}\left[\Gamma(-\alpha + \mathrm{j}\omega)\right] \leq \bar{\sigma}\left[\delta_P(-\alpha + \mathrm{j}\omega)\right] \bar{\sigma}\left[V_0(-\alpha + \mathrm{j}\omega)\right] \leq \varepsilon \tag{17}$$

Denote by $\gamma_{h,k}(s)$ the $h$-th row and $k$-th column entry of $\Gamma(s)$. Since

$$\max_{h,\,k} |\gamma_{h,k}(-\alpha + \mathrm{j}\omega)| \leq \bar{\sigma}\left[\Gamma(-\alpha + \mathrm{j}\omega)\right]\,, \quad \forall \omega \in \mathbb{R}^+\text{-}\{\omega_1, \ldots, \omega_r\} \tag{18}$$

from (17) and (18) it follows that $\Gamma(s)$ belongs to $\mathcal{RL}_{-\alpha,\infty}$. Indeed, if $\Gamma(s)$ had a pole in $-\alpha+\mathrm{j}\bar{\omega}$, or were not proper, then the left-hand side of (18) should be unbounded as $\omega$ tends to $\bar{\omega}$ or to infinity, in contradiction with (17) and (18). Hence, $\bar{\sigma}\,[\Gamma(-\alpha+\mathrm{j}\omega)]$ is a continuous function of $\omega\in\mathbb{R}^+$, and therefore

$$\bar{\sigma}\,[\Gamma(-\alpha + \mathrm{j}\omega)] \leq \varepsilon\,, \quad \forall\omega \in \mathbb{R}^+ \tag{19}$$

This implies

$$\det\left(I - \lambda\delta_P(-\alpha + \mathrm{j}\omega)V_0(-\alpha + \mathrm{j}\omega)\right) \neq 0\,, \quad \forall\omega \in \mathbb{R}, \quad \forall\lambda \in [0, 1]$$

Then the proof can be completed through arguments wholly similar to Reference [7, pp.278-279], provided suitable amendments of notations. □

In order to attest the improvement of Theorem 1 as compared with Lemma 3, it is stressed that, if Theorem 1 is applied to Example 4, the robust stability region thus computed matches the one computed through Lemma 1. The following remarks are concerned with the applicability of Theorem 1.

**Remark 7** Although (16) allows the bound $l_a(\omega)$ to be infinite for all $\omega$'s in a certain finite set, wholly similar limitations to those stated by Proposition 3 for Lemma 3, continue to hold also for Theorem 1, since it can be shown that, under the hypotheses (i) and (ii) of Theorem 1, no perturbed system $\mathcal{P}$ exists satisfying the conditions (a) and (b) of the same theorem, and such that the poles of $P_0(s)$ on the $-\alpha+\mathrm{j}\omega$-axis disappear in $P(s)$, or reduce their algebraic multiplicities as eigenvalues of $\mathcal{P}$. This might not be restrictive, if such poles are actually independent of the uncertainty (as in Example 4), e.g. when, for $\alpha=0$, they are due to an internal model of exogenous signals provided by a subcompensator which is considered as embedded in $\mathcal{P}$ (see the following remark). □

**Remark 8** Hypothesis (ii) and condition (16) of Theorem 1 imply

$$\bar{\sigma}\,[V_0(-\alpha+\mathrm{j}\omega_i)] = 0, \quad \forall\, i = 1, \ldots, r$$

Hence, the only possible $\omega$'s such that $l_a(\omega)$ can assume infinite value (thus allowing $\delta_P(s)$ to have a pole at $-\alpha+\mathrm{j}\omega$) correspond to blocking zeros of $V_0(s)$ (i.e. zeros common to all non zero elements of $V_0(s)$) on the $\mathbb{C}_g$-boundary. It is then interesting to investigate when a pole $s_0$ of $P_0(s)$ lying on the $-\alpha+\mathrm{j}\omega$-axis is also a blocking zero of $V_0(s)$, so that the use of Theorem 1 instead of Lemma 3 might be advantageous.

