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Synthesis of robust controllers for uncertain plants with rank one real perturbations

Ph.D. Thesis

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Notation

$\mathbb{R}^n$	:	real n-space
$v \in \mathbb{R}^n$	:	vector of $\mathbb{R}^n$
v'	:	transpose of v
$\|v\|$	:	norm of v
$\|v\|_p$	:	p-norm of v
$\mathrm{span}\{v_1, \ldots, v_m\}$	:	vector space generated by $v_1, \ldots, v_m$
$\mathbb{C}$	:	the complex plane
$s \in \mathbb{C}$	:	complex number
$\mathrm{Re}[s], \mathrm{Im}[s]$	:	real and imaginary parts of s
$\arg[s]$	:	argument of s
$\mathrm{Res}[\Phi(s), s_0]$	:	residue of the function $\Phi(s)$ in $s_0 \in \mathbb{C}$
$\mathcal{S}$	:	the open left half of the complex plane a.k.a. the Hurwitz stability region
$\partial\mathcal{S}$	:	the boundary of the Hurwitz stability region i.e., the imaginary axis
$\mathcal{U}$	:	the closed right half of the complex plane, i.e., $\mathcal{U} = \mathbb{C} - \mathcal{S}$
$\mathcal{U}_0$	:	the open right half of the complex plane
$P(s)$	:	a polynomial in the complex variable s
∂P	:	degree of a polynomial $P(s)$
$[P(s)]_o$	:	polynomial containing only the odd powers of $P(s)$
$\mathcal{H}$	:	the set of Hurwitz polynomials
$\mathcal{RH}_\infty$	:	the set of proper (bounded at infinity) rational functions with real coefficients and without poles in the right half plane

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Chapter 1

Introduction

1.1 A little history

The aim of robust control theory is that of providing a set of design techniques ensuring the invariance of certain essential properties of the control system under the effect of perturbations, disturbances, and model uncertainty. Most classical methods such as frequency domain design may inherently guarantee limited robustness with respect to small perturbations. With these methods, however, no bounds are provided such that stability and performance requirements are preserved as long as the uncertainty lies within them. The need to fill this void has led to the development of new techniques which explicitly take suitable measures of the perturbation into account.

H_∞ optimal control, which was first proposed by Zames [18],[42], explicitly deals with robustness issues. This approach basically assumes that the disturbance signals which affect a system are actually the *worst* disturbances belonging to a prescribed class, and exploits the fact that the H_∞ norm of the sensitivity operator represents the energy gain from the disturbances to the system outputs. By means of this technique, a controller can be designed so that an a-priori bound is enforced on the effects of exogenous signals.

Along with the H_∞ theory, a number of frequency domain results concerning robust stabilization with respect to norm-bounded perturbations was given by Kimura and Glover [24],[39],[19]. Anyway, the so-called *nonparametric unstructured uncertainty model* employed in this context completely disregards all relationship between bounds on the real parameters of the system transfer function and the norm bounds in the frequency domain.

Dealing with *real parametric uncertainty* in robustness analysis and robust control synthesis was considered an extremely difficult issue until the advent of Kharitonov's result concerning Hurwitz stability of interval polynomials [23]. Until then, the only approaches to this kind of problems were based on standard optimization techniques [32] or conservative parameter overbounding. Kharitonov's theorem showed that the stability of a family of polynomials of arbitrary degree corresponding to a box in coefficient space ("interval polynomial") is equivalent to the stability of four prescribed vertex polynomials only. This was indeed a breakthrough because the apparently impossible task of verifying the stability of a continuum of systems was shown to boil down to the simple application of the Routh-Hurwitz criterion to a number of polynomials which is finite and independent of the order.

The appearance of Kharitonov's theorem led to a renewed interest in the study of robustness with respect to real parametric uncertainty; it finally became clear that the robust control problem could be approached with computational ease and without conservatism or overbounding, since easy-to-handle geometrical and algebraic properties of the stability region in parameter space could be exploited to develop efficient methods, as opposed to blind optimization problems.

An approach somehow opposite to Kharitonov's was proposed by Soh et al. [33]: in this setting, the largest stability ball for polynomials in the coefficient space is computed around a nominal stable polynomial $P_0(s)$. The vector space of all polynomials of degree n is identified with $\mathbb{R}^n$ equipped with its standard Euclidean norm, and the largest stability ball around $P_0(s)$ is defined as the hyphersphere containing all stable polynomials and with at least one unstable polynomial lying on its boundary. Biernacki et al. [10] extended these results by computing the largest stability ball in the space of parameters appearing linearly or affinely in the expression of a transfer function (the so-called *parametric stability margin*). A numerical procedure was also given to calculate such stability radius. Other important analysis results dealing with stability margin computation were given by Fan et al. [17] and Qiu et al. [29].

Another significant result in this field was the Edge Theorem by Bartlett et al. [3], in which a family of polynomials whose coefficients belong to an arbitrary polytope in $\mathbb{R}^n$ was considered. It was proved that the root space of the family is bounded by the root loci of the exposed edges of the polytope. In particular, the family is stable if and only if all the edges are stable.

Finally, a generalized version of the Kharitonov theorem [12] provided necessary and sufficient conditions for robust stability of feedback interval control systems. Moreover it was shown that all the relevant parameters of the behaviour of the uncertain family (stability, frequency domain plot envelopes, mixed uncertainty stability and performance margins) are characterized by an extremal set of systems.

1.2 Parametric robust control problems

The above results have laid down the foundation of the *parametric approach* to robustness in control system analysis and design. Apart from the contributions discussed in the previous section, which are mainly concerned with analysis issues such as stability margin computation, several approaches aimed at the design of robust controllers with respect to parametric uncertainty are present in the literature. Some heuristic methods such as the so-called *D-K iteration* [15] and the *Quantitative feedback theory* [20] have been proposed. Moreover, some problems involving a single uncertain parameter have been approached by means of Nevanlinna-Pick interpolation using a conformal mapping [34],[22],[16]. This way, gain and phase margin maximization for scalar systems can be carried out.

Within the framework of parametric robust synthesis, a problem which has received considerable attention is that of designing controllers maximizing the stability margin. It has been shown that such optimization over the class of controllers stabilizing the nominal plant is in general a non convex problem. In [31], Rantzer and Megretski proved that for a significant set of uncertain systems, such optimization is indeed convex for a suitable controller parameterization. Despite this remarkable result, several design issues still deserve some attention. In particular, since the optimization problem in [31] is infinite dimensional, suboptimal solutions must be looked for through finite dimensional convex programming. Hence, a suitable structure for the approximating solution must be found. This choice obviously affects the computational burden and the complexity of the resulting controller. Moreover, for implementation reasons, it is often needed to select the "optimal" controller within a prescribed set such as the class of PID or lag-lead compensators.

In this thesis we consider the class of uncertain linear systems with rank one real perturbations. For this class, we investigate the problem of maximizing the l_2 parametric stability margin over a set of controllers described by a limited number of free parameters. Many widely used controller structures are indeed included in this set.

The main contribution of this work is to show that such a problem can be easily solved provided that a suitable overparameterization is employed. Accomplishing this task basically relies on the possibility of designing a filter ensuring the robust strict positive realness ($RSPR$) property to a family of uncertain transfer functions; once a procedure for computing such a filter is known, we will show that the maximization of the stability margin can be carried out via the solution of a sequence of LMI feasibility problems with respect to the controller parameters.

The above mentioned problem of robust SPR synthesis will also be addressed in this thesis. As opposed to the results available in the literature, it will be shown that filters satisfying the $RSPR$ requirements for systems with l_2 real parametric perturbations can be easily designed in closed form. The proposed procedure exploits the properties of a suitable polynomial factorization to derive the solution.

1.3 Outline of the work

We will first present a brief review of some fundamental results of the parametric theory of robust control, with emphasis on the characterization of robust stability and the computation of the l_2 parametric stability margin (Chapter 2). We will then focus on the class of uncertain control systems with rank-one real perturbations. For this class of systems, a characterization of robustly stabilizing controllers will be discussed and the problem of l_2 parametric stability margin maximization via control design will be analyzed (Chapter 3). It will be shown that this problem enjoys the convexity property, though its solution in the general case presents noticeable computational difficulties and practical implementation problems. In Chapter 4, we will present a new approach aimed at overcoming such issues. The l_2 stability margin maximization problem within a class of restricted complexity controllers will be considered and its solution will be presented in the form of a simple iterative procedure involving, at each step, the solution of a LMI feasibility problem and the synthesis of a robust strictly positive-real filter. Finally, some analysis will be conducted on convergence and optimality properties of the proposed algorithm. The results related to l_2 robust SPR synthesis will be reported and discussed in Appendix A. Appendix B reports the proofs of some technical results.

Chapter 2

Robustness problems. The parametric approach

2.1 Characterization of robust stability of uncertain polynomials

For the purpose of this work, we will first focus on robust Hurwitz stability of a family of real polynomials described by a real parameter vector.
Let

$$\mathcal{P}_\delta = \{P(s;\delta) \quad : \quad \delta \in \mathcal{D} \subseteq \mathbb{R}^n\} \tag{2.1}$$

be a set of real polynomials of degree l in the variable s depending continuously on the real parameter vector δ. We recall that robust Hurwitz stability of the set $\mathcal{P}_\delta$ is chatacterized as

$$P(s;\delta) \in \mathcal{H} \;\; \forall \delta \in \mathcal{D} \tag{2.2}$$

which is to say that $P(s;\delta)$ has all its roots belonging to the stability region $\mathcal{S}$ for all $\delta \in \mathcal{D}$, i.e.,

$$\forall \delta \in \mathcal{D}, \quad P(s;\delta) = 0 \quad \Rightarrow \quad s \in \mathcal{S}. \tag{2.3}$$

Given a set $\mathcal{P}_\delta$, we want to characterize the presence of unstable polynomials in the family.

2.1.1 The Boundary Crossing Theorem

Let $P(s;\lambda)$ be a family of polynomials, depending on a scalar parameter λ, satisfying the following assumption:

Assumption 2.1 $P(s;\lambda)$ is such that

1. $\partial P(s;\lambda) = l \;\; \forall \lambda$,

2. $P(s;\lambda)$ is continuous with respect to λ on a fixed interval $I = [a, b]$.

It can be shown (see e.g. [5], p. 32-34) that for any open subset $\mathcal{O}$ of the complex plane, the set of polynomials of fixed degree l with a number $r \leq l$ of roots in $\mathcal{O}$ is itself an open set. This means in particular that if, for some $\lambda \in I$, $P(s;\lambda)$ has all its roots in $\mathcal{S}$, then there exists $\varepsilon > 0$ such that $P(s;\lambda')$ has all roots in $\mathcal{S}$ for all $\lambda' \in (\lambda - \varepsilon, \lambda + \varepsilon)$. This leads to the following result.

Theorem 2.1 (Boundary Crossing Theorem) *Let $P(s;\lambda)$ satisfy Assumption 2.1. Suppose $P(s;a) \in \mathcal{H}$ (i.e., it has all its roots in the stability region $\mathcal{S}$) and $P(s;b)$ has at least one root in the instability region $\mathcal{U}$. Then, there exists at least one $\bar{\lambda} \in (a,b]$ such that*

1. *$P(s;\bar{\lambda})$ has all its roots in $\mathcal{S} \cup \partial\mathcal{S}$,*
2. *$P(s;\bar{\lambda})$ has at least one root in $\partial\mathcal{S}$.*

Proof: See Appendix B.

The above result basically states that when going from the open set $\mathcal{S}$ to the open set $\mathcal{U}_0$, disjoint from the first, the root set of a polynomial $P(s;\lambda)$ depending continuously on the parameter λ must intersect the boundary of $\mathcal{S}$. This result does not hold in general (i.e., boundary crossing may not occur) if $P(s;\lambda)$ loses degree for some $\lambda \in [a,b]$, since some roots may move from $\mathcal{S}$ to $\mathcal{U}_0$ through the point at infinity.

2.1.2 The Zero Exclusion Principle

The Boundary Crossing Theorem can be applied to a family of uncertain polynomials to detect the presence of polynomials with unstable roots.
Consider the polynomial family $\mathcal{P}_\delta$ in (2.1). Let us first introduce the notion of *value set*.

Definition 2.1 Given the polynomial family $\mathcal{P}_\delta$ in (2.1) and a complex number s, the *value set* of $\mathcal{P}_\delta$ evaluated in s is a subset of the complex plane defined as

$$\Delta(s) = \{P(s;\delta) \ : \ \delta \in \mathcal{D}\}. \tag{2.4}$$

Let us assume that $\mathcal{P}_\delta$ contains at least one stable polynomial $P(s;\delta_0) \in \mathcal{H}$, $\delta_0 \in \mathcal{D}$. Since all polynomials in the family are supposed to be of the same degree l, if $P(s;\delta_1) \in \mathcal{P}_\delta$ is an unstable polynomial (i.e., it has some roots in $\mathcal{U}$), then it follows from Theorem 2.1 that on any continuous path connecting δ_0 to δ_1 there exists $\bar{\delta}$ such that $P(s;\bar{\delta})$ has at least one root on the stability boundary $\partial\mathcal{S}$. If $\mathcal{D}$ is pathwise connected (i.e., for any

$\delta_1, \delta_2 \in \mathcal{D}$ there exists a continuous path from δ_1 to δ_2 entirely contained in $\mathcal{D}$), then $\bar{\delta} \in \mathcal{D}$. Therefore, the family $\mathcal{P}_\delta$ contains unstable polynomials if and only if it contains at least one polynomial with roots on the stability boundary. If s^* is any root of a polynomial in the family (i.e., $P(s^*; \delta) = 0$ for some δ), then the value set $\Delta(s^*)$ must contain the origin. Hence, recalling Theorem 2.1, we get that detecting the presence of unstable polynomials in $\mathcal{P}_\delta$ amounts to evaluating the value set $\Delta(s)$ along the stability boundary $\partial\mathcal{S}$ and checking if the zero exclusion condition $0 \notin \Delta(s)$ is violated for some $s \in \partial\mathcal{S}$. In other words, we have the following result:

Theorem 2.2 (Zero exclusion principle) *Given the degree-invariant uncertain polynomial family $\mathcal{P}_\delta$ in (2.1) with $\mathcal{D}$ pathwise connected, suppose there exists $\delta_0 \in \mathcal{D}$ such that $P(s; \delta_0) \in \mathcal{H}$. Then, the entire family is stable ($\mathcal{P}_\delta \subseteq \mathcal{H}$) if and only if*

$$0 \notin \Delta(s) \quad \forall s \in \partial\mathcal{S}. \tag{2.5}$$

2.1.3 Z.E.P. - The affine case

We will now derive a formulation of the Zero Exclusion Principle in the case of $\mathcal{P}_\delta$ being an uncertain polynomial family with invariant degree l depending affinely on a parameter vector. For this purpose, let

$$\mathcal{P}_\delta = \left\{ P(s;\delta) = P_0(s) + \sum_{i=1}^{n} \delta_i P_i(s) \quad : \quad \delta \in \mathcal{D} \subseteq \mathbb{R}^n \right\} \tag{2.6}$$

where $P_0(s), P_1(s), \ldots, P_n(s)$ are given real polynomials such that $\partial P_0 = l$, $\partial P_i < l$ for all $i = 1, \ldots, n$ and $\mathcal{D}$ is pathwise connected. The polynomial $P_0(s)$ will be referred to as the *nominal polynomial* of the family and $P(s;\delta) - P_0(s)$ as the *perturbation term*. Enforce the following assumption on $\mathcal{P}_\delta$:

Assumption 2.2 The nominal polynomial of $\mathcal{P}_\delta$ is Hurwitz, i.e., $P_0(s) \in \mathcal{H}$.

Obviously, $P(s;\delta)$ depends continuously on δ. The value set is given by

$$\Delta(s) = \left\{ P_0(s) + \sum_{i=1}^{n} \delta_i P_i(s) \quad : \quad \delta \in \mathcal{D} \right\}. \tag{2.7}$$

Let us introduce the vector

$$G(s) = -\left[\frac{P_1(s)}{P_0(s)}, \ldots, \frac{P_n(s)}{P_0(s)} \right]. \tag{2.8}$$

The Zero Exclusion Principle holds and since $P_0(s) \in \mathcal{H}$ (and therefore it has no zeroes for $s \in \partial\mathcal{S}$), condition (2.5) in Theorem 2.2 characterizing the stability of the whole family can be expressed as

$$1 - \delta' G(s) \neq 0 \quad \forall s \in \partial\mathcal{S} \quad \forall \delta \in \mathcal{D}. \tag{2.9}$$

Since all polynomials are real, the above condition is equivalent to

$$1 - \delta' G(j\omega) \neq 0 \quad \forall \omega \geq 0 \quad \forall \delta \in \mathcal{D}. \tag{2.10}$$

2.2 *The stability ball in coefficient space*

A central problem in parametric robust stability is that of evaluating, in the parameter space, the largest region of a prescribed shape where the stability property is preserved, around a given nominal stable polynomial. The problem can be approached with ease by invoking the Boundary Crossing Theorem and the Zero Exclusion Principle as long as the parameter region of interest can be associated with a norm.

Consider the set $\mathcal{P}_l$ of the real polynomials of degree at most l. Such a set is a vector space which is isomorphic to $\mathbb{R}^{l+1}$. Let $\|\cdot\|$ be any norm on $\mathcal{P}_l$.

Definition 2.2 Given a *nominal* polynomial $P_0(s) \in \mathcal{P}_l$ and a scalar $\rho > 0$, the set

$$\mathcal{B}(P_0, \rho) = \{P(s) \in \mathcal{P}_l \quad : \quad \|P(s) - P_0(s)\| < \rho\} \tag{2.11}$$

is called the *open ball* of radius ρ centered in $P_0(s)$ induced by the norm.

Suppose $P_0(s) \in \mathcal{H}$ and has degree l. Since the set of polynomials of degree l with all roots in $\mathcal{S}$ is an open set, there exists $\varepsilon > 0$ such that the open ball $\mathcal{B}(P_0, \varepsilon)$ satisfies the following property:

Property 2.1 $\mathcal{B}(P_0, \varepsilon)$ is such that

1. Every $P(s) \in \mathcal{B}(P_0, \varepsilon)$ has degree l,

2. $\mathcal{B}(P_0, \varepsilon) \subseteq \mathcal{H}$.

