RECALL: \( \dot{x} = f(x) \quad x \in \mathbb{R}^n \) (1)

A solution \( x(t) \) is a function \( x : I \to \mathbb{R}^n \) satisfying (1).

\( f(x) \) is a function \( f : U \to \mathbb{R}^n \) \( U \subset \mathbb{R}^n \)

\( f(x) \) defines a flow on \( \mathbb{R}^n \) such that for any \( x \in \mathbb{R}^n \) the velocity field is defined by the vector \( \dot{x} = (\dot{x}_1, \dot{x}_2, ..., \dot{x}_n) = f(x) \).

Given an initial condition \( x(t_0) = x_0 \) (usually \( t_0 = 0 \)) and under the hypothesis that \( f \) is differentiable with continuous derivative a solution of (1) exists and this solution is unique.

Equilibria or steady states of (1) are constant solutions \( x^* \) such that

\[ \dot{x}^* = f(x^*) = 0 \]

An equilibrium solution is said Lyapunov stable if all trajectories that start sufficiently close to \( x^* \) remain close to it.
for all time.
If $x^*$ is also attracting ($x(t) \to x^*$, $t \to \infty$) we call it asymptotically stable (or stable).
$x^*$ is unstable if it is not attracting nor Lyapunov stable.

Examples in 2 order systems.
the role of nullclines.
The nullclines are the curves along which $\dot{x} = 0$ or $\dot{y} = 0$. Along the nullclines the flow is purely horizontal ($\dot{y} = 0$) or vertical ($\dot{x} = 0$).
EXAMPLE OF STABLE NODE (negative eigenvalues)

IN BLACK: THE AXES (st. variables)
IN GREEN: NULLCLINES
IN RED: EIGENVALUES (invariant subspaces)
Ordine Lineare

\[
\begin{align*}
\dot{x}_1 &= f_1(x_1, x_2) \\
\dot{x}_2 &= f_2(x_1, x_2)
\end{align*}
\]
\[
(x_1^*, x_2^*) \text{ s.s.}
\]

\[
j = d
\]
\[
f_1(x_1^*, x_2^*) = 0 \\
f_2(x_1^*, x_2^*) = 0
\]

\[
u = x_1 - x_1^* \\
v = x_2 - x_2^*
\]

\[
\dot{u} = u \\
\dot{v} = v
\]

\[
f_1(x_1, x_2) = f_1(u + x_1^*, v + x_2^*) =
\]
\[
= f_1(x_1^*, x_2^*) + u \frac{df_1}{dx_1} \bigg|_{x_1^*} + v \frac{df_1}{dx_2} \bigg|_{x_2^*} + O(x_1^2, x_2^2, u, v)
\]

\[
= u \frac{df_1}{dx_1} \bigg|_{x_1^*} + v \frac{df_1}{dx_2} \bigg|_{x_2^*}
\]

\[
f_2(x_1, x_2) = f_2(x_1^*, x_2^*) + u \frac{df_2}{dx_1} \bigg|_{x_1^*} + v \frac{df_2}{dx_2} \bigg|_{x_2^*} + O(x_1^2, x_2^2, u, v)
\]

\[
= u \frac{df_2}{dx_1} \bigg|_{x_1^*} + v \frac{df_2}{dx_2} \bigg|_{x_2^*}
\]

\[
\begin{pmatrix}
\dot{u} \\
\dot{v}
\end{pmatrix} = \begin{pmatrix}
\frac{df_1}{dx_1} & \frac{df_1}{dx_2} \\
\frac{df_2}{dx_1} & \frac{df_2}{dx_2}
\end{pmatrix} \begin{pmatrix}
x_1^* \\
x_2^*
\end{pmatrix} \begin{pmatrix}
u \\
v
\end{pmatrix}
\]
There are **ROBUST CASES**

**REPELERS** (both eigenvalues have positive real parts)

**ATTRACTION** (both eigenvalues have negative real parts)

**SADDLES** one eigen. is positive and one is negative

There are **MARGINAL CASES**

**CENTERS** both eigen. are purely imaginary

**NON ISOLATED FIXED POINTS**.