In the SISO case, it is trivial to verify that, if $P_0(s)$ has a pole $s_0{=}{-}\alpha{+}j\omega_0$ on the $\mathbb{C}_g$-boundary, and $\Sigma_0$ is $\mathbb{C}_g$-stable, then $s_0$ is a zero of $V_0(s)$. Recall that, for $\alpha{=}0$, the pole of $P_0(s)$ at $s_0$ implies that $\mathcal{P}_0$ has an internal model of the sinusoidal (or constant, if $\omega_0{=}0$) exogenous signals with angular frequency $\omega_0$. Since $s_0$ is still a pole of the transfer matrix $P(s)$ of all the perturbed systems $\mathcal{P}$ for which the asymptotic stability of $\Sigma$ is preserved by virtue of Theorem 1 (see Remark 7), the property of asymptotic tracking of reference signals, as well as asymptotic rejection of disturbances, of the considered type is robustly maintained by $\Sigma$ for all those systems $\mathcal{P}$ (as in Example 4).

In the MIMO case, the existence of a pole of $P_0(s)$ at $s_0{=}{-}\alpha{+}j\omega_0$ in general does not imply that $s_0$ is also a blocking zero of $V_0(s)$, but it is stressed that, for $\alpha{=}0$, this occurs when $\mathcal{P}_0$ has a complete internal model of the exogenous signals of the above specified type. Notice that a complete internal model of the exogenous signals corresponding to the eigenvalue $s_0{=}j\omega_0$ can be provided by a suitable subcompensator which is connected in series with the controlled system and is considered as embedded in $\mathcal{P}_0$, as well as in $\mathcal{P}$. In this case, $\mathcal{K}$ in Figure 1 only denotes the stabilizing subcompensator. □

# 5 Conclusions

In the literature several conditions for stability robustness within various uncertainty descriptions can be found. In order to use such conditions properly, so as to meet robustness requirements in realistic cases, it is useful to have knowledge of both their advantages and limitations. In this paper two kinds of classical robust stability conditions (based on the Nyquist criterion and on the small-gain theorem, respectively), which apply in case of either additive or multiplicative perturbations, were analyzed, and some properties of the related families of perturbed systems were formally derived. In particular, a proof was given of the invariance - in locations and algebraic multiplicities - of the eigenvalues outside $\mathbb{C}_g$ of the perturbed systems whose $\mathbb{C}_g$-stabilization is guaranteed by the conditions based on the small-gain theorem, whereas for the perturbed systems whose $\mathbb{C}_g$-stabilization is guaranteed by the conditions based on the Nyquist criterion a proof was given of the invariance - in locations and algebraic multiplicities - of the only eigenvalues lying on the $\mathbb{C}_g$-boundary. The mentioned conditions were then compared pairwise, and simple examples were exhibited in order to illustrate both the presented properties and comparisons. The conditions based on the Nyquist criterion appeared to be advantageous. Some advantages exhibited by this type of conditions only in the case of multiplicative perturbations may be recovered also in the case of additive perturbations through the use of the presented extension of Lemma 3, namely Theorem 1, which allows to take into account additive perturbations having poles on the $\mathbb{C}_g$-boundary.

The analysis proposed in the paper can be useful in order to choose which type of perturbation and which type of condition (as well as which value of $\alpha \geq 0$ characterizing $\mathbb{C}_g$) are more suitable in each particular application, in view of the properties of the nominal controlled system.

Finally, it is stressed that results and discussions wholly similar to those presented in this paper, can be developed also for discrete-time systems, considering as $\mathbb{C}_g$-boundary a circle of radius $R$, $0<R\leq 1$, depending on the strength of the stability requirement (which corresponds to asymptotic stability if $R=1$) and/or on motivations similar to those contained in Section 3.2.

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