It can be easily checked that the set

$$R(P_0) = \{\rho > 0 \quad : \quad \mathcal{B}(P_0, \rho) \ \text{ satisfies Property 2.1}\} \tag{2.12}$$

is an interval $(0, \rho^*(P_0)]$ where

$$\rho^*(P_0) = \sup_{\rho \in R(P_0)} \rho. \tag{2.13}$$

Hence, we have the following result characterizing the *stability ball* around any stable polynomial (see [5], p. 123 for a complete proof).

Theorem 2.3 *Given a polynomial $P_0(s) \in \mathcal{H}$ with degree l and a norm on $\mathcal{P}_l$, there exists $\rho(P_0) > 0$ such that*

1. *Every polynomial $P(s) \in \mathcal{B}(P_0, \rho(P_0))$ is Hurwitz and is of degree l.*

2. *At least one polynomial on the boundary $\partial\mathcal{B}(P_0, \rho(P_0))$ has one of its roots in $\partial\mathcal{S}$ or is of degree less than l.*

3. *No polynomial $P(s) \in \partial\mathcal{B}(P_0, \rho(P_0))$ has roots in the open right half plane $\mathcal{U}_0$.*

2.2.1 Computing the l_2 Hurwitz Stability Ball

Let $\mathcal{P}_l$ be the set of polynomials of degree at most l with the usual inner product $< P(s), R(s) >$ and associated Euclidean (l_2) norm $\|P(s)\|_2$, i.e., if

$$P(s) = \sum_{j=0}^{l} p_j s^j \quad ; \quad R(s) = \sum_{j=0}^{l} r_j s^j \tag{2.14}$$

then

$$< P(s), R(s) > \; = \; \sum_{j=0}^{l} p_j r_j \tag{2.15}$$

and

$$\|P(s)\|_2^2 \; = \; \sum_{j=0}^{l} p_j^2. \tag{2.16}$$

Let us introduce the following subspaces of $\mathcal{P}_l$:

- The subspace Δ_0 of dimension l of all $P(s) \in \mathcal{P}_l$ such that $P(0) = 0$, i.e.,

$$\Delta_0 = \text{span}\left\{s, s^2, s^3, \ldots, s^l\right\}; \tag{2.17}$$

- The subspace Δ_l, also of dimension l, of all $P(s) \in \mathcal{P}_l$ having degree less than l, i.e.,

$$\Delta_l = \text{span}\left\{1, s, s^2 \ldots, s^{l-1}\right\}. \tag{2.18}$$

- If $l \geq 2$ and for each $\omega \geq 0$, the subspace Δ_ω of all $P(s) \in \mathcal{P}_l$ such that $s^2 + \omega^2$ is a factor of $P(s)$. Equivalently, Δ_ω is the subset of all elements in $\mathcal{P}_l$ containing $\pm j\omega$ among their roots. This subspace has dimension $l-1$ and

$$\Delta_\omega = \text{span}\left\{s^2 + \omega^2, s(s^2+\omega^2), s^2(s^2+\omega^2), \ldots, s^{l-2}(s^2+\omega^2)\right\}. \tag{2.19}$$

Since $\mathcal{P}_l$ is an Euclidean vector space, for any $P(s) \in \mathcal{P}_l$ and any subspace $\Delta \subseteq \mathcal{P}_l$, there exists a unique polynomial $\pi_{P|\Delta}(s) \in \Delta$, called the orthogonal projection of $P(s)$ on Δ, at which the distance from $P(s)$ to all elements of Δ is minimized. Hence, $\|P(s) - \pi_{P|\Delta}(s)\|_2$ is called the distance from $P(s)$ to the subspace Δ.
Consider a Hurwitz polynomial $P_0(s) \in \mathcal{P}_l$

$$P_0(s) = \sum_{j=0}^{l} p_j^0 s^j \tag{2.20}$$

and let d_0, d_l, and d_ω be the distances from $P_0(s)$ to Δ_0, Δ_l, and Δ_ω, respectively, i.e.,

$$\begin{aligned} d_0 &= \|P_0(s) - \pi_{P_0|\Delta_0}(s)\|_2 \\ d_l &= \|P_0(s) - \pi_{P_0|\Delta_l}(s)\|_2 \\ d_\omega &= \|P_0(s) - \pi_{P_0|\Delta_\omega}(s)\|_2. \end{aligned} \tag{2.21}$$

Moreover, let

$$\bar{d} = \inf_{\omega \geq 0} d_\omega \tag{2.22}$$

By Theorem 2.3, we have that every polynomial in $\mathcal{B}(P_0, \rho(P_0))$ is stable and of degree l, whereas there exists at least one polynomial in $\partial\mathcal{B}(P_0, \rho(P_0))$ with degree less than l or with at least one root on the imaginary axis. This fact leads to the following characterization of the stability radius $\rho(P_0)$.

Theorem 2.4 *The radius of the largest stability ball around a Hurwitz polynomial $P_0(s)$ is given by*

$$\rho(P_0) = \min\{d_0, d_l, \bar{d}\}. \tag{2.23}$$

It is quite straightforward to verify that

$$d_0 = |p_0^0| \quad ; \quad d_l = |p_l^0|. \tag{2.24}$$

To compute $\bar{d}$, let us first characterize d_ω. Rewrite $P_0(s)$ as

$$P_0(s) = P_0^e(s) + P_0^o(s) \tag{2.25}$$

where $P_0^e(s)$ and $P_0^o(s)$ denote the even and odd degree terms in $P_0(s)$, respectively.

Lemma 2.1 *The distance* d_ω *between* $P_0(s)$ *and* Δ_ω *is given by*

$$d_\omega = \begin{cases} \dfrac{[P_0^e(\omega)]^2}{1+\omega^4+\cdots+\omega^{4k}} + \dfrac{[P_0^o(\omega)]^2}{1+\omega^4+\cdots+\omega^{4(k-1)}} & \text{if } \; l = 2k \\[2ex] \dfrac{[P_0^e(\omega)]^2 + [P_0^o(\omega)]^2}{1+\omega^4+\cdots+\omega^{4k}} & \text{if } \; l = 2k+1 \end{cases} \tag{2.26}$$

Proof: See Appendix B.

Lemma 2.1 provides a closed form expression for d_ω given the coefficients of $P_0(s)$. Finally,

$$\bar{d} = \inf_{\omega \geq 0} d_\omega. \tag{2.27}$$

It can be shown that for computing $\bar{d}$ there is no need to solve the minimization problem (2.27) over the infinite frequency range $[0, +\infty)$. A simple manipulation (see [5], p. 127) indeed yields the equivalence of (2.27) to the following optimization:

$$\bar{d}^2 = \min \left\{ \inf_{\omega \in [0,1]} d_\omega^2, \inf_{\omega \in [0,1]} d_{\frac{1}{\omega}}^2 \right\} \tag{2.28}$$

where $d_{\frac{1}{\omega}}^2$ is d_ω^2 computed for the polynomial $s^l P_0(1/s)$, whose coefficients are those of $P_0(s)$ in reverse order.

Note that for $l = 1$ the set Δ_ω is not properly defined, since perturbing the coefficients cannot yield imaginary roots. In that case

$$\rho(P_0) = \min\{d_0, d_l\} = \min\{|p_0^0|, |p_1^0|\}. \tag{2.29}$$

2.3 The parametric stability margin

In the previous section we have introduced and characterized the notion of stability ball in coefficient space under the assumption that all coefficients of the uncertain polynomial can be perturbed independently. Such a framework is of little use in practical control problems. In general, testing robust stability of a control system reduces to the analysis of the root location of a closed-loop characteristic polynomial whose coefficients are not the uncertain parameters themselves but, more likely, linear or affine functions of such parameters. The results in the previous section can be extended in order to take into account interdependent perturbations.

Consider the standard feedback control system configuration in Fig. 2.1. Let the

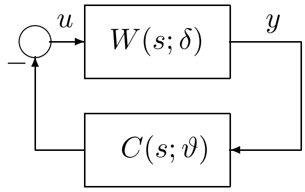


Figure 2.1: Standard control loop

uncertain plant $W(s;\delta)$ be defined by the real perturbation vector $\delta \in \mathcal{D}$ and let $W_0(s) = W(s;0)$ be the nominal plant model. Moreover, let the controller $C(s;\vartheta)$ have a given structure and be defined by a vector of tunable parameters $\vartheta \in \Theta$. Suppose a given controller $C(s;\vartheta)$ stabilizes the closed loop system in nominal conditions. When perturbations are present, a natural question that arises is how large such perturbations can be in order to preserve closed loop stability. In other words, it is extremely useful to give a bound on the size of δ for which stability is not compromised. Such a bound can be provided by evaluating the stability ball in the uncertain parameter space. Accordingly, for a fixed controller $C(s;\vartheta)$, we define the *parametric stability margin* of the control system as a suitable norm of the smallest perturbation δ which destabilizes the closed loop. The computation of the stability margin provides an analysis tool for measuring the robustness of the control system designed in nominal conditions as well as comparing the robust performance of several proposed controllers. In the synthesis context, on the other hand, one may have to deal with the problem of designing the structure of the controller and adjusting its tunable parameters, in order to achieve an increase or the maximization of the stability margin.

We start with the characterization of the stability margin of an uncertain polynomial depending on a parameter vector [5].
Let $P(s;\delta)$ be a polynomial of degree l whose coefficients are parameterized by a real vector $\delta \in \mathcal{D} \subseteq \mathbb{R}^n$

$$P(s;\delta) = \sum_{j=0}^{l} p_j(\delta) s^j. \tag{2.30}$$

where $p_i(\delta),\ i = 0,\ldots,l$ are continuous functions of δ. Refer to $P_0(s) = P(s;0)$ as the nominal polynomial and suppose $P_0(s) \in \mathcal{H}$. Let $\|\cdot\|$ be a norm in parameter space.

Introduce the set of polynomials of degree l

$$\mathcal{P}_\rho = \{P(s;\delta) \quad : \quad \|\delta\| < \rho\}\,. \tag{2.31}$$

Definition 2.3 The real parametric stability margin ρ^* is defined as the largest ρ such that $P(s;\delta)$ is stable whenever $\|\delta\| < \rho$, i.e.,

$$\rho^* = \sup_{\mathcal{P}_\rho \subseteq \mathcal{H}} \rho. \tag{2.32}$$

As the above definition states, the parametric stability margin is nothing but the maximal stability ball in parameter space, as opposed to the stability ball in coefficient space introduced in the previous section. As such, a result characterizing the stability margin can be given [5] which parallels Theorem 2.3.

Theorem 2.5 *Given the uncertain polynomial $P(s;\delta)$ in (2.30), the parametric stability margin is characterized as follows:*

1. *There exists a maximal ρ^* such that*
   - *For all $\|\delta\| < \rho^*$, $P(s;\delta) \in \mathcal{H}$ and is of degree l;*
   - *There exists at least one $\bar{\delta}$ such that $\|\bar{\delta}\| = \rho^*$ and the polynomial $P(s;\bar{\delta})$ is either unstable or of degree less than l;*
2. *If for some $\bar{\delta}$ with $\|\bar{\delta}\| = \rho^*$ the polynomial $P(s;\bar{\delta})$ is unstable, then its unstable roots must lie on the stability boundary $\partial\mathcal{S}$.*

2.3.1 Computing the parametric stability margin

Consider the polynomial family $\mathcal{P}_\rho$ in (2.31) and define the value set

$$\Delta_\rho(s) = \{P(s;\delta) \quad : \quad \|\delta\| < \rho\}\,. \tag{2.33}$$

Let the nominal polynomial $P_0(s)$ be Hurwitz. If the family $\mathcal{P}_\rho$ is of constant degree l, then, by continuity of the roots of $P(s;\delta)$ with respect to δ, the Zero Exclusion Principle applies and the stability of $\mathcal{P}_\rho$ is equivalent to the condition

$$0 \notin \Delta_\rho(s) \quad \forall s \in \partial\mathcal{S}. \tag{2.34}$$

Such a condition can be successfully exploited in order to compute the value of the stability margin ρ^*. Since $P_0(s)$ is Hurwitz and the stability region $\mathcal{S}$ is an open set, for small values of ρ the value set $\Delta_\rho(s)$ does not contain the origin for any $s \in \partial\mathcal{S}$. By increasing ρ, a value ρ^* may be reached for which either one of the polynomials in $\mathcal{P}_{\rho^*}$ loses degree or acquires a root on the stability boundary $\partial\mathcal{S}$. According to Theorem 2.5, this limiting value is indeed the parametric stability margin. If such a value is not reached, the stability margin is equal to infinity.

Conversely, let s belong to the stability boundary and let ρ_s denote the limiting value of ρ such that the value set $\Delta_\rho(s)$ contains the origin, i.e.,

$$\rho_s = \inf_{0 \in \Delta_\rho(s)} \rho. \tag{2.35}$$

Define

$$\bar{\rho} = \inf_{s \in \partial\mathcal{S}} \rho_s. \tag{2.36}$$

Clearly, $\bar{\rho}$ represents the limiting value of ρ for which some polynomial in the family acquires a root on the stability boundary. Moreover, let ρ_d be the limiting value of ρ for which some polynomial in $\mathcal{P}_\rho$ has degree less than l, i.e.,

$$\begin{aligned} \rho_d &= \inf \rho \\ &\text{s.t.} \\ p_l(\delta) = 0 \quad &; \quad \|\delta\| = \rho. \end{aligned} \tag{2.37}$$

We have the following result.

Theorem 2.6 *The parametric stability margin of the polynomial family $\mathcal{P}_\rho$ is given by*

$$\rho^* = \min\{\bar{\rho}, \rho_d\}. \tag{2.38}$$

According to Theorem 2.6, computing the stability margin ρ^* amounts to going through the following steps:

1. calculate the value of ρ_s at each $s \in \partial\mathcal{S}$;

2. compute $\bar{\rho}$ by taking the minimum of ρ_s over the stability boundary $\partial\mathcal{S}$;

3. compute ρ_d;

4. set $\rho^* = \min\{\bar{\rho}, \rho_d\}$.

In general, calculating ρ_s in step 1 is not an easy task. However, a simple closed form solution can be given provided that the parameters δ enter the coefficients of $P(s;\delta)$ in a linear or affine fashion. Moreover, We will see how this problem reduces to a simple least square computation in the case when the l_2 parameter norm is considered. The dependency of ρ_s on s is in general strongly nonlinear. However, the minimization in step 2 can be accomplished with relative ease since it reduces to a one-dimensional sweep along the stability boundary $\partial\mathcal{S}$.

2.3.2 Computing the parametric stability margin: linear case

Consider an uncertain polynomial family of degree l of the form (2.31) and assume the coefficients $p_i(\delta)$ to be linear or affine functions of the parameter vector $\delta \in \mathbb{R}^n$. Clearly, $\mathcal{P}_\rho$ can be rewritten in a form similar to that in (2.6)

$$\mathcal{P}_\rho = \left\{ P(s;\delta) = P_0(s) + \sum_{i=1}^{n} \delta_i P_i(s) \quad : \quad \|\delta\| < \rho \right\}. \tag{2.39}$$

As usual, let the nominal polynomial of the family $P_0(s)$ be stable and of degree l.
In order for the origin to belong to the value set $\Delta_\rho(s)$ with $s \in \partial\mathcal{S}$, there must exist δ with $\|\delta\| < \rho$ such that

$$P_0(s) + \sum_{i=1}^{n} \delta_i P_i(s) = 0 \tag{2.40}$$

or equivalently

$$1 - \delta' G(s) = 0 \tag{2.41}$$

with $G(s)$ defined as in (2.8). Hence,

$$\begin{gathered} \rho_s = \inf \rho \\ \text{s.t.} \\ 1 - \delta' G(s) = 0 \ \ ; \ \ \|\delta\| = \rho. \end{gathered} \tag{2.42}$$

Similarly, corresponding to a loss of degree in $P(s;\delta)$ we have

$$p_l(\delta) = 0. \tag{2.43}$$

Let

$$P_i(s) = \sum_{j=0}^{l} p_j^i s^j, \tag{2.44}$$

equation (2.43) can be rewritten as

$$p_l^0 + \sum_{i=1}^{n} \delta_i p_l^i = 0. \tag{2.45}$$

Hence, from (2.37) we get the minimization problem

$$\begin{gathered} \rho_d = \inf \rho \\ \text{s.t.} \\ 1 - \delta' g = 0 \;\;;\;\; \|\delta\| = \rho \end{gathered} \tag{2.46}$$

where

$$g = -\left[\frac{p_l^1}{p_l^0}, \ldots, \frac{p_l^n}{p_l^0}\right]' \tag{2.47}$$

which has the same form as (2.42) except for the fact that (2.42) is complex and (2.46) is real.

2.3.3 Computing the l_2 parametric stability margin

We now focus on the case of $\|\cdot\|$ being the l_2 norm in parameter space.

Let us evaluate $G(s)$ along the Hurwitz stability boundary. Since all coefficients are real, it suffices to examine the upper half of the imaginary axis, i.e., we assume

$$\partial\mathcal{S} = \{s = j\omega \;\;:\;\; \omega \geq 0\}. \tag{2.48}$$

Introduce the functions

$$R(\omega) = \mathrm{Re}[G(j\omega)] \;\;;\;\; I(\omega) = \mathrm{Im}[G(j\omega)]. \tag{2.49}$$

With $\hat{\rho}(\omega)$ playing the role of ρ_s, the optimization problem (2.42) becomes

$$\begin{gathered} \hat{\rho}(\omega) = \inf \rho \\ \text{s.t.} \\ A(\omega)\delta = b \;\;;\;\; \|\delta\| = \rho \end{gathered} \tag{2.50}$$

where

$$A(\omega) = \begin{bmatrix} R'(\omega) \\ I'(\omega) \end{bmatrix} \;\;;\;\; b = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \tag{2.51}$$

Moreover, define the two sets of frequencies

$$\begin{aligned} \Omega_0 &= \{\omega \geq 0 \;\;:\;\; I(\omega) = 0\} \\ \bar{\Omega}_0 &= \{\omega \geq 0 \;\;:\;\; I(\omega) \neq 0\} \end{aligned}. \tag{2.52}$$

The following situations may occur.

