Exercise 1
Consider the 2 following dynamical systems:

\[
\begin{align*}
\dot{x}_1 &= -y_1 - z_1 \\
\dot{y}_1 &= x_1 + ay_1 \\
\dot{z}_1 &= b + z_1(x_1 - c)
\end{align*}
\]

and

\[
\begin{align*}
\dot{x}_2 &= dx_2 - y_2 + sx_2(x_2^2 + y_2^2) \\
\dot{y}_2 &= x_2 + dy_2 + sy_2(x_2^2 + y_2^2)
\end{align*}
\]

where for system 1, \(b = 2\), \(c = 4\) and \(a \in [0.1, 0.4]\), for system 2, \(s = -1\) and \(d \in [-0.5, 0.5]\), and initial conditions are \(x_1(0) = 1\), \(y_1(0) = 0.4\), \(z_1(0) = 0.4\), \(x_2(0) = 1\) and \(y_2(0) = 0.6\).

Answer to the following questions:
1) Report a brief description of the two systems;
Answer. The first system is the Rossler system (RS), the second system is the normal form system of the Hopf bifurcation (HS).

2) Introduce a coupling term between the variables \(x_1\) and \(x_2\) of the two systems, as follows:

\[
\begin{align*}
\dot{x}_1 &= \cdots + h(x_2 - x_1) \\
\dot{x}_2 &= \cdots + h(x_1 - x_2)
\end{align*}
\]

For each \(h = \{0, 1, 2, 3, 4, 5, 6\}\), report a brief description of the resulting dynamics in the following conditions by varying parameters \(a\) and \(d\):

- System 1 is stable, System 2 is stable;
  Answer. Systems RS and HS remain stable and tend to synchronize. The steady state of HS moves from zero to a positive value.

- System 1 is stable, System 2 is oscillating;
  Answer. System HS is stabilized and tends to synchronize to RS. The steady state of HS moves from zero to a positive value.

- System 1 is oscillating, System 2 is stable;
  Answer. System RS is stabilized and tends to synchronize to HS. The steady state of HS moves from zero to a positive value.
• System 1 is oscillating, System 2 is oscillating;  
  **Answer.** Both RS and HS are stabilized and tend to synchronize for enough high values of $h$. The steady state of HS moves from zero to a positive value.

• If possible, use for System 1 oscillating solutions of period $2nT$;  
  **Answer.** Both RS and HS are stabilized and tend to synchronize for enough high values of $h$. The steady state of HS moves from zero to a positive value.

• System 1 is chaotic, System 2 is stable;  
  **Answer.** System RS is stabilized and tends to synchronize to HS. The steady state of HS moves from zero to a positive value.

• System 1 is chaotic, System 2 is oscillating.  
  **Answer.** Both RS and HS are stabilized and tend to synchronize. The steady state of HS moves from zero to a positive value. For some combinations of the parameters both systems oscillate in a synchronous way.

3) Repeat some of the previous questions by varying also the initial conditions.  
**Answer.** The results are independent on the initial conditions in the neighborhoods of the steady states.

4) Can you summarize all the results obtained in a single sentence?  
**Answer.** The coupling stabilizes the two systems and eliminates oscillations.

5) Discuss the obtained results by comparing them with the solutions of the uncoupled systems.  
**Answer.** Stable steady states are stronger than oscillations and period one oscillations are stronger than period $2nT$ oscillations and chaotic oscillations.