1. $\omega \in \bar{\Omega}_0$, rank $A = 2$: the solution of (2.50) can be calculated by computing the pseudo-inverse of $A(\omega)$ yielding

$$\delta = A'(\omega)[A(\omega)A'(\omega)]^{-1}b \tag{2.53}$$

and

$$\hat{\rho}(\omega) = \frac{\|I(\omega)\|_2}{\left[\|I(\omega)\|_2^2 \, \|R(\omega)\|_2^2 - (R'(\omega)I(\omega))^2\right]^{1/2}} \tag{2.54}$$

2. $\omega \in \bar{\Omega}_0$, rank $A = 1$: (2.50) has no solution, therefore

$$\hat{\rho}(\omega) = \infty \tag{2.55}$$

3. $\omega \in \Omega_0$: in this case rank $A(\omega) = 1$ and the solution of (2.50) is given by

$$\delta = R'(\omega)[R(\omega)R'(\omega)]^{-1} \tag{2.56}$$

therefore

$$\hat{\rho}(\omega) = \frac{1}{\|R(\omega)\|_2}. \tag{2.57}$$

Note that if only one scalar parameter is present, i.e., $\delta \in \mathbb{R}$, case 1 can never occur. The solution for ρ_d in (2.46) is analogous to that for $\hat{\rho}(\omega)$ in case 3, hence

$$\rho_d = \frac{1}{\|g\|_2}. \tag{2.58}$$

We have thus proved the following result.

Theorem 2.7 *Given the l_2 uncertain polynomial family*

$$\mathcal{P}_\rho = \left\{ P(s;\delta) = P_0(s) + \sum_{i=1}^{n} \delta_i P_i(s) \quad : \quad \|\delta\|_2 < \rho \right\} \tag{2.59}$$

suppose $P_0(s) = P(s;0)$ is stable and of degree l. The l_2 parametric stability margin ρ^* of $\mathcal{P}_\rho$ is given by

$$\rho^* = \begin{cases} \min\{\rho_d, \rho_0\} & \text{if } \ n = 1 \\ \min\{\rho_d, \rho_0, \bar{\rho}\} & \text{if } \ n > 1 \end{cases} \tag{2.60}$$

where

$$\rho_d = \frac{1}{\|g\|_2} \tag{2.61}$$

$$\rho_0 = \inf_{\omega \in \Omega_0} \frac{1}{\|R(\omega)\|_2} \tag{2.62}$$

$$\bar{\rho} = \inf_{\omega \in \bar{\Omega}_0} \hat{\rho}(\omega) \tag{2.63}$$

being

$$\hat{\rho}(\omega) = \begin{cases} \dfrac{\|I(\omega)\|_2}{\left[\|I(\omega)\|_2^2 \, \|R(\omega)\|_2^2 - (R'(\omega)I(\omega))^2\right]^{1/2}} & \text{if} \quad \omega \in \bar{\Omega}_s \\ \infty & \text{if} \quad \omega \notin \bar{\Omega}_s \end{cases} \tag{2.64}$$

and

$$\begin{aligned} \bar{\Omega}_s = \left\{\omega \in \bar{\Omega}_0 \quad : \quad \text{rank } A(\omega) = 2\right\} = \\ = \left\{\omega \in \bar{\Omega}_0 \quad : \quad \|I(\omega)\|_2^2 \, \|R(\omega)\|_2^2 - (R'(\omega)I(\omega))^2 \neq 0\right\}. \end{aligned} \tag{2.65}$$

Remark 2.1 Since the set Ω_0 plays an important role in the development of what follows, we briefly discuss its structure. It is easily verified that Ω_0 contains at most a finite number k of frequencies in addition to $\omega = 0$, i.e.,

$$\Omega_0 = \{0, \omega_1, \ldots, \omega_k\}. \tag{2.66}$$

Furthermore, since any frequency $\omega_i \in \Omega_0$, $i = 1, \ldots, k$ must be a common root of n polynomials in ω of degree less than $2l$, we observe that the existence of such frequencies is not generic, especially for large n. Therefore, the case $\Omega_0 = \{0\}$ can be considered as the generic case.

Chapter 3

Uncertain control systems with rank one real perturbations

3.1 Rank one SISO control systems

In this section we will introduce the general context of robust analysis and synthesis for linear control systems involving rank one real perturbations.

A pretty standard framework for robust control is the representation of uncertain systems via Linear Fractional Transformations (LFT) [43]. Indeed, any system containing a set of uncertainty blocks $\{\Delta_1(s), \Delta_2(s), \dots\}$ including (but not limited to) uncertain real parameters, can in general be represented in the feedback form depicted in Fig. 3.1, where $\Delta(s)$ is a block-diagonal matrix containing all the uncertain blocks and $M(s)$ is a suitable transfer matrix.

$$\begin{aligned} \begin{bmatrix} z \\ y \end{bmatrix} &= M(s) \begin{bmatrix} w \\ u \end{bmatrix} \\ w &= \Delta(s) z \\ \Delta(s) &= \operatorname{diag}\{\Delta_1(s), \Delta_2(s), \dots\}. \end{aligned} \tag{3.1}$$

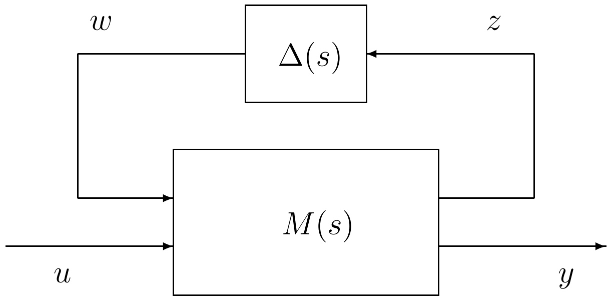


Figure 3.1: LFT representation

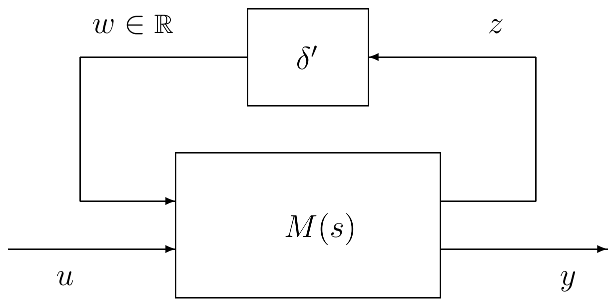


Figure 3.2: Rank one *LFT* representation

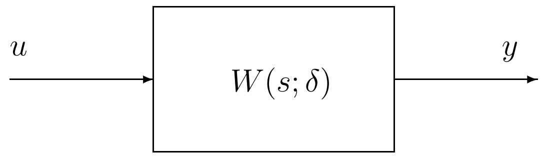


Figure 3.3: Uncertain *SISO* plant

Definition 3.1 An uncertain linear system is said to be affected by *rank one real perturbations* if it admits a standard (*LFT*) feedback representation in which the uncertainty block is represented by a real parameter vector $\delta \in \mathcal{D} \subseteq \mathbb{R}^n$, i.e., $w \in \mathbb{R}$ (Fig. 3.2).

The study and development of analysis and synthesis techniques for the class of rank one uncertain systems is strongly motivated by the fact that such a class suitably represents a large number of well known and widely used system structures. In many practical problems, the uncertain system can be modeled with a *SISO* linear transfer function whose coefficients depend affinely on a real parameter vector. The rank one representation properly fits this context.

Consider the following uncertain *SISO* plant family

$$\mathcal{W} = \left\{ \begin{array}{c} W(s;\delta) = \dfrac{B_0(s) + \delta' \bar{B}(s)}{A_0(s) + \delta' \bar{A}(s)} \ : \ \delta \in \mathcal{D} \subseteq \mathbb{R}^n \\ \\ \bar{B}(s) = [B_1(s) \dots B_n(s)]' \ ; \ \bar{A}(s) = [A_1(s) \dots A_n(s)]' \end{array} \right\} \tag{3.2}$$

where $\delta = [\delta_1 \dots \delta_n]' \in \mathcal{D} \subseteq \mathbb{R}^n$ is the uncertain parameter vector and $B_1(s), \dots, B_n(s)$, $A_1(s), \dots, A_n(s)$ are given polynomials (see Fig. 3.3).

Clearly, any *SISO* linear system in which parameter uncertainty enters the numerator and denominator of the transfer function affinely can be represented this way.

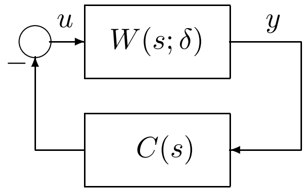


Figure 3.4: Standard control loop

In order to avoid introducing technicalities which are not crucial for the purpose of this work, in the sequel the following simplifying assumption on the degree of the involved polynomials will be enforced on $\mathcal{W}$.

Assumption 3.1

$$\partial B_0 < \partial A_0 \quad ; \quad \partial A_i < \partial A_0 \quad ; \quad \partial B_i < \partial B_0 \quad i = 1, \ldots, n. \tag{3.3}$$

As we will clarify later, this assumption basically implies that parameter variations will not affect the degree of the characteristic polynomial of the controlled system, thus allowing to disregard the role of ρ_d in the computation of the stability margin. We will refer to

$$W_0(s) = \frac{B_0(s)}{A_0(s)} \tag{3.4}$$

as the *nominal plant* of the family.

By performing a linear fractional transformation as described in [43], a rank one standard feedback representation (see Fig. 3.2) can be obtained for the system $W(s;\delta)$ in (3.2)

$$\begin{bmatrix} z \\ y \end{bmatrix} = M \begin{bmatrix} w \\ u \end{bmatrix} \qquad w = \delta' z \tag{3.5}$$

where

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} = \begin{bmatrix} -\dfrac{\bar{A}}{A_0} & \bar{B} - \dfrac{\bar{A}B_0}{A_0} \\ \dfrac{1}{A_0} & \dfrac{B_0}{A_0} \end{bmatrix}. \tag{3.6}$$

Now consider the feedback interconnection in Fig.3.4 where

$$C(s) = \frac{N(s)}{D(s)} \tag{3.7}$$

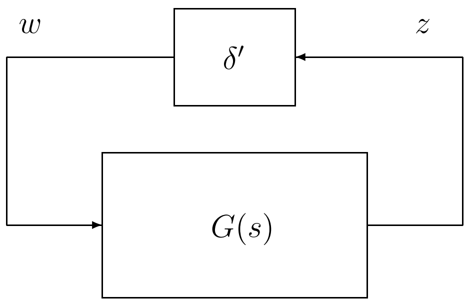


Figure 3.5: Closed loop *LFT* representation

is any linear controller transfer function. Taking the representation (3.5),(3.6) into account, a simple computation allows for expressing the control loop in *LFT* form as well (Fig. 3.5) where

$$G(s) = -\frac{\bar{A}(s)D(s) + \bar{B}(s)N(s)}{A_0(s)D(s) + B_0(s)N(s)}. \tag{3.8}$$

3.2 Robustly stabilizing controller parameterization

Consider the *SISO* feedback interconnection in Fig. 3.4 with the uncertain plant $W(s;\delta)$ defined as in (3.2). In this section we will provide a parameterization of all controllers which stabilize the closed loop system for all possible parameter combinations. In the sequel we will show that, provided a bound is enforced on the norm of the perturbation vector δ, such a parameterization can be expressed by means of a convex condition, thus allowing to treat robust control problems, such as that of maximizing the parametric stability margin, in terms of *quasi-convex optimization*.
Obviously, robustly stabilizing controllers must be looked for in the set $\mathcal{C}_0$ of all $C(s)$ which stabilize the nominal plant $W_0(s)$. The parameterization of all controllers stabilizing a given plant is based on the notion of *coprime factorization* over $\mathcal{RH}_\infty$. Although the results related to this topic are well known [38], we briefly recall them here for the sake of completeness.

Definition 3.2 Two transfer functions $\tilde{B}(s), \tilde{A}(s)$ are said to be coprime over $\mathcal{RH}_\infty$ if

1. $\tilde{B}(s), \tilde{A}(s) \in \mathcal{RH}_\infty$

2. All common divisors are invertible in $\mathcal{RH}_\infty$ (i.e., they are stable, minimum phase transfer functions with equal numerator and denominator degrees).

Theorem 3.1 *The following statements hold.*

1. *Two transfer functions $\tilde{B}(s), \tilde{A}(s) \in \mathcal{RH}_\infty$ are coprime over $\mathcal{RH}_\infty$ if and only if there exists a solution $\tilde{X}(s), \tilde{Y}(s) \in \mathcal{RH}_\infty$ of the diophantine equation*

$$\tilde{B}(s)\tilde{X}(s) + \tilde{A}(s)\tilde{Y}(s) = 1. \tag{3.9}$$

2. *Let the plant $W_0(s)$ be given by*

$$W_0(s) = \frac{B_0(s)}{A_0(s)} = \frac{\tilde{B}(s)}{\tilde{A}(s)} \tag{3.10}$$

with $\tilde{B}(s), \tilde{A}(s)$ coprime over $\mathcal{RH}_\infty$. Then, a compensator $C_0(s)$ stabilizes the closed loop system if and only if

$$C_0(s) = \frac{\tilde{X}(s)}{\tilde{Y}(s)} \tag{3.11}$$

where $\tilde{X}(s), \tilde{Y}(s) \in \mathcal{RH}_\infty$ are solutions of (3.9).

3. *The set of all stabilizing controllers for the plant $W_0(s)$ has the following parameterization*

$$\mathcal{C}_0 = \left\{ C(s) = \frac{\tilde{X}(s) - \tilde{Q}(s)\tilde{A}(s)}{\tilde{Y}(s) + \tilde{Q}(s)\tilde{B}(s)} \quad : \quad \tilde{B}(s)\tilde{X}(s) + \tilde{A}(s)\tilde{Y}(s) = 1 \ ; \ \tilde{Q}(s) \in \mathcal{RH}_\infty \right\}. \tag{3.12}$$

The transfer function $\tilde{Q}(s)$ is commonly referred to as the Youla parameter.
To obtain the set (3.12) of controllers stabilizing the nominal plant $W_0(s)$ it suffices to compute the coprime factorization (3.10) and solve the polynomial equation (3.9) for $\tilde{X}(s), \tilde{Y}(s)$. For a complete discussion of the properties of coprime factorization over $\mathcal{RH}_\infty$, refer to [38].
Taking into account the stabilizing controller parameterization (3.12), the transfer function $G(s)$ in (3.8) can be rewritten in the standard form [41]

$$G(s) = T_1(s) + T_2(s)\tilde{Q}(s) \tag{3.13}$$

where

$$T_1(s) = -\frac{\bar{A}(s)\tilde{Y}(s) + \bar{B}(s)\tilde{X}(s)}{A_0(s)\tilde{Y}(s) + B_0(s)\tilde{X}(s)} \quad ; \quad T_2(s) = -\frac{\bar{A}(s)\tilde{B}(s) - \bar{B}(s)\tilde{A}(s)}{A_0(s)\tilde{Y}(s) + B_0(s)\tilde{X}(s)}. \tag{3.14}$$

Note that $T_1(s), T_2(s) \in \mathcal{RH}_\infty$.

Let us recall the robust stability condition for the closed loop system.

Fact 3.1 *The closed loop system is robustly stable if and only if*

$$[1 - \delta' G(s)]^{-1} \in \mathcal{RH}_\infty \quad \forall \delta \in \mathcal{D}. \tag{3.15}$$

Since the controller $C(s)$ is chosen so that it stabilizes the nominal plant $W_0(s)$ and the poles of the closed loop are continuous with δ, $[1 - \delta' G(s)]^{-1}$ is indeed stable for sufficiently small δ. The Zero Exclusion Principle applies to the characteristic polynomial, hence the robust stability condition can be reformulated as follows.

Fact 3.2 *The closed loop system is robustly stable if and only if*

$$1 - \delta' G(j\omega) \neq 0 \quad \forall \omega \geq 0 \quad \forall \delta \in \mathcal{D}. \tag{3.16}$$

By taking (3.13) into account, this can be expressed as a condition on the Youla parameter $\tilde{Q}(s)$ thus yielding a parameterization of all robustly stabilizing controllers:

$$1 - \delta'[T_1(j\omega) + T_2(j\omega)\tilde{Q}(j\omega)] \neq 0 \quad \forall \omega \geq 0 \quad \forall \delta \in \mathcal{D} \quad ; \quad \tilde{Q}(s) \in \mathcal{RH}_\infty. \tag{3.17}$$

Unfortunately, the above parameterization is not convex in $\tilde{Q}(s)$ and as such it cannot be successfully employed for computing $\tilde{Q}(s)$ in order to optimize a given robust performance index. For example, suppose we want to compute a controller $C(s)$ maximizing the parametric stability margin of the closed loop system. In this case, given a suitable norm $\|\cdot\|$ in coefficient space, the set $\mathcal{D}$ is a ball of radius ρ, i.e., $\mathcal{D} = \{\delta : \|\delta\| < \rho\}$. Once a stabilizing controller for the nominal plant $C_0(s)$ is computed as in Theorem 3.1 and $T_1(s), T_2(s)$ are calculated according to (3.14), the stability margin maximization problem (SMM) can be stated as follows.

SMM **problem**

$$\begin{gathered} \rho^* = \sup_{\tilde{Q}(s) \in \mathcal{RH}_\infty} \rho \\ \text{s.t.} \\ 1 - \delta'[T_1(j\omega) + T_2(j\omega)\tilde{Q}(j\omega)] \neq 0 \quad \forall \omega \geq 0 \\ \|\delta\| < \rho. \end{gathered} \tag{3.18}$$

This problem, which will be recalled in deeper detail later, is extremely difficult to approach from a computational point of view, since the constraint on $\tilde{Q}(s)$ is non-convex.

3.3 A convex parameterization of robustly stabilizing controllers

As we have pointed out in the previous section, the robust stability condition (3.16) is not directly exploitable in robust performance optimization problems since it yields a non-convex constraint on the Youla parameter which characterizes the controller.

In the fundamental paper [31], an overparameterization is introduced in order to devise a convex condition which is equivalent to robust stability. The approach is similar to that used for reducing conservatism in passivity arguments and is based on the introduction of a "multiplier" entering the robust stability condition as an additional free parameter [1].

From now on, we will assume the set $\mathcal{D}$ to be a ball of radius ρ in coefficient space with respect to the norm $\|\cdot\|$, i.e., $\mathcal{D} = \{\delta : \|\delta\| < \rho\}$.

Let $\|\cdot\|_d$ denote the dual norm of $\|\cdot\|$, defined as follows.

Definition 3.3

$$\|x\|_d = \max\left\{x'y \quad : \quad \|y\| < 1\right\}. \tag{3.19}$$

It can be easily shown that the dual of the l_2 norm is the l_2 norm itself.

The following result provides a convex condition which is equivalent to (3.16).

Theorem 3.2 *The following two statements are equivalent:*

1. *The closed loop system is stable for all $\delta : \|\delta\| < \rho$, i.e.,*

$$1 - \delta' G(j\omega) \neq 0 \quad \forall \omega \geq 0 \quad \forall \delta : \|\delta\| < \rho, \tag{3.20}$$

2. *There exists a transfer function $\Phi(s) \in \mathcal{RH}_\infty$ which satisfies*

$$\mathrm{Re}\left[\Phi(j\omega)(1 - \delta' G(j\omega))\right] > 0 \quad \forall \omega \geq 0 \quad \forall \delta : \|\delta\| < \rho. \tag{3.21}$$

Proof: We give the main idea, the complete proof can be found in [31].

Fix $\omega \geq 0$. The value set

$$\Delta_G(\omega) = \left\{1 - \delta' G(j\omega) : \|\delta\| < \rho\right\} \tag{3.22}$$

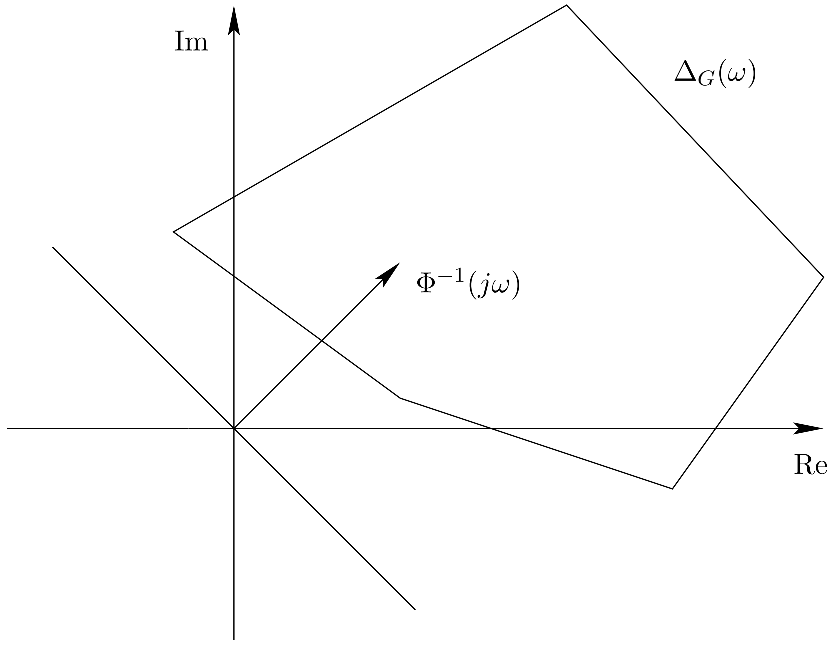


Figure 3.6: Graphical illustration of Theorem 3.2

is a convex subset of the complex plane. In order for the system to be robustly stable, $\Delta_G(\omega)$ must exclude zero for all $\omega \geq 0$. For each ω, this is equivalent to the existence of a line in the complex plane which contains the origin and does not intersect $\Delta_G(\omega)$ (see Fig. 3.6). Choose the complex number $\Phi^{-1}(j\omega)$ as a vector normal to such a line and contained in the half plane containing $\Delta_G(\omega)$. It is easy to see that for any element d of $\Delta_G(\omega)$, $\mathrm{Re}\,[\Phi(j\omega)d] > 0$, hence (3.21) holds.
Since only the argument of $\Phi(j\omega)$ is relevant to this inequality, it can be shown [31] that the amplitude can be adjusted in order to make $\Phi(s)$ a stable transfer function. $\diamondsuit$

By taking $\delta = 0$ in (3.21), it turns out that $\Phi(s)$ must be strictly positive real (see Definition A.2, Appendix A). Hence, $\Phi^{-1}(s)$ is in turn stable (see Property A.1, Appendix A). This allows, without loss of generality, to further express the Youla parameter $\tilde{Q}(s)$ as

$$\tilde{Q}(s) = \frac{Q(s)}{\Phi(s)} \quad ; \quad Q(s) \in \mathcal{RH}_\infty. \tag{3.23}$$

The robust stability condition (3.21) can thus be rewritten in terms of $T_1(s), T_2(s), \Phi(s)$ and $Q(s)$ as

$$\mathrm{Re}\left[\Phi(j\omega)\right] - \delta' \mathrm{Re}\left[T_1(j\omega)\Phi(j\omega) + T_2(j\omega)Q(j\omega)\right] > 0 \quad \forall\omega \geq 0 \quad \forall\delta : \|\delta\| < \rho. \tag{3.24}$$

This yields a parameterization of all robustly stabilizing controllers which is indeed convex in the pair $[\Phi(s), Q(s)]$. In fact, directly from the definition of dual norm, (3.24) can be rewritten as

$$\|\mathrm{Re}\left[T_1(j\omega)\Phi(j\omega) + T_2(j\omega)Q(j\omega)\right]\|_d < \frac{1}{\rho}\mathrm{Re}\left[\Phi(j\omega)\right] \quad \forall\omega \geq 0. \tag{3.25}$$

It is straightforward to check that the set of $[\Phi(s), Q(s)]$ for which condition (3.25) holds is convex. Summing up, we have the following result

Corollary 3.1 *Given the uncertain plant family (3.2) and a controller $C_0(s)$ computed according to Theorem 3.1 stabilizing the nominal plant, let $T_1(s), T_2(s)$ be as in (3.14). The set of controllers robustly stabilizing the control system for all δ such that $\|\delta\| < \rho$ is convex and given by*

$$\mathcal{C}_\rho = \left\{ \begin{array}{c} C(s) = \dfrac{\tilde{X}(s) - \tilde{Q}(s)\tilde{A}(s)}{\tilde{Y}(s) + \tilde{Q}(s)\tilde{B}(s)} \quad ; \quad \tilde{Q}(s) = \dfrac{Q(s)}{\Phi(s)} \\ \|\mathrm{Re}\left[T_1(j\omega)\Phi(j\omega) + T_2(j\omega)Q(j\omega)\right]\|_d < \dfrac{1}{\rho}\mathrm{Re}\left[\Phi(j\omega)\right] \quad \forall\omega \geq 0 \\ \Phi(s) \in \mathcal{RH}_\infty \quad ; \quad Q(s) \in \mathcal{RH}_\infty \end{array} \right\} \tag{3.26}$$

By introducing the additional free parameter $\Phi(s) \in \mathcal{RH}_\infty$, we have just derived a convex parameterization of all robustly stabilizing controllers for the plant family (3.2) provided that the parameter uncertainty set is defined as a ball of radius ρ in coefficient space. This allows to formulate some optimization problems such as the maximization of the parametric stability margin of the closed loop system in terms of the minimization of a quasi-convex functional $\phi(\Phi, Q)$ of the free parameters. The functional $\phi(\Phi, Q)$ being quasi-convex means that the set

$$\{(\Phi, Q) \in \mathcal{RH}_\infty \times \mathcal{RH}_\infty \quad : \quad \phi(\Phi, Q) < \lambda\} \tag{3.27}$$

is convex for all $\lambda > 0$.

In the next section we will formulate the stability margin maximization (SMM) problem using the convex parameterization (3.26) and give a sketch of its solution when $\Phi(s)$ and $Q(s)$ are free to vary over $\mathcal{RH}_\infty$ by employing the Ritz method.

3.4 Stability margin maximization problem: the general case

Consider the plant family (3.2), with the uncertainty set $\mathcal{D}$ defined as a ball of given radius ρ in coefficient space. As usual, compute a stabilizing controller for the nominal plant as in Theorem 3.1 and let $T_1(s), T_2(s)$ be as in (3.14). Let $\Phi(s), Q(s) \in \mathcal{RH}_\infty$ and define the following functional

$$\phi_\omega(\Phi, Q) = \frac{\|\mathrm{Re}\,[T_1(j\omega)\Phi(j\omega) + T_2(j\omega)Q(j\omega)]\,\|_d}{\mathrm{Re}\,[\Phi(j\omega)]} \tag{3.28}$$

Now recall the parametric stability margin maximization problem introduced in (3.18). By taking the controller parameterization (3.26) into account, such a problem can be reformulated as follows.

SMM **problem**

$$\begin{aligned} \rho^{*-1} &= \inf_{(\Phi,Q)\in\mathcal{RH}_\infty\times\mathcal{RH}_\infty} \lambda \\ &\text{s.t.} \\ &\phi_\omega(\Phi, Q) < \lambda \;\; \forall\omega \geq 0 \end{aligned} \tag{3.29}$$

or equivalently

$$\rho^{*-1} = \inf_{(\Phi,Q)\in\mathcal{RH}_\infty\times\mathcal{RH}_\infty} \; \sup_{\omega\geq 0} \; \phi_\omega(\Phi, Q). \tag{3.30}$$

Since the set $\{(\Phi, Q) \in \mathcal{RH}_\infty \times \mathcal{RH}_\infty \;\; : \;\; \phi_\omega(\Phi, Q) < \lambda\}$ is convex with respect to (Φ, Q) for all $\lambda > 0$ (see 3.25)), the SMM optimization problem (3.29),(3.30) admits a global minimum.

Although the above formulation enjoys the nice property of the existence of a global optimum, two questions arise as far as the computation of such optimum is concerned.

1. $\mathcal{RH}_\infty \times \mathcal{RH}_\infty$ is an infinite dimensional space. Hence, the solution for $[\Phi(s), Q(s)]$ and the corresponding controller $C(s)$ cannot be expected to belong, for instance, to the set of rational transfer functions of finite order.

2. The problem can only be approached by means of approximation techniques involving the optimization over finite dimensional subspaces of $\mathcal{RH}_\infty \times \mathcal{RH}_\infty$ with the use of a suitable functional basis and by tuning a number of free parameters. Clearly, the choice of the approximating strategy and of the basis functions affects the structure of the resulting controller as well as the computational burden.

The Ritz method for infinite dimensional programming consists of solving a sequence of optimizations defined over larger and larger finite dimensional subspaces. In this case, applying such a method amounts to solve, for increasing values of N, the following problem

$$\rho^{*-1} = \inf_{x\in\mathbb{R}^{2N+1}} \sup_{\omega\geq 0} \phi_\omega(\Phi_x, Q_x) \tag{3.31}$$

where

$$\begin{aligned} \Phi_x(s) &= \Phi_0(s) + x'\bar{\Phi}(s) \\ Q_x(s) &= x'\bar{Q}(s) \end{aligned} \tag{3.32}$$

and

$$\begin{aligned} \Phi_0(s) &\in \mathcal{RH}_\infty \\ \bar{\Phi}(s) &= [\Phi_1(s), \ldots, \Phi_N(s), 0_{N+1}, \ldots, 0_{2N+1}]' \\ \bar{Q}(s) &= [0, \ldots, 0_N, \bar{Q}_{N+1}(s), \ldots, \bar{Q}_{2N+1}(s)]' \end{aligned} \tag{3.33}$$

form a suitable functional basis. Under some conditions on the sequence $\{(\Phi_1, Q_1), (\Phi_2, Q_2), \ldots\}$, ensuring that any pair $(\Phi, Q) \in \mathcal{RH}_\infty \times \mathcal{RH}_\infty$ can be approximated sufficiently well by a pair (Φ_x, Q_x) of the form (3.32), this method gives a globally convergent algorithm.

Remark 3.1 Recalling the controller parameterization in (3.26), it turns out that for any given N, the above procedure involves a finite dimensional optimization over the following class of controllers parameterized by the real vector x.

$$\mathcal{C}_N = \left\{ C(s) = \frac{\tilde{X}(s) + x'\Phi_0^{-1}(s)[\tilde{X}(s)\bar{\Phi}(s) - \tilde{A}(s)\bar{Q}(s)]}{\tilde{Y}(s) + x'\Phi_0^{-1}(s)[\tilde{Y}(s)\bar{\Phi}(s) + \tilde{B}(s)\bar{Q}(s)]} \;\; ; \;\; x \in \mathbb{R}^{2N+1} \right\}. \tag{3.34}$$

Provided that the basis functions $(\bar{\Phi}, \bar{Q})$ are rational and of finite order, the controller class $\mathcal{C}_N$ has the same form as the class of uncertain plants itself, with the vector x playing the role of the plant parameter vector δ.

Chapter 4

Restricted complexity l_2 stability margin maximization

As it has been pointed out in the previous chapter, the problem of designing a controller maximizing the parametric stability margin of a class of uncertain $SISO$ systems depending affinely on a parameter vector enjoys the existence of a global optimum. Unfortunately, this involves the solution of an infinite dimensional optimization problem over $\mathcal{RH}_\infty$. Hence, the optimal controller is in general not even a rational transfer function and some approximation has to be found by means of finite programming once a suitable approximating functional basis is chosen. Besides the influence of this choice on the computational aspect of the problem, the relationship between the complexity of the resulting controller and its degree of robust performance (measured in terms of guaranteed stability margin) is strongly dependent on the selected structure. It is not clear how approximating solutions should be chosen in order to ensure a given performance level within the degree of controller complexity imposed, for instance, by implementation constraints. Moreover, it is often required to compute the "optimal" controller within a prescribed low-complexity controller class such as PID or lag-lead compensators.

In this chapter we will discuss a new approach to the stability margin maximization problem aimed at finding an optimal solution within a prescribed class of controllers described by an a-priori fixed number of tunable parameters. For the case of uncertainty being characterized by the l_2 norm of the parameter vector, we will propose an algorithm for the computation of a controller belonging to such restricted complexity class maximizing the stability margin. Each step of this algorithm involves the solution of a Linear Matrix Inequality feasibility problem and the synthesis of a transfer function ensuring the robust strict positive realness property to a family of polynomials.

4.1 A restricted complexity controller class

Consider the plant family $\mathcal{W}$ of the form (3.2) and let the uncertainty be described by an l_2 ball in parameter space, i.e, let $\mathcal{D} = \{\delta : \|\delta\|_2 < \rho\}$

$$\mathcal{W} = \left\{ \begin{array}{c} W(s;\delta) = \dfrac{B_0(s) + \delta' \bar{B}(s)}{A_0(s) + \delta' \bar{A}(s)} \quad : \quad \|\delta\|_2 < \rho \\ \\ \bar{B}(s) = [B_1(s) \dots B_n(s)]' \ ; \ \bar{A}(s) = [A_1(s) \dots A_n(s)]' \\ \\ \partial B_0 < \partial A_0 \quad ; \quad \partial A_i < \partial A_0 \quad ; \quad \partial B_i < \partial B_0 \quad i = 1, \dots, n \end{array} \right\}. \tag{4.1}$$

Let $C_0(s) \in \mathcal{C}_0$ be a stabilizing controller for the nominal plant $W_0(s)$ computed according to Theorem 3.1 and introduce the class of controllers

$$\mathcal{C} = \left\{ \begin{array}{c} C_\vartheta(s) = \dfrac{N_\vartheta(s)}{D_\vartheta(s)} = \dfrac{N_0(s) + \vartheta' \bar{N}(s)}{D_0(s) + \vartheta' \bar{D}(s)} \quad : \quad \vartheta \in \mathbb{R}^m \\ C_0(s) = \dfrac{N_0(s)}{D_0(s)} \\ \\ \bar{N}(s) = [N_1(s) \dots N_m(s)]' \quad ; \quad \bar{D}(s) = [D_1(s) \dots D_m(s)]' \end{array} \right\} \tag{4.2}$$

where $\vartheta = [\vartheta_1 \dots \vartheta_m]'$ is a tunable controller parameter vector and $N_1(s), \dots, N_m(s)$, $D_1(s) \dots D_m(s)$ are given polynomials. We call the set $\mathcal{C}$ *class of restricted complexity controllers*.

Enforce the following condition on the degrees of the polynomials defining the controller family.

Assumption 4.1

$$\partial N_0 \leq \partial D_0 \quad ; \quad \partial N_i \leq \partial N_0 \quad ; \quad \partial D_i \leq \partial D_0 \qquad i = 1, \dots, m. \tag{4.3}$$

Remark 4.1 The choice of a controller class of the form (4.2) is motivated mainly by practical reasons, since many widely used structures can be seen as members of such a class. As an example, consider a PI controller of the form

$$C(s) = (K_P^0 + \tilde{K}_P) + \frac{K_I^0 + \tilde{K}_I}{s} \tag{4.4}$$

where

$$C_0(s) = K_P^0 + \frac{K_I^0}{s} \tag{4.5}$$

is a stabilizing nominal PI compensator and $\tilde{K}_P, \tilde{K}_I$ are tunable parameters. Clearly, $C(s)$ belongs to the class $\mathcal{C}$ with

$$N_0(s) = K_I^0 + K_P^0 s \ ; \ D_0(s) = s \ ; \ \bar{N}(s) = [1 \ s]' \ ; \ \bar{D}(s) = [0 \ 0] \ ; \ \vartheta = [\tilde{K}_I \ \tilde{K}_P]'. \quad (4.6)$$

It is easy to see that many other controller structures can be included in $\mathcal{C}$, see the example section in this chapter.

Remark 4.2 For an arbitrary choice of $\bar{N}(s)$ and $\bar{D}(s)$, the class $\mathcal{C}$ is not in general guaranteed to stabilize $W_0(s)$ for all values of ϑ, i.e., $\mathcal{C} \nsubseteq \mathcal{C}_0$. On the other hand, since the stability domain is defined by an open set and $\vartheta = 0$ defines $C_0(s)$, which asymptotically stabilizes the nominal control system, there exist a neighbourhood Θ of $\vartheta = 0$ and a neighbourhood Δ of $\delta = 0$ such that the uncertain control system is asymptotically stable for all $\vartheta \in \Theta$ and $\delta \in \Delta$. This is indeed true for any choice of $\bar{N}(s)$ and $\bar{D}(s)$.

Given ϑ such that the controller $C_\vartheta(s) \in \mathcal{C}$ stabilizes the nominal plant, let us introduce the parametric stability margin achieved by $C_\vartheta(s)$.

Definition 4.1 For a fixed stabilizing controller $C_\vartheta(s) \in \mathcal{C}$, we define the l_2 parametric stability margin ρ_ϑ as the maximal ρ such that the feedback system is stable for all δ such that $\|\delta\|_2 < \rho$.

4.2 The RCSMM problem

We can now state the stability margin maximization problem over the class of restricted complexity controllers introduced above ($RCSMM$ problem).

$RCSMM$ **problem.** Given the uncertain plant family $\mathcal{W}$ and a controller $C_0(s)$ stabilizing the nominal plant $W_0(s)$, find a parameter value ϑ^* such that the controller $C_{\vartheta^*}(s) \in \mathcal{C}$ achieves the maximum of the closed loop stability margin over the class $\mathcal{C}$, i.e.,

$$\rho_{\vartheta^*} = \sup_{\vartheta \in \mathbb{R}^m} \rho_\vartheta. \quad (4.7)$$

By the properties pointed out in Remark 4.2, this problem is well posed for any $\bar{N}(s)$ and $\bar{D}(s)$ defining the class $\mathcal{C}$.

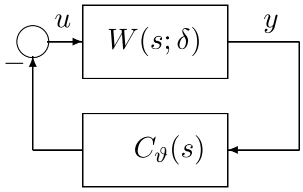


Figure 4.1: Control loop

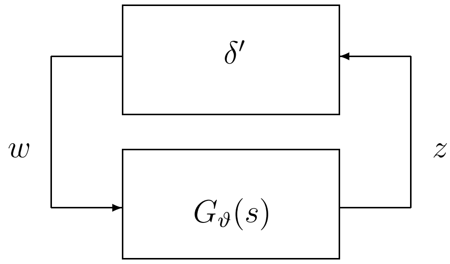


Figure 4.2: LFT representation of the control loop

We will now derive some properties of the *RCSMM* problem. As in the previous section, consider the feedback interconnection of $W(s;\delta)$ and $C_\vartheta(s)$ (Fig. 4.1). The closed loop system can be represented in *LFT* form (Fig. 4.2) where

$$G_\vartheta(s) = -\frac{\bar{A}(s)D_\vartheta(s) + \bar{B}(s)N_\vartheta(s)}{A_0(s)D_\vartheta(s) + B_0(s)N_\vartheta(s)} \tag{4.8}$$

(see (3.8)).
The following robust stability condition parallels (3.16).

Fact 4.1 *The closed loop system is robustly stable if and only if*

$$1 - \delta' G_\vartheta(j\omega) \neq 0 \;\; \forall\omega \geq 0 \;\; \forall\delta : \|\delta\|_2 < \rho. \tag{4.9}$$

Hence, the l_2 parametric stability margin of the closed loop is given by

$$\begin{gathered} \rho_\vartheta = \sup \rho \\ \text{s.t.} \\ 1 - \delta' G_\vartheta(j\omega) \neq 0 \quad \forall\omega \geq 0 \quad \forall\delta : \|\delta\|_2 < \rho. \end{gathered} \tag{4.10}$$

The transfer function $G_\vartheta(s)$ can be rewritten as

$$G_\vartheta(s) = -\left[\frac{P_{\vartheta,1}(s)}{P_{\vartheta,0}(s)}, \ldots, \frac{P_{\vartheta,n}(s)}{P_{\vartheta,0}(s)}\right]' \tag{4.11}$$

where the polynomials $P_{\vartheta,i}(s)$ are given by

$$P_{\vartheta,i}(s) = A_i(s)D_\vartheta(s) + B_i(s)N_\vartheta(s) \quad ; \quad i = 0, \dots, n. \tag{4.12}$$

The characteristic polynomial of the closed loop system has the following expression:

$$P_\vartheta(s;\delta) = P_{\vartheta,0}(s) + \sum_{i=1}^{n} \delta_i P_{\vartheta,i}(s). \tag{4.13}$$

Since ϑ is chosen such that the controller $C_\vartheta(s)$ stabilizes the nominal plant, the polynomial $P_\vartheta(s,0) = P_{\vartheta,0}(s)$ is Hurwitz. Recalling Assumption 3.1 on the plant family and Assumption 4.1 on the controller family, we get that the degree of $P_\vartheta(s;\delta)$ is invariant with respect to the parameters δ and ϑ. Introduce the two functions

$$R_\vartheta(\omega) = \mathrm{Re}[G_\vartheta(j\omega)] \quad ; \quad I_\vartheta(\omega) = \mathrm{Im}[G_\vartheta(j\omega)] \tag{4.14}$$

and define the sets

$$\begin{array}{l} \Omega_0 = \{\omega \geq 0 \quad : \quad I_\vartheta(\omega) = 0\} \\ \bar{\Omega}_0 = \{\omega \geq 0 \quad : \quad I_\vartheta(\omega) \neq 0\} \end{array} . \tag{4.15}$$

A frequency domain characterization of the l_2 stability margin ρ_ϑ can be given based on Theorem 2.7, in which the role of ρ_d can be disregarded because of degree-invariance.

Theorem 4.1 *Let ϑ be such that $C_\vartheta(s)$ stabilizes the nominal plant. Then, the l_2 parametric stability margin ρ_ϑ is given by*

$$\rho_\vartheta = \begin{cases} \rho_\vartheta^0 & \text{if} \quad n = 1 \\ \min\{\rho_\vartheta^0, \bar{\rho}_\vartheta\} & \text{if} \quad n > 1 \end{cases} \tag{4.16}$$

where

$$\rho_\vartheta^0 = \inf_{\omega \in \Omega_0} \frac{1}{\|R_\vartheta(\omega)\|_2} \tag{4.17}$$

$$\bar{\rho}_\vartheta = \inf_{\omega \in \bar{\Omega}_0} \hat{\rho}_\vartheta(\omega) \tag{4.18}$$

being

$$\hat{\rho}_\vartheta(\omega) = \begin{cases} \dfrac{\|I_\vartheta(\omega)\|_2}{\left[\|I_\vartheta(\omega)\|_2^2 \, \|R_\vartheta(\omega)\|_2^2 - \left(R'_\vartheta(\omega) I_\vartheta(\omega)\right)^2\right]^{1/2}} & \text{if} \quad \omega \in \bar{\Omega}_s \\ \infty & \text{if} \quad \omega \notin \bar{\Omega}_s \end{cases} \tag{4.19}$$

and

$$\bar{\Omega}_s = \left\{\omega \in \bar{\Omega}_0 \quad : \quad \|I_\vartheta(\omega)\|_2^2 \, \|R_\vartheta(\omega)\|_2^2 - \left(R'_\vartheta(\omega) I_\vartheta(\omega)\right)^2 \neq 0\right\}. \tag{4.20}$$

The following result parallels Theorem 3.2 and relates robust stability of the controlled system with the existence of a transfer function satisfying some robust strict positive realness properties. This fact is of fundamental importance in our approach.

Theorem 4.2 *The following statements are equivalent.*

1. *The closed loop uncertain system with controller $C_\vartheta(s)$ is robustly stable within l_2 uncertainty radius ρ, i.e.,*

$$1 - \delta' G_\vartheta(j\omega) \neq 0 \quad \forall \omega \geq 0 \quad \forall \delta : \|\delta\|_2 < \rho, \tag{4.21}$$

2. *There exists a transfer function $\Phi_\vartheta(s) \in \mathcal{RH}_\infty$ such that*

$$\mathrm{Re}\left[\Phi_\vartheta(j\omega)\left(1 - \delta' G_\vartheta(j\omega)\right)\right] > 0 \quad \forall \omega \geq 0 \quad \forall \delta : \|\delta\|_2 < \rho, \tag{4.22}$$

3. *There exists a transfer function $\Phi_\vartheta(s) \in \mathcal{RH}_\infty$ such that*

$$\begin{cases} \mathrm{Re}[\Phi_\vartheta(j\omega)] > 0 \\ \|R_\vartheta(\omega) - \gamma_{\Phi_\vartheta}(\omega) I_\vartheta(\omega)\|_2^2 < \dfrac{1}{\rho^2} \end{cases} \quad \forall \omega \geq 0 \tag{4.23}$$

   where

$$\gamma_{\Phi_\vartheta}(\omega) = \frac{\mathrm{Im}[\Phi_\vartheta(j\omega)]}{\mathrm{Re}[\Phi_\vartheta(j\omega)]}. \tag{4.24}$$

Proof: The equivalence between statements 1 and 2 is the same as Theorem 3.2. Taking $\delta = 0$ in (4.22) yields the first of (4.23). Therefore, (4.22) can be rewritten equivalently as

$$\begin{array}{ll} (a) & \mathrm{Re}[\Phi_\vartheta(j\omega)] > 0 \\ (b) & \delta' \left[R_\vartheta(\omega) - \gamma_{\Phi_\vartheta}(\omega) I_\vartheta(\omega)\right] < 1 \end{array} \quad \forall \omega \geq 0 \quad \forall \delta : \|\delta\|_2 \leq \rho \tag{4.25}$$

By the definition of dual norm and recalling that the dual of the l_2 norm is the l_2 norm itself, we get that (4.25b) holds for all δ such that $\|\delta\|_2 \leq \rho$ if and only if the second of (4.23) holds. ♢

Remark 4.3 Condition (4.22) has a close relationship with the well known problem of designing linear filters preserving the strict positive realness property of systems under the influence of parametric uncertainty ($RSPR$ problem). This problem, which is of major interest in fields such as parametric model identification, adaptive control and nonlinear system stability analysis, will be extensively discussed as a standalone topic in Appendix A, where a new filter design approach is proposed for the case of the uncertainty being represented by a l_2 ball in parameter space. This technique will be used in the sequel as a tool for deriving our restricted complexity stability margin optimization procedure.

We recall that a rational function $\Phi(s)$ is said to be strictly positive real (SPR) if the following two conditions hold:

1. $\Phi(s) \in \mathcal{RH}_\infty$

2. $\mathrm{Re}[\Phi(j\omega)] > 0 \;\; \forall \omega \geq 0.$

It can be easily checked that (4.22) is equivalent to the following condition

$$\Phi_\vartheta(s)\frac{P_\vartheta(s;\delta)}{P_{\vartheta,0}(s)} \;\text{ is } SPR \;\; \forall \delta : \|\delta\|_2 < \rho. \tag{4.26}$$

In turn, finding $\Phi_\vartheta(s)$ satisfying (4.22) is equivalent to solving the $RSPR$ problem in its standard formulation for l_2 parametric uncertainty (see Appendix A): given the polynomial $P_\vartheta(s;\delta)$, compute a rational function $F_\vartheta(s) \in \mathcal{RH}_\infty$ such that

$$\frac{P_\vartheta(s;\delta)}{F_\vartheta(s)} \;\text{ is } SPR \;\; \forall \delta : \|\delta\|_2 < \rho. \tag{4.27}$$

What follows provides a closed form expression for $\Phi_\vartheta(s)$ satisfying (4.22).

Introduce the polynomial

$$\Pi_\vartheta(s) = \sum_{i=1}^{n} P_{\vartheta,0}(s)P_{\vartheta,i}(-s)\left[P_{\vartheta,0}(-s)P_{\vartheta,i}(s)\right]_o. \tag{4.28}$$

Theorem 4.3 *Suppose for a given ϑ it results $I_\vartheta(\omega) \neq 0$ for all $\omega > 0$, i.e, $\Omega_0 = \{0\}$. Then,*

1. *The following factorization holds for $\Pi_\vartheta(s)$:*

$$\Pi_\vartheta(s) = A_\vartheta s^{r_\vartheta}\bar{\Pi}_{\vartheta,1}(s)\bar{\Pi}_{\vartheta,2}(-s) \tag{4.29}$$

where A_ϑ is a real constant, $r_\vartheta \geq 1$ is an integer and $\bar{\Pi}_{\vartheta,1}(s)$ and $\bar{\Pi}_{\vartheta,2}(s)$ are uniquely determined monic Hurwitz polynomials.

2. Let $\rho < \rho_\vartheta$. For sufficiently small ε, τ, the transfer function

$$\Phi_\vartheta(s) = \begin{cases} \dfrac{\bar{\Pi}_{\vartheta,1}(s)}{\bar{\Pi}_{\vartheta,2}(s)}(1+\tau s)^{\partial\bar{\Pi}_{\vartheta,2}-\partial\bar{\Pi}_{\vartheta,1}} & \text{for even } r_\vartheta \\ \\ \dfrac{\bar{\Pi}_{\vartheta,1}(s)}{\bar{\Pi}_{\vartheta,2}(s)}(s+\varepsilon)^{\mathrm{sgn} A_\vartheta \ (-1)^{(r_\vartheta-1)/2}} \cdot & \text{for odd } r_\vartheta \\ \cdot(1+\tau s)^{\partial\bar{\Pi}_{\vartheta,2}-\partial\bar{\Pi}_{\vartheta,1}-\mathrm{sgn} A_\vartheta \ (-1)^{(r_\vartheta-1)/2}} & \end{cases} \tag{4.30}$$

satisfies (4.22).

Proof: This result is a direct consequence of Theorem A.2 stated in Appendix A in the general context of l_2 SPR robust synthesis (the notation is self-explanatory). As a matter of fact, Theorem A.2 provides the solution to the $RSPR$ problem (4.27). ♢

Remark 4.4 We require $\Omega_0 = \{0\}$ as it indeed represents the generic situation (see Remark 2.1), although a solution for $\Phi_\vartheta(s)$ can be derived as well in the singular case of Ω_0 containing other frequencies (see Theorem A.4 in Appendix A).

Once $\Phi_\vartheta(s)$ is computed according to Theorem 4.3, a new characterization of the parametric stability margin ρ_ϑ can be given.

Theorem 4.4 *Let $\Phi_\vartheta(s)$ be computed as in Theorem 4.3. The parametric stability margin ρ_ϑ achieved by the controller $C_\vartheta(s)$ is given by*

$$\begin{gathered} \rho_\vartheta = \lim_{\varepsilon,\tau\to 0} \sup \rho \\ \text{s.t.} \\ \|R_\vartheta(\omega) - \gamma_{\Phi_\vartheta}(\omega) I_\vartheta(\omega)\|_2^2 < \frac{1}{\rho^2} \quad \forall \omega \geq 0. \end{gathered} \tag{4.31}$$

Proof: It follows from condition (4.23) and from the observation that $\gamma_\Phi(\omega)$ is continuous with respect to ω in $\omega = 0$ for all $\varepsilon, \tau > 0$ and $\gamma_{\Phi_\vartheta}(0) = 0$. ♢

Introducing the function

$$r(\vartheta, \omega) = \frac{1}{\|R_\vartheta(\omega) - \gamma_{\Phi_\vartheta}(\omega) I_\vartheta(\omega)\|_2} \tag{4.32}$$

the stability margin ρ_ϑ can be calculated as

$$\rho_\vartheta = \lim_{\varepsilon,\tau\to 0} \inf_{\omega\geq 0} r(\vartheta, \omega). \tag{4.33}$$

The solution of the $RCSMM$ problem is then given by

$$\rho^* = \sup_{\vartheta} \rho_{\vartheta}. \tag{4.34}$$

where ρ_{ϑ} is computed as in (4.33).

Unfortunately, ρ_{ϑ} may have local maxima. Moreover, $r(\vartheta, \omega)$ is in general a non-convex function of ω. Hence, employing this procedure would require the computation of $\Phi_{\vartheta}(s)$ and a sweep along the ω axis at each optimization step, plus some blind optimum search or heuristic on ϑ.

4.3 The surrogate stability margin function

Fix some value $\bar{\vartheta}$ of the controller parameters providing a stability margin $\rho_{\bar{\vartheta}}$. Let ϑ vary in a neighbourhood $\Theta_{\bar{\vartheta}}$ of $\bar{\vartheta}$ in which $\rho_{\vartheta} > 0$. Let $\Phi_{\bar{\vartheta}}(s)$ be computed as in Theorem 4.3 and define the rational function

$$\Psi_{\bar{\vartheta},\vartheta}(s) = \Phi_{\bar{\vartheta}}(s) \frac{P_{\vartheta,0}(s)}{P_{\bar{\vartheta},0}(s)} \quad ; \quad \vartheta \in \Theta_{\bar{\vartheta}}. \tag{4.35}$$

Let

$$\tilde{r}(\bar{\vartheta}; \vartheta, \omega) = \frac{1}{\|R_{\vartheta}(\omega) - \gamma_{\Psi_{\bar{\vartheta},\vartheta}}(\omega) I_{\vartheta}(\omega)\|_2} \tag{4.36}$$

and define

$$\tilde{\rho}(\bar{\vartheta}; \vartheta) = \inf_{\omega \geq 0} \tilde{r}(\bar{\vartheta}; \vartheta, \omega). \tag{4.37}$$

By the same argument as in Theorem 4.2, with $\Psi_{\bar{\vartheta},\vartheta}(s)$ playing the role of $\Phi_{\vartheta}(s)$, it follows that

$$\begin{gathered} \tilde{\rho}(\bar{\vartheta}; \vartheta) = \sup \rho \\ \text{s.t.} \\ \mathrm{Re}[\Psi_{\bar{\vartheta},\vartheta}(s)(1 - \delta' G_{\vartheta}(j\omega))] > 0 \quad \forall \omega \geq 0 \quad \forall \delta : \|\delta\|_2 < \rho. \end{gathered} \tag{4.38}$$

We call $\tilde{\rho}(\bar{\vartheta}; \vartheta)$ the *surrogate stability margin function* computed around $\bar{\vartheta}$.

Lemma 4.1 *The following properties hold for $\tilde{\rho}(\bar{\vartheta}; \vartheta)$.*

1.

$$\lim_{\varepsilon, \tau \to 0} \tilde{\rho}(\bar{\vartheta}; \bar{\vartheta}) = \rho_{\bar{\vartheta}} \tag{4.39}$$

2. *For sufficiently small* $\varepsilon, \tau > 0$,

$$\tilde{\rho}(\bar{\vartheta};\vartheta) \leq \rho_\vartheta \quad \forall \vartheta \in \Theta_{\bar{\vartheta}} \tag{4.40}$$

3. *Given a value of* $\bar{\vartheta}$, *suppose* ρ_ϑ *is a smooth function of* ϑ *in a neighbourhood of* $\bar{\vartheta}$. *Moreover, suppose there exists a unique value of* ω, *say* $\bar{\omega}$, *in which* $\tilde{r}(\bar{\vartheta};\bar{\vartheta},\omega)$ *takes its minimum value, i.e.,*

$$\exists!\ \bar{\omega} \geq 0\ :\ \tilde{\rho}(\bar{\vartheta};\bar{\vartheta}) = \min_{\omega\geq 0} \tilde{r}(\bar{\vartheta};\bar{\vartheta},\omega) = \tilde{r}(\bar{\vartheta};\bar{\vartheta},\bar{\omega}) \tag{4.41}$$

then, $\tilde{\rho}(\bar{\vartheta};\vartheta)$ *is a smooth function of* ϑ *in a neighbourhood of* $\vartheta = \bar{\vartheta}$.

Proof:

1. It follows directly from (4.36) and (4.37) by observing that $\gamma_{\Psi_{\bar{\vartheta},\bar{\vartheta}}}(\omega) = \gamma_{\Phi_{\bar{\vartheta}}}(\omega)$ is continuous with respect to ω in $\omega = 0$ for all $\varepsilon, \tau > 0$.

2. Since for $\varepsilon, \tau \to 0$, $\gamma_{\Phi_\vartheta(\omega)}$ achieves the minimum of the quadratic expression $\|R_\vartheta(\omega) - \gamma I_\vartheta(\omega)\|_2^2$ with respect to γ for all $\omega \geq 0$, we get $\tilde{r}(\bar{\vartheta};\vartheta,\omega) \leq r(\vartheta,\omega)$ for all $\omega \geq 0$ and hence (4.40).

3. This is a consequence of the fact that $\tilde{r}(\bar{\vartheta};\vartheta,\omega)$ is a rational function in ϑ and ω. The value of its minimum with respect to ω is a smooth function of ϑ as long as it is determined by only one relative minimum. $\diamondsuit$

In some singular cases, the function $\tilde{\rho}(\bar{\vartheta};\vartheta)$ can be proved to be non-smooth in $\vartheta = \bar{\vartheta}$. Anyway, both smoothness of ρ_ϑ and (4.41) are satisfied in the generic case.
If such conditions are met, then $\tilde{\rho}(\bar{\vartheta};\vartheta)$ is a smooth function which approximates ρ_ϑ from below in a neighbourhood of $\vartheta = \bar{\vartheta}$ (Fig. 4.3).

4.4 A LMI-based optimization procedure

Lemma 4.1 suggests the following idea to solve the problem of maximizing ρ_ϑ. Given a value $\bar{\vartheta}$ for which $\rho_{\bar{\vartheta}} > 0$ is well defined and condition 3 holds, pick a value $\rho > \rho_{\bar{\vartheta}}$ and find, if possible, some value ϑ_f of ϑ ensuring that $\tilde{\rho}(\bar{\vartheta};\vartheta_f) \geq \rho$. If such ϑ_f exists, then by condition 2 we have that ρ is a lower bound for ρ_{ϑ_f}. If possible, repeat the procedure with $\bar{\vartheta}$ equal to the value ϑ_f previously found and a larger value of ρ, otherwise try with

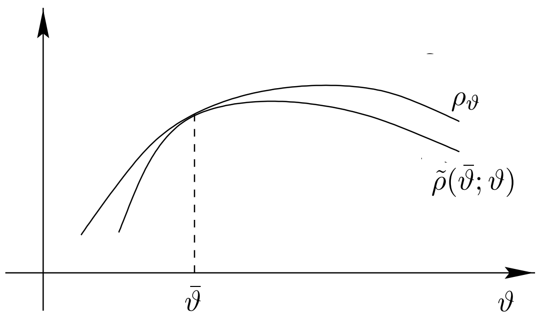


Figure 4.3: Surrogate stability margin function

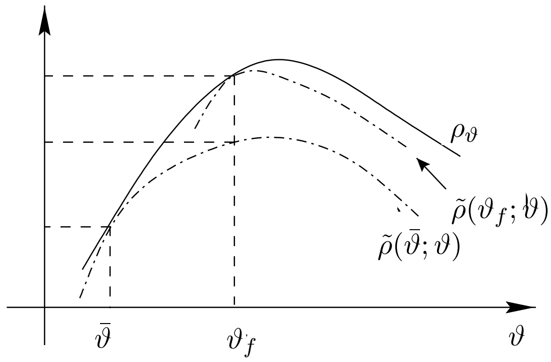


Figure 4.4: Optimization based on the surrogate stability margin function

a smaller ρ, until some optimality criterion is met. As we will further clarify later, this procedure is in general convergent to a local optimum of ρ_ϑ (Fig. 4.4).

Remark 4.5 The smoothness of $\tilde{\rho}(\bar{\vartheta};\vartheta)$ in $\vartheta = \bar{\vartheta}$, ensured if condition 3 of Lemma 4.1 holds, guarantees, as long as $\bar{\vartheta}$ is not a stationary point for ρ_ϑ, that a value of ϑ exists such that $\tilde{\rho}(\bar{\vartheta};\vartheta) > \tilde{\rho}(\bar{\vartheta};\bar{\vartheta})$. If that condition holds at each step of the procedure, there always exists ϑ_f such that $\tilde{\rho}(\bar{\vartheta};\vartheta_f) \geq \rho$ for some $\rho > \rho_{\bar{\vartheta}}$. If condition 3 is not satisfied, such ϑ_f may not exist, eventually causing the procedure to fail (see Fig. 4.5).

The main problem here is to determine, provided it exists, a value of ϑ such that $\tilde{\rho}(\bar{\vartheta};\vartheta) \geq \rho$ for given ρ. In this section we will show how it is possible to accomplish this by means of the solution of a *LMI feasibilty problem* [11].

Lemma 4.2 *The following two statements are equivalent*

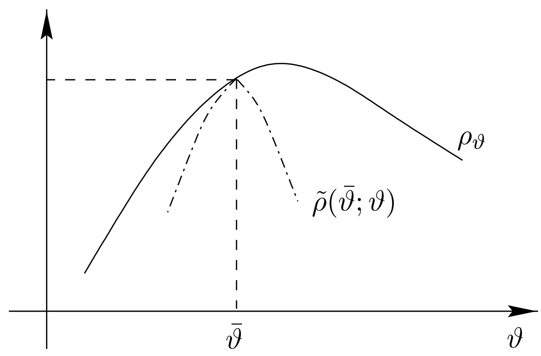


Figure 4.5: Non-smoothness of $\tilde{\rho}(\bar{\vartheta},\vartheta)$

1. *A transfer function* $\Phi(s) \in \mathcal{RH}_\infty$ *satisfies*

$$\mathrm{Re}\left[\Phi(j\omega)\left(1-\delta' G_\vartheta(j\omega)\right)\right] > 0 \quad \forall \omega \geq 0 \quad \forall \delta : \|\delta\|_2 < \rho, \tag{4.42}$$

2. *The transfer function* $\Phi(s) \in \mathcal{RH}_\infty$ *is such that*

$$\mathrm{Re}[T(\Phi,\rho,\vartheta;j\omega)] > 0 \quad \forall \omega \geq 0 \tag{4.43}$$

where

$$T(\Phi,\rho,\vartheta;s) = \Phi(s)\begin{bmatrix} I & \rho G_\vartheta(s) \\ \rho G'_\vartheta(s) & 1 \end{bmatrix}. \tag{4.44}$$

Proof: See Appendix B.

For a generic $\bar{\vartheta}$, let $\Phi_{\bar{\vartheta}}(s)$ be computed according to Theorem 4.3. Consider the corresponding $\Psi_{\bar{\vartheta},\vartheta}(s)$ and define $T(\Psi_{\bar{\vartheta},\vartheta},\rho,\vartheta;s)$ as in Lemma 4.2.
We get

$$T(\Psi_{\bar{\vartheta},\vartheta},\rho,\vartheta;s) = \frac{\Phi_{\bar{\vartheta}}(s)}{P_{\bar{\vartheta},0}(s)}\begin{bmatrix} P_{\vartheta,0}(s)I & \rho P_{\vartheta,0}(s)G_\vartheta(s) \\ \rho P_{\vartheta,0}(s)G'_\vartheta(s) & P_{\vartheta,0}(s) \end{bmatrix}. \tag{4.45}$$

Note that $T(\Psi_{\bar{\vartheta},\vartheta},\rho,\vartheta;s)$ depends affinely on ϑ and ρ. In particular, the poles of $T(\Psi_{\bar{\vartheta},\vartheta},\rho,\vartheta;s)$ do not depend on ϑ. This observation suggests that a canonical controllable state space realization of $T(\Psi_{\bar{\vartheta},\vartheta},\rho,\vartheta;s)$ has the form

$$[A(\bar{\vartheta}), B, C(\vartheta,\bar{\vartheta},\rho), D(\vartheta,\bar{\vartheta},\rho)] \tag{4.46}$$

where $C(\vartheta,\bar{\vartheta},\rho)$ and $D(\vartheta,\bar{\vartheta},\rho)$ are affine functions of ϑ and ρ. In particular we get

$$\begin{array}{c} C(\vartheta,\bar{\vartheta},\rho) = C_0^0(\bar{\vartheta}) + \sum_{i=1}^m \vartheta_i C_i^0(\bar{\vartheta}) + \rho[C_0^\rho(\bar{\vartheta}) + \sum_{i=1}^m \vartheta_i C_i^\rho(\bar{\vartheta})] \\ D(\vartheta,\bar{\vartheta},\rho) = D_0^0(\bar{\vartheta}) + \sum_{i=1}^m \vartheta_i D_i^0(\bar{\vartheta}) + \rho[D_0^\rho(\bar{\vartheta}) + \sum_{i=1}^m \vartheta_i D_i^\rho(\bar{\vartheta})]. \end{array} \tag{4.47}$$

Given $\bar{\vartheta}$, the values of matrices $A(\bar{\vartheta})$ and B and the coefficients $C_i^0(\bar{\vartheta})$, $C_i^\rho(\bar{\vartheta})$, $D_i^0(\bar{\vartheta})$, $D_i^\rho(\bar{\vartheta})$ can be explicitly computed.

The following result is closely related to the well-known Kalman-Yacubovich-Popov lemma and allows for expressing the strict positive realness condition on $T(\Psi_{\bar{\vartheta},\vartheta},\rho,\vartheta;s)$ in the form of a Linear Matrix Inequality. The proof can be found in [40].

Lemma 4.3 *Let $[A(\bar{\vartheta}),B,C(\vartheta,\bar{\vartheta},\rho),D(\vartheta,\bar{\vartheta},\rho)]$ be a state-space realization of $T(\Psi_{\bar{\vartheta},\vartheta},\rho,\vartheta;s)$ in controllability canonical form, i.e.,*

$$T(\Psi_{\bar{\vartheta},\vartheta},\rho,\vartheta;s) = C(\vartheta,\bar{\vartheta},\rho)(sI - A(\bar{\vartheta}))^{-1}B + D(\vartheta,\bar{\vartheta},\rho). \tag{4.48}$$

Then,

$$\mathrm{Re}[T(\Psi_{\bar{\vartheta},\vartheta},\rho,\vartheta;j\omega)] > 0 \quad \forall\omega \geq 0 \tag{4.49}$$

if and only if there exists a positive definite symmetric matrix X ($X = X' > 0$) such that

$$\left[\begin{array}{cc} A'(\bar{\vartheta})X + XA(\bar{\vartheta}) & XB - C'(\vartheta,\bar{\vartheta},\rho) \\ B'X - C(\vartheta,\bar{\vartheta},\rho) & -D(\vartheta,\bar{\vartheta},\rho) - D'(\vartheta,\bar{\vartheta},\rho) \end{array}\right] < 0. \tag{4.50}$$

Lemma 4.3 and the properties of $[A(\bar{\vartheta}),B,C(\vartheta,\bar{\vartheta},\rho),D(\vartheta,\bar{\vartheta},\rho)]$ show how it is possible to look for values of ϑ ensuring strictly positive real character to $T(\Psi_{\bar{\vartheta},\vartheta},\rho,\vartheta;s)$ through the solution of standard LMI feasibility problems. Indeed, given a value of ρ, efficient numerical methods exist to determine ϑ such that (4.50) is satisfied for some $X = X' > 0$. It is easy to see that, since $\tilde{\rho}(\bar{\vartheta};\vartheta)$ is only dependent on the filter $\Phi_{\bar{\vartheta}}(s)$, if the expression of such a filter for $\bar{\vartheta}$ corresponding to the optimal ϑ were known, the global optimum could be found at once. As a matter of fact, the feasibility problem (4.50) allows to determine if a value of ϑ exists such that $\tilde{\rho}(\bar{\vartheta};\vartheta) \geq \rho$ for given ρ, as the following result clarifies. This result is a straightforward consequence of Lemmas 4.2, 4.3 and is fundamental for defining our optimization procedure.

Theorem 4.5 *Let the closed loop stability margin ρ_ϑ be defined in a neighbourhood $\vartheta_{\bar{\vartheta}}$ of $\bar{\vartheta}$. Compute $\Phi_{\bar{\vartheta}}(s)$ as in Theorem 4.3 and consider the corresponding $\Psi_{\bar{\vartheta},\vartheta}(s)$. Given*

$\rho > 0$, compute $T(\Psi_{\bar{\vartheta},\vartheta}, \rho, \vartheta; s)$ and the coefficients of a canonical controllable state space realization $[A(\bar{\vartheta}), B, C(\vartheta, \bar{\vartheta}, \rho), D(\vartheta, \bar{\vartheta}, \rho)]$ as affine functions of ϑ.

1. *There exists ϑ such that*

$$\tilde{\rho}(\bar{\vartheta}; \vartheta) \geq \rho \tag{4.51}$$

if and only if ϑ is a solution of the following LMI feasibility problem

$$\begin{bmatrix} A'(\bar{\vartheta})X + XA(\bar{\vartheta}) & XB - C'(\vartheta, \bar{\vartheta}, \rho) \\ B'X - C(\vartheta, \bar{\vartheta}, \rho) & -D(\vartheta, \bar{\vartheta}, \rho) - D'(\vartheta, \bar{\vartheta}, \rho) \end{bmatrix} < 0 \quad ; \quad X = X' > 0. \tag{4.52}$$

2. *The parametric stability margin $\rho_{\bar{\vartheta}}$ is given by the solution of the following LMI optimization problem*

$$\rho_{\bar{\vartheta}}^{-1} = \lim_{\varepsilon,\tau \to 0} \min \rho^{-1}$$
$$\text{s.t.}$$
$$\begin{bmatrix} A'(\bar{\vartheta})X + XA(\bar{\vartheta}) & XB - C'(\bar{\vartheta}, \bar{\vartheta}, \rho) \\ B'X - C(\bar{\vartheta}, \bar{\vartheta}, \rho) & -D(\bar{\vartheta}, \bar{\vartheta}, \rho) - D'(\bar{\vartheta}, \bar{\vartheta}, \rho) \end{bmatrix} < 0 \quad ; \quad X = X' > 0. \tag{4.53}$$

The last result makes it clear how it is possible to obtain a solution to the restricted complexity stability margin optimization problem by means of a procedure involving, at each step, the computation of a rational function $\Phi_{\vartheta}(s)$ according to Theorem 4.3 and the solution of a LMI feasibility problem of the form (4.52). This procedure can be formalized with the following algorithm.

Algorithm 4.1 Given the uncertain plant family and the parameters $A_0(s)$, $\bar{A}(s)$, $B_0(s)$, $\bar{B}(s)$, a tolerance value $\sigma > 0$ and the maximum number of iterations $k_{\max}$.

1. Compute a nominal stabilizing controller $C_0(s)$;

2. Choose the controller class parameterization $\bar{N}(s), \bar{D}(s)$;

3. Set $\bar{\vartheta} = 0$;

   Compute the vector $G_0(s)$ and the transfer function $\Phi_0(s)$ as in Theorem 4.3;

   Compute (an approximation of) ρ_0 via the solution of the LMI optimization problem (4.53) in Theorem 4.5;

4. Set $k = 0$, $\vartheta_f = \bar{\vartheta} = 0$, $\rho_f = \rho_0$, $\Phi_{\bar{\vartheta}}(s) = \Phi_0(s)$;

   Set $\rho_c = 2\rho_0$;

5. Repeat

   (a) Compute the coefficients of the state space realization of $T(\Psi_{\bar{\vartheta},\vartheta}, \rho, \vartheta; s)$;

   Solve the LMI feasibility problem (4.52) in Theorem 4.5 for $\rho = \rho_c$;

   (b) If the problem has a feasible solution ϑ_f, then

      i. set $\bar{\vartheta} = \vartheta_f$;

      ii. set $\rho_f = \rho_c$;

      iii. set $\rho_c = 2\rho_c$;

      iv. compute a new $\Phi_{\bar{\vartheta}}(s)$ as in Theorem 4.3;

   else

      i. if $|\rho_c - \rho_f| < \sigma$ then exit: $\bar{\vartheta}, \rho_f$ is the solution.

      ii. set $\rho_c = (\rho_c + \rho_f)/2$;

   (c) set $k = k + 1$;

   until $k \geq k_{\max}$;

6. Maximum number of iterations reached.

Remark 4.6 The algorithm described above exploits the properties of $\tilde{\rho}(\bar{\vartheta};\vartheta)$ stated in Lemma 4.1 and the LMI formulation of the robust SPR condition on $T(\Psi_{\bar{\vartheta},\vartheta},\rho,\vartheta;s)$ in Lemma 4.3. The main idea lies on the fact that, given $\bar{\vartheta}$ providing a stability margin $\rho_{\bar{\vartheta}}$ and a value of $\rho > \rho_{\bar{\vartheta}}$, a standard procedure can be employed to determine, if it exists, a value of ϑ such that $\tilde{\rho}(\bar{\vartheta};\vartheta) \geq \rho$. Since $\tilde{\rho}(\bar{\vartheta};\vartheta)$, under the hypotheses of Lemma 4.1, is a locally smooth function approximating the true parametric stability margin ρ_ϑ from below in a neighbourhood of $\bar{\vartheta}$ (see again Fig. 4.3), the value of ϑ solving the LMI feasibility problem in Theorem 4.5 for a given ρ is such that $\rho_\vartheta > \rho$ (Fig. 4.4). Hence, by computing $[A(\bar{\vartheta}), B, C(\vartheta,\bar{\vartheta},\rho), D(\vartheta,\bar{\vartheta},\rho)]$ at every step, solving the corresponding LMI feasibility problem and proceeding by bisection on ρ, it is possible to converge to a local maximum of ρ_ϑ. More precisely, the proposed procedure may stop only in the following situations:

1. At a local maximum for ρ_ϑ;

2. At a value of ϑ for which condition 3 in Lemma 4.1 is not met. In this case, $\tilde{\rho}(\bar{\vartheta};\vartheta)$ may be a non-smooth approximation of ρ_ϑ in $\vartheta = \bar{\vartheta}$ (see Remark 4.5) and the LMI problem (4.52) may turn out to be non-feasible for whatever choice of $\rho > \rho_{\bar{\vartheta}}$.

3. At a saddle point of the function ρ_ϑ. Also in this case, there may not exist ϑ such that $\tilde{\rho}(\bar{\vartheta};\vartheta) > \tilde{\rho}(\bar{\vartheta};\bar{\vartheta})$ and (4.52) may be non-feasible as well.

An interesting property of the proposed algorithm, which obviously cannot be guaranteed to occur in general, derives from the fact that the numerical procedure for solving the LMI problem (4.52) *will always find* a feasible solution when it exists. Consider the situation depicted in Fig. 4.6: although $\bar{\vartheta}$ is (close to) a local maximum of ρ_ϑ, the surrogate stability margin function $\tilde{\rho}(\bar{\vartheta};\vartheta)$ approximates ρ_ϑ in a way such that a feasible solution ϑ_f exists for $\rho > \rho_{\bar{\vartheta}}$, thus allowing the algorithm to *escape* from the local maximum. The same argument applies to values of $\bar{\vartheta}$ for which ρ_ϑ or $\tilde{\rho}(\bar{\vartheta};\vartheta)$ are non-smooth, and is most likely to hold in saddle points of ρ_ϑ.

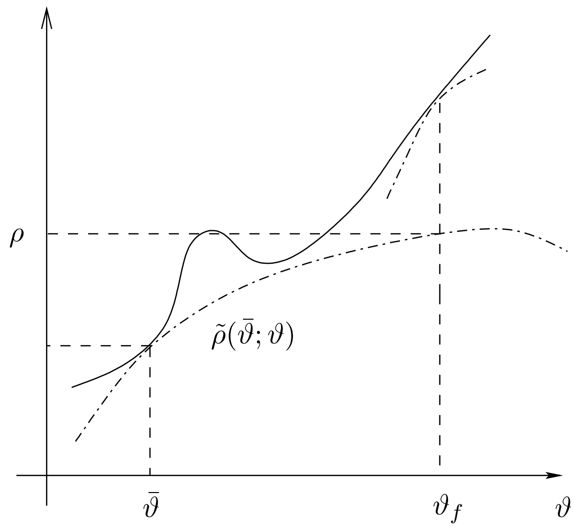


Figure 4.6: The algorithm “escapes” from a local maximum

4.5 *Examples*

4.5.1 Stabilty margin optimization using PID and lag-lead like compensators

We will briefly describe how controller structures such as PID compensators and lag-lead like networks can be parameterized in order to be employed in the $RCSMM$ procedure.

- PID **controller**

$$C(s) = (K_P^0 + \tilde{K}_P) + \frac{K_I^0 + \tilde{K}_I}{s} + \frac{(K_D^0 + \tilde{K}_D)s}{1 + Ts} \tag{4.54}$$

where

$$C_0(s) = K_P^0 + \frac{K_I^0}{s} + \frac{K_D^0 s}{1 + Ts} \tag{4.55}$$

is a stabilizing PID compensator for the nominal plant and $\tilde{K}_P$, $\tilde{K}_I$ and $\tilde{K}_D$ are the tunable parameters.

$C(s)$ belongs to the restricted complexity controller class $\mathcal{C}$ with

$$N_0(s) = K_P^0 s(1 + Ts) + K_I^0(1 + Ts) + K_D^0 s^2 \ ; \ D_0(s) = s(1 + Ts) \ ;$$
$$\bar{N}(s) = [1 + Ts \quad s(1 + Ts) \quad s^2]' \ ; \ \bar{D}(s) = [0 \ \ 0 \ \ 0]' \ ; \ \vartheta = [\tilde{K}_I \quad \tilde{K}_P \quad \tilde{K}_D]'. \tag{4.56}$$

- **Lag-lead like network**

  The controller

$$C(s) = \frac{K_C^0}{s^h}\frac{1+\tau_1 s}{1+\tau_2 s}\frac{1+\tau_3 s}{1+\tau_4 s} \tag{4.57}$$

  can be represented as a member of $\mathcal{C}$ with

$$\begin{gathered} N_0(s) = b_2 s^2 + b_1 s + b_0 \ ; \ D_0(s) = s^h(a_2 s^2 + a_1 s + a_0) \ ; \\ \bar{N}(s) = [0\ 0\ 0\ 1\ s\ s^2]' \ ; \ \bar{D}(s) = s^h[1\ s\ s^2\ 0\ 0\ 0]' \ ; \ \vartheta = [\vartheta_0\ \vartheta_1\ \vartheta_2\ \vartheta_3\ \vartheta_4\ \vartheta_5]'. \end{gathered} \tag{4.58}$$

  As usual, $C_0(s)$ is a stabilizing controller for the nominal plant.

Example 4.1 Consider the simple uncertain plant

$$W(s;\delta) = \frac{1+\delta_1}{s+1+\delta_2}$$

and a proportional controller of the form

$$C_\vartheta(s) = -0.5 + \vartheta.$$

Clearly, $C_0(s) = -0.5$ stabilizes the nominal plant and ϑ is the tunable parameter (the nominal closed loop is stable for all $\vartheta > -0.5$). The characteristic polynomial is given by

$$P_\vartheta(s;\delta) = s + (-0.5+\vartheta)\delta_1 + \delta_2 + 0.5 + \vartheta.$$

The l_2 parametric stability margin can be simply computed as a function of ϑ in closed form, since there is only one root which can cross the stability boundary only at $s = 0$.

$$\begin{gathered} \rho_\vartheta^2 = \min \delta_1^2 + \delta_2^2 \\ \text{s.t} \\ (-0.5+\vartheta)\delta_1 + \delta_2 + 0.5 + \vartheta = 0. \end{gathered}$$

Thus we get

$$\rho_\vartheta = \frac{0.5+\vartheta}{\sqrt{(-0.5+\vartheta)^2+1}}.$$

The stabilty margin is maximized for $\vartheta = \vartheta^* = 1.5$ with $\rho_{\vartheta^*} = \sqrt{2}$ (see Fig. 4.7). We have

$$G_\vartheta(s) = -\frac{[-0.5+\vartheta\ \ 1]'}{s+0.5+\vartheta}.$$

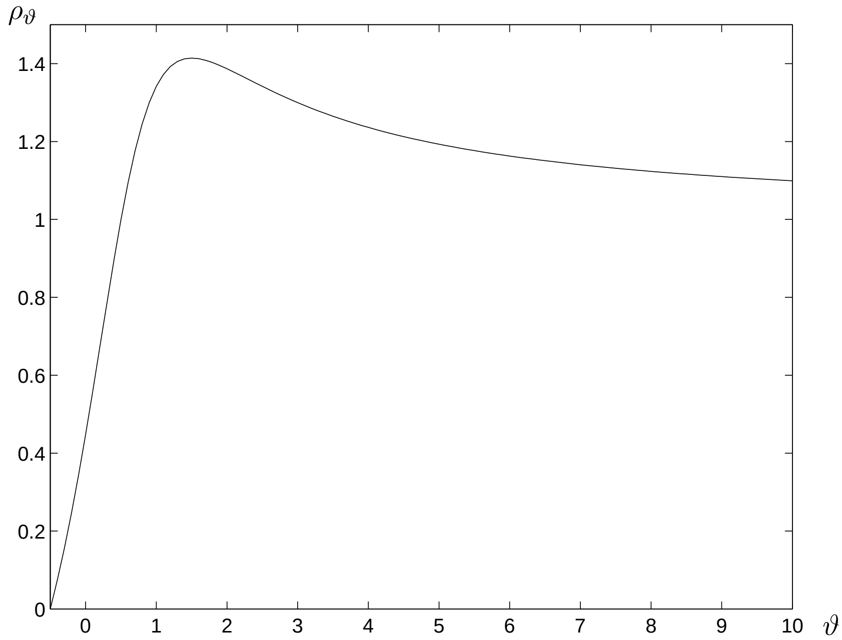


Figure 4.7: Example 4.1: Parametric stability margin ρ_ϑ

The filter $\Phi_{\bar{\vartheta}}(s)$ can easily be computed for generic $\bar{\vartheta}$ yielding

$$\Phi_{\bar{\vartheta}}(s) = \frac{s + 0.5 + \bar{\vartheta}}{s + \varepsilon}$$

hence

$$\Psi_{\bar{\vartheta},\vartheta}(s) = \frac{s + 0.5 + \vartheta}{s + \varepsilon}.$$

We note that in this case $\Psi_{\bar{\vartheta},\vartheta}(s)$ does not depend on $\bar{\vartheta}$ (the filter $\Phi_{\bar{\vartheta}}(s)$ is somehow "universal"). This means that also $\tilde{\rho}(\bar{\vartheta};\vartheta)$ is independent of $\bar{\vartheta}$ and hence $\rho_\vartheta = \tilde{\rho}(\bar{\vartheta};\vartheta)$ (for $\varepsilon \to 0$). Therefore, the $RCSMM$ procedure must converge to the global optimum. Anyway, this particular case is quite trivial since only one maximum is actually present. In Fig. 4.8 the sequence of feasible values of ϑ is shown; in Fig. 4.9, the solid line shows the value of ρ for which the LMI problem is found feasible at each optimization step and the dashed line shows the analytically computed "true" stability margin at the same step.

Example 4.2 Consider the uncertain plant

$$W(s;\delta) = \frac{10 + \delta_1}{(s+1)(s^2 + (2 + 0.1\delta_2)s + 10)}$$

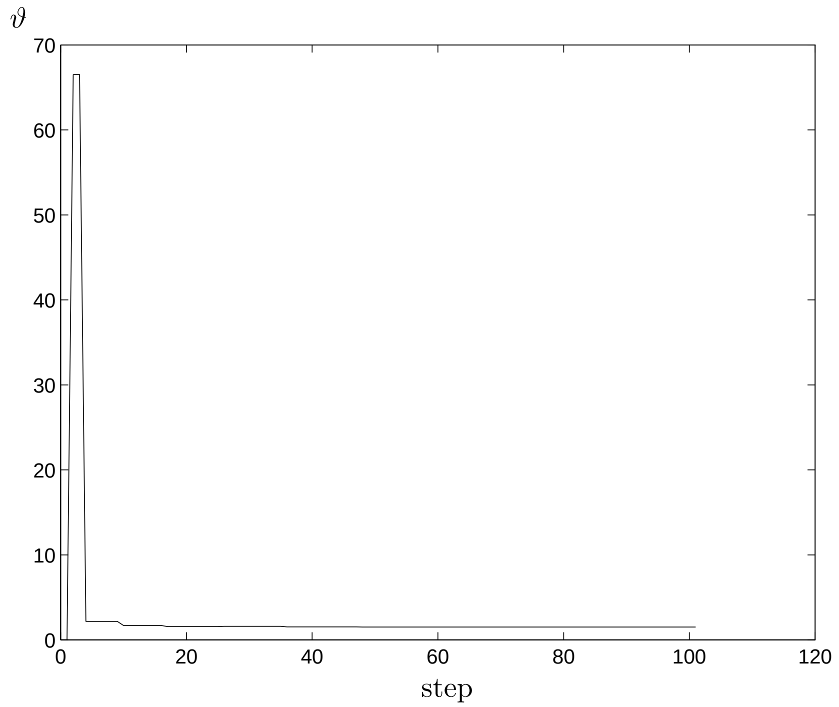


Figure 4.8: Example 4.1: $RCSMM$ procedure: parameter value ϑ vs. optimization step.

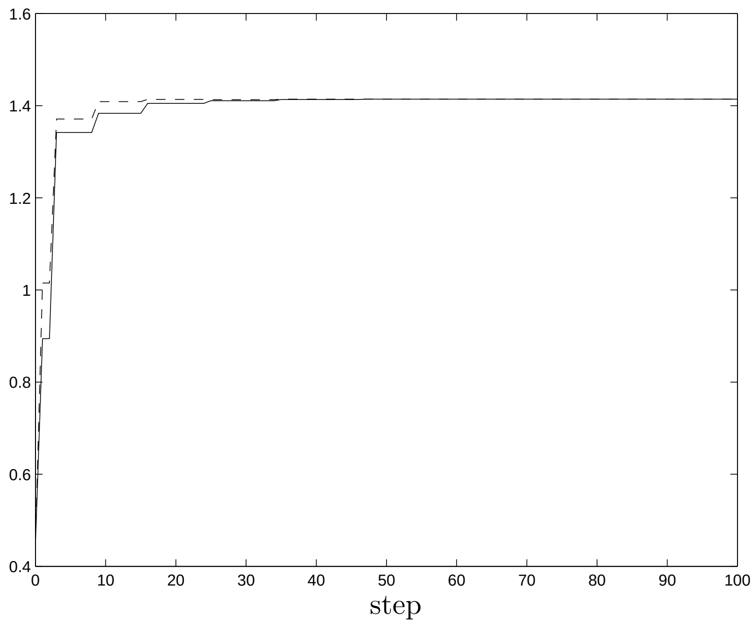


Figure 4.9: Example 4.1: feasible stability margin (solid line), "real" stability margin (dashed line) during $RCSMM$ procedure.

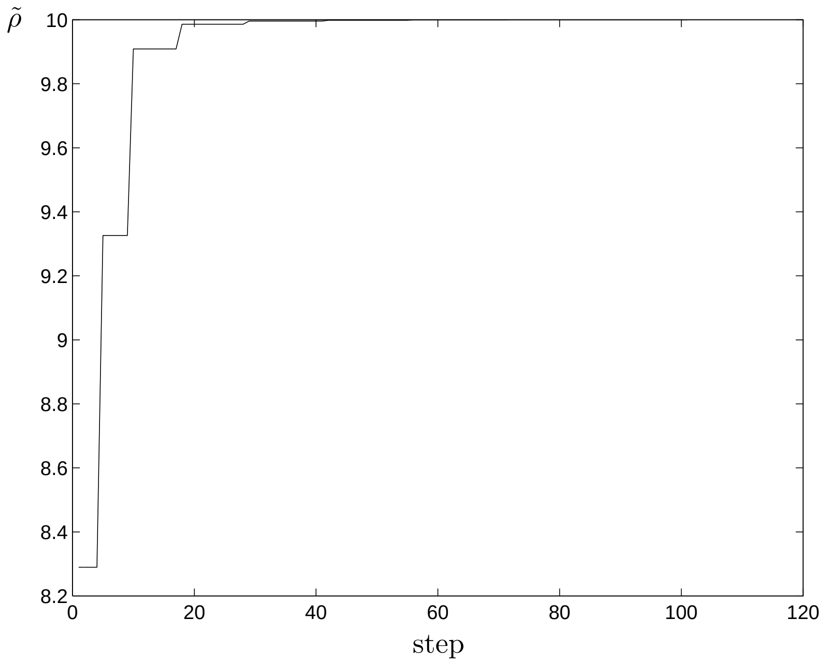


Figure 4.10: Example 4.2: feasible surrogate stability margin vs. optimization step.

and the PI controller

$$C_\vartheta(s) = 1 + \vartheta_1 + \frac{0.5 + \vartheta_2}{s}.$$

Again, $C_0(s)$ stabilizes the nominal plant and $\vartheta = [\vartheta_1 \ \vartheta_2]'$ is the tunable parameter vector.

The $RCSMM$ procedure progress is depicted in Fig. 4.10 (feasible surrogate stability margin), Fig. 4.11 (ϑ_1), Fig. 4.12 (ϑ_2). The optimal values of the controller parameters turn out to be

$$\vartheta_1^* = -0.54 \quad ; \quad \vartheta_2^* = -0.45$$

and the optimal PI controller is given by

$$C_{\vartheta^*}(s) = 0.46 + \frac{0.05}{s}.$$

Example 4.3 Let

$$W(s;\delta) = \frac{1+\delta_1}{s - 4 + \delta_2}.$$

A stabilizing controller for the nominal (unstable) plant is given by

$$C_0(s) = 20\frac{s+4}{s+10}$$

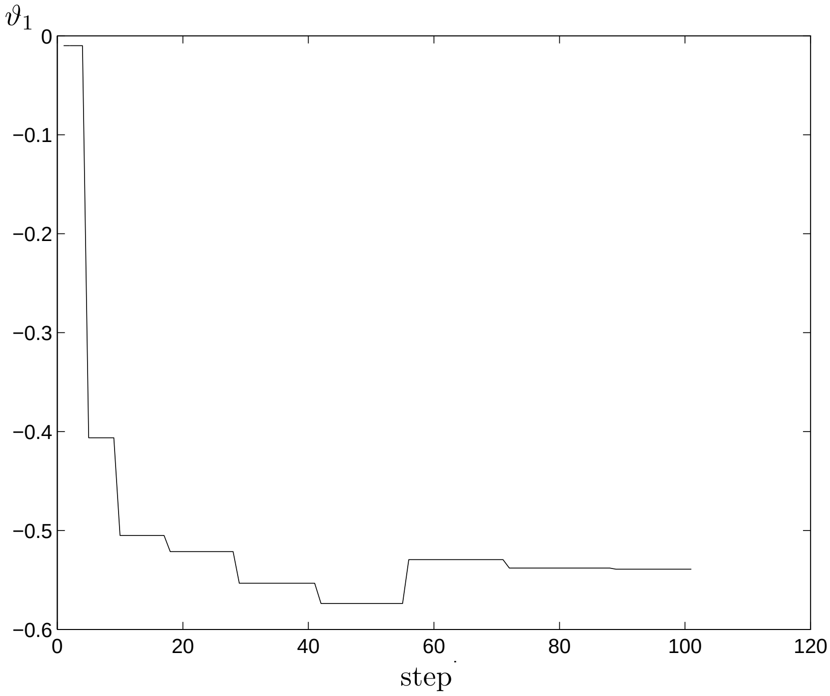


Figure 4.11: Example 4.2: ϑ_1 vs. optimization step

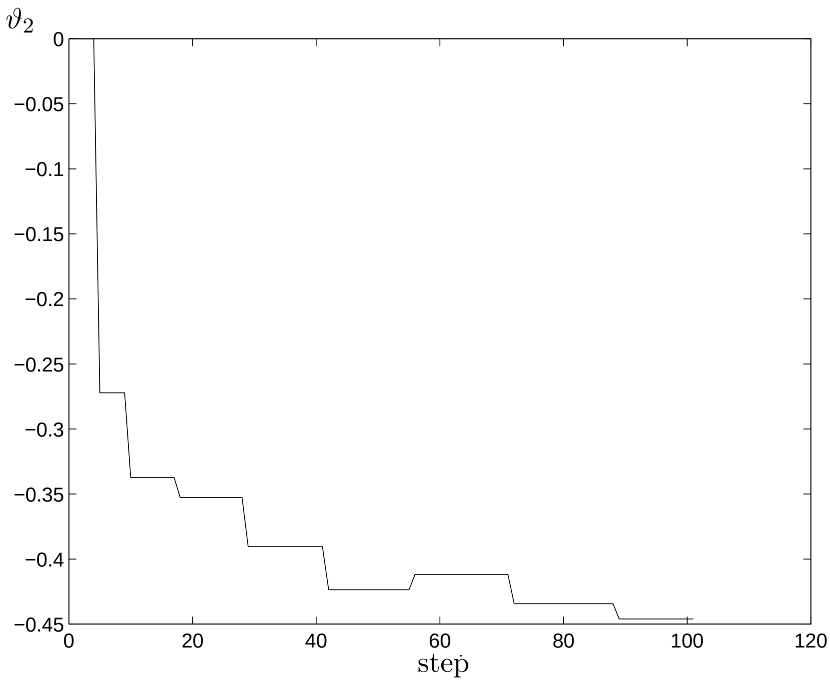


Figure 4.12: Example 4.2: ϑ_2 vs. optimization step.

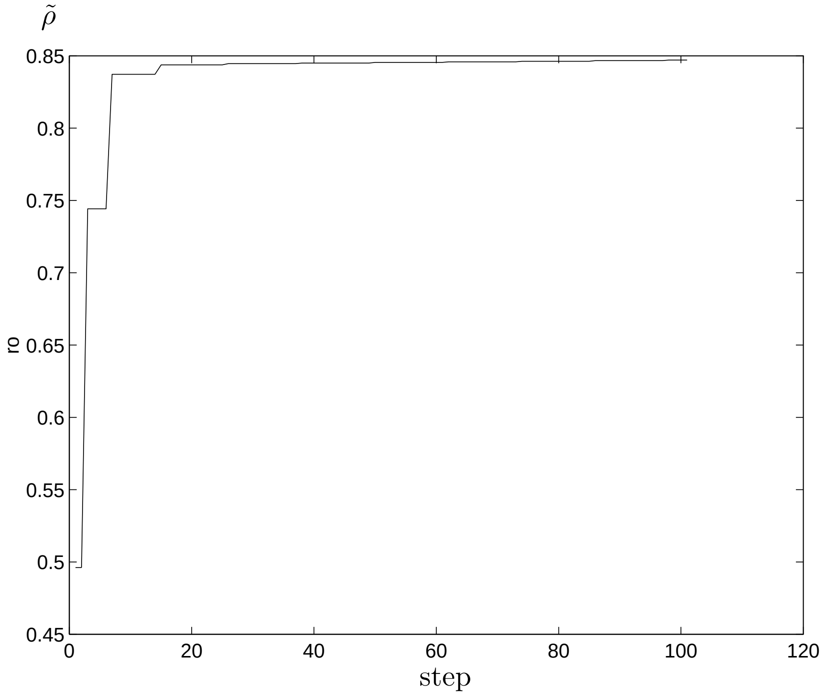


Figure 4.13: Example 4.3: feasible surrogate stability margin vs. optimization step.

hence, we can apply the $RCSMM$ procedure to the following controller class

$$C_\vartheta(s) = 20\frac{s + 4 + \vartheta_1}{s + 10 + \vartheta_2}.$$

The optimal parameter values turn out to be

$$\vartheta_1^* = -2.7 \;\; ; \;\; \vartheta_2^* = -9$$

(see Fig. 4.13,4.14).

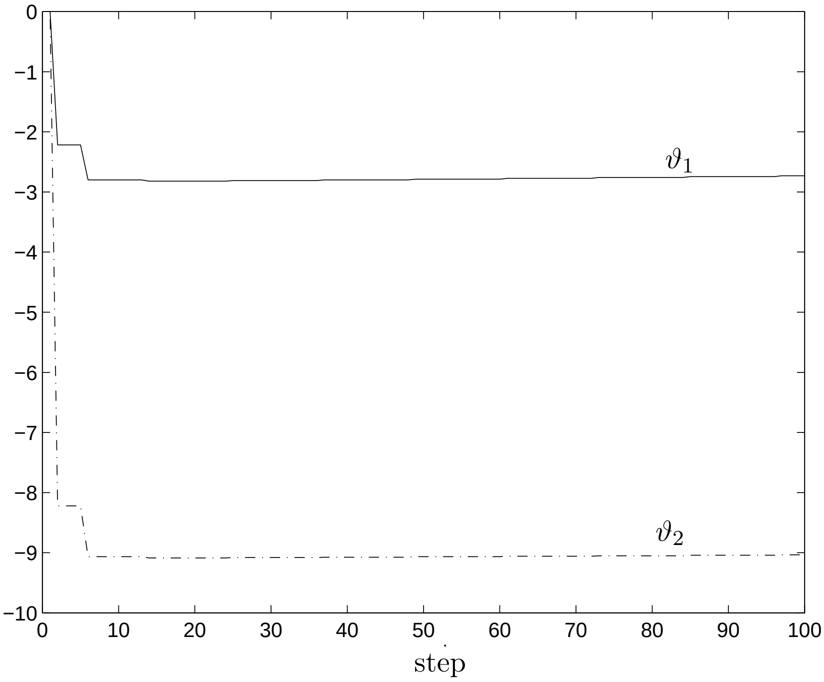


Figure 4.14: Example 4.3: ϑ_1 (solid) and ϑ_2 (dashed) vs. optimization step.

Appendix A

l_2 robust SPR synthesis

A.1 Introduction

The solution of the Restricted Complexity Stability Margin Maximization problem, addressed in Chapter 4 of this thesis, relies on the possibility of designing a rational filter ensuring the Strict Positive Realness (SPR) property to a family of transfer functions described by a parameter vector (see Remark 4.3). Besides this, the study of invariance of the SPR property with respect to perturbations is a relevant issue in the analysis of absolute stability of nonlinear Lur'e systems and the design of adaptive schemes (see, e.g., [13],[28]). As a matter of fact, a sufficient condition for the convergence of several recursive algorithms of adaptive schemes is indeed the SPR of a suitable family of transfer functions (see, e.g., [26],[25]).

The robust SPR filter synthesis problem in its most general formulation is the following: given a set of polynomials $\mathcal{P}$ and a region Λ of the complex plane, determine if there exists a polynomial (or rational) filter F such that each transfer function P/F, $P \in \mathcal{P}$ is strictly positive real over Λ. For instance, in the context of recursive identification schemes, the set $\mathcal{P}$ can be viewed as a model of the uncertainty about the true plant and Λ is the region of the complex plane where the power spectral density of the regressor is concentrated.

Several useful results are available on the existence and construction of F for different choices of $\mathcal{P}$ and Λ. In [14],[4] the continuous-time and discrete-time robust SPR problems are considered when $\mathcal{P}$ is a polyhedron in the coefficient space, while in [37],[35] the set $\mathcal{P}$ is described in terms of root location regions and Λ is some subset of the complement of the unit disk.

In [1] an important result stating necessary and sufficient conditions for the existence of the sought filter is given when $\mathcal{P}$ is assumed to be a polyhedron in the coefficient space. Such condition simply requires that all the polynomials of the set $\mathcal{P}$ are stable. The corresponding filter turns out to be a polynomial in the discrete-time case and in general a rational function in the continuous-time case. In addition, a procedure for constructing the filter F as a series expansion is given. However, this technique does not provide the filter F in closed form, i.e., F may have an arbitrarily high degree. On the other hand, some sufficient conditions have been given to ensure the existence of a polynomial filter [4] and a finite degree rational filter [27] when $\mathcal{P}$ is an interval polynomial. Finally, a finite degree rational filter can also be designed when the set $\mathcal{P}$ contains only two discrete time polynomials [28].

We consider the robust SPR problem in the context used for the $RCSMM$ procedure, i.e., the set $\mathcal{P}$ is supposed to be an ellipsoid in the coefficient space and the involved transfer functions are assumed continuous-time [9]. An extension of the presented results to the discrete-time case is also possible [8]. First, exploiting the results in [1], it is shown that the stability of the polynomials of $\mathcal{P}$ is a necessary and sufficient condition for the existence of the filter F. Then, a completely different analysis is performed in order to construct a solution of the problem in closed form and with an a-priori bounded degree. More specifically, it turns out that the filter F is a rational function having a degree less than twice the degree of the polynomials of the set $\mathcal{P}$. Moreover, F can be obtained in closed form via a suitable polynomial factorization problem.

We recall the definitions of positive realness (PR) and strict positive realness (SPR) of a rational function [21].

Definition A.1 A rational function $\Phi(s)$ is positive real if

1. $\Phi(s)$ is real for real s;

2. $\Phi(s)$ is analytic in $\mathrm{Re}[s] > 0$ and the poles on the imaginary axis are simple and such that the associated residue is non-negative;

3. for any real value of ω for which $s = j\omega$ is not a pole of $\Phi(s)$, $\mathrm{Re}\,[\Phi(j\omega)] \geq 0$.

Definition A.2 A rational function $\Phi(s)$ is said to be strictly positive real if

1. $\Phi(s) \in \mathcal{RH}_\infty$;

2. $\mathrm{Re}\,[\Phi(j\omega)] > 0 \quad \forall \omega \geq 0$.

Property A.1 $\Phi(s)$ is strictly positive real if and only if $\Phi^{-1}(s)$ is [21].

The following result relates PR and SPR.

Lemma A.1 *Let* $\Phi^*(s) = \dfrac{P_1(s)}{P_2(s)}$ *be positive real. Then, for sufficiently small* $\varepsilon, \tau > 0$, *the function*

$$\Phi(s) = \Phi^*(s+\varepsilon)(1+\tau s)^{\partial P_2 - \partial P_1} \tag{A.1}$$

is strictly positive real.

Proof: See Appendix B.

Property A.2 From a well-known property concerning the relative degree of a positive real rational function, it follows that ∂P_1 and ∂P_2 in (A.1) satisfy the relation:

$$-1 \leq \partial P_2 - \partial P_1 \leq 1. \tag{A.2}$$

A.2 The l_2 continuous-time Robust SPR (RSPR) problem

The robust SPR problem in the continuous-time case can be stated as follows [14],[1]. Given a set of polynomials $\mathcal{P}$, determine, if it exists, a polynomial (or in general a rational function) $F(s)$ such that for any $P(s;\delta) \in \mathcal{P}$ the function $P(s;\delta)/F(s)$ is strictly positive real over the closed right half plane.

We address the robust SPR problem for a set of polynomials described by the l_2 norm of a parameter vector, i.e., an ellipsoid in parameter space centered at a given nominal polynomial.

Definition A.3 An ellipsoidal set of polynomials of degree l is the set

$$\mathcal{P}_\rho := \left\{ P(s;\delta) = P_0(s) + \sum_{i=1}^{n} \delta_i P_i(s) : \|\delta\|_2 < \rho \right\} \tag{A.3}$$

where $P_0(s)$, $P_1(s)$, $\ldots$,$P_n(s)$ are such that $\partial P_0 = l$, $\partial P_i < l$ for all $i = 1, \ldots, n$, $\delta = (\delta_1 \ldots \delta_n)' \in \mathbb{R}^n$ is the parameter vector, and $\rho > 0$.

Note that the characteristic polynomial of the closed loop system considered in the $RCSMM$ problem belongs to a family of this form (see (4.13)).
Recalling Definition A.2, we can state our robust SPR ($RSPR$) problem in the following way.

$RSPR$ **problem.** Given the set $\mathcal{P}_\rho$, determine a transfer function $F(s)$, if it exists, such that the SPR conditions

1.
$$\frac{P(s;\delta)}{F(s)} \in \mathcal{RH}_\infty \tag{A.4}$$

2.
$$\text{Re}\left[\frac{P(j\omega;\delta)}{F(j\omega)}\right] > 0 \ \ \forall\omega \geq 0. \tag{A.5}$$

hold for all $P(s;\delta) \in \mathcal{P}_\rho$.

Remark A.1 We recall that condition (A.5) is equivalent to the phase condition

$$|\arg[P(j\omega;\delta)] - \arg[F(j\omega)]| < \pi/2 \quad \forall\omega \geq 0. \tag{A.6}$$

Clearly, for the solvability of the $RSPR$ problem, conditions (A.4) and (A.5) must hold for $P_0(s)$. Therefore, recalling that the numerator of any SPR function is necessarily a Hurwitz polynomial, the following requirement on the set $\mathcal{P}_\rho$ can be enforced without loss of generality.

Assumption A.1 The nominal polynomial $P_0(s)$ is Hurwitz.

A preliminary result for the $RSPR$ problem can be obtained quite readily. To this purpose, let ρ^* denote the l_2 parametric stability margin of $\mathcal{P}_\rho$, i.e., the maximal ρ such that $\mathcal{P}_\rho$ cointains all Hurwitz polynomials

$$\rho^* = \sup_{\mathcal{P}_\rho \subseteq \mathcal{H}} \rho. \tag{A.7}$$

According to condition (A.4), it follows that the condition $\rho < \rho^*$ is necessary for the solution of the $RSPR$ problem. Exploiting convexity of $\mathcal{P}_\rho$ and the results in [1], it turns out that such a condition is also sufficient.

Theorem A.1 *Consider the set $\mathcal{P}_\rho$ of uncertain polynomials and suppose that $\rho < \rho^*$. Then, there exist a non-negative integer M and a Hurwitz polynomial $R(s)$ of degree $l + M$ such that the rational function*

$$F(s) = \frac{R(s)}{(s+1)^M} \tag{A.8}$$

solves the $RSPR$ problem.

Proof: It follows from a straightforward extension of Theorem 3.1 in [1], once the finite set $\{n_i(s)\}$ is replaced by the convex set $\mathcal{P}_\rho$. Indeed, let

$$\overline{\phi}(\omega) =: \sup_{\|\delta\|_2<\rho} \arg[P(j\omega;\delta)] \quad ; \quad \underline{\phi}(\omega) =: \inf_{\|\delta\|_2<\rho} \arg[P(j\omega;\delta)]. \tag{A.9}$$

Since $\mathcal{P}_\rho$ is a convex degree-invariant set of Hurwitz polynomials, the following condition is true (see [5])

$$\overline{\phi}(\omega) - \underline{\phi}(\omega) < \pi \quad \forall\omega \geq 0. \tag{A.10}$$

Now, introducing the function

$$\phi^*(\omega) = \frac{\overline{\phi}(\omega) + \underline{\phi}(\omega)}{2}, \tag{A.11}$$

it can be easily checked that the relation

$$|\arg[P(j\omega;\delta)] - \phi^*(\omega)| < \frac{\pi}{2} \quad \forall\omega \geq 0 \tag{A.12}$$

holds for each polynomial $P(s;\delta) \in \mathcal{P}_\rho$.

Thus, from Remark A.1 it turns out that the $RSPR$ problem is solved if a function $F^*(s)$ is determined such that $F^{*^{-1}}(s) \in \mathcal{RH}_\infty$ and its phase on the imaginary axis satisfies

$$\arg[F^*(j\omega)] = \phi^*(\omega). \tag{A.13}$$

Employing a series expansion as in [1], it can be shown that $F^*(s)$ can be arbitrarily approximated via a rational function of the form (A.8) for suitable $R(s)$ and M. ◇

Although quite interesting from a conceptual viewpoint, Theorem A.1 only provides a partial solution to the $RSPR$ problem. Indeed, since $F(s)$ is computed via a procedure based on a series expansion, there is no a-priori knowledge of the degree of such filter. In the next section, we will overcome this drawback by introducing a completely new approach to the $RSPR$ problem, which allows us to construct a rational filter F with a finite known degree via the solution of a suitable factorization problem.

A.3 RSPR problem solution

To solve the robust SPR problem, we first recall the expression of the l_2 stability margin given in Theorem 2.7 in the case of $\mathcal{P}_\rho$ being a degree-invariant polynomial family.

Let

$$G(s) = -\left[\frac{P_1(s)}{P_0(s)} \cdots \frac{P_n(s)}{P_0(s)}\right]', \tag{A.14}$$

recall the two functions

$$R(\omega) = \mathrm{Re}[G(j\omega)], \quad I(\omega) = \mathrm{Im}[G(j\omega)] \tag{A.15}$$

and the two complementary sets of frequencies

$$\Omega_0 \;=\; \{\omega \geq 0 \;:\; I(\omega) = 0\} \tag{A.16}$$

$$\overline{\Omega}_0 \;=\; \{\omega \geq 0 \;:\; I(\omega) \neq 0\}. \tag{A.17}$$

Lemma A.2 *Let*

$$\rho_0 = \min_{\omega \in \Omega_0} \frac{1}{\|R(\omega)\|_2} \tag{A.18}$$

$$\bar{\rho} = \inf_{\omega \in \bar{\Omega}_0} \tilde{\rho}(\omega) \tag{A.19}$$

where

$$\tilde{\rho}(\omega) = \begin{cases} \dfrac{\|I(\omega)\|_2}{\left[\|I(\omega)\|_2^2 \, \|R(\omega)\|_2^2 - (R'(\omega)I(\omega))^2\right]^{1/2}} & \text{if } \omega \in \bar{\Omega}_s \\ +\infty & \text{if } \omega \notin \bar{\Omega}_s \end{cases} \tag{A.20}$$

being

$$\bar{\Omega}_s = \left\{\omega \in \bar{\Omega}_0 \;:\; \|I(\omega)\|_2^2 \, \|R(\omega)\|_2^2 - \left(R'(\omega)I(\omega)\right)^2 \neq 0\right\}. \tag{A.21}$$

Then, the l_2 parametric stability margin of $\mathcal{P}_\rho$ is given by

$$\rho^* = \begin{cases} \rho_0 & \text{if } n = 1 \\ \min\{\rho_0, \bar{\rho}\} & \text{if } n > 1 \end{cases}. \tag{A.22}$$

It is straightforward to check that the $RSPR$ problem can be restated equivalently as follows. Determine a function $\Phi(s)$ such that

$$\Phi(s) \in \mathcal{RH}_\infty \tag{A.23}$$

and

$$\mathrm{Re}\left[\Phi(j\omega)\left(1 - \delta' G(j\omega)\right)\right] > 0 \quad \forall \omega \geq 0 \quad \forall \delta : \|\delta\|_2 < \rho. \tag{A.24}$$

Obviously, once $\Phi(s)$ has been determined, $F(s)$ is readily obtained via the relation

$$F(s) = \frac{P_0(s)}{\Phi(s)}. \tag{A.25}$$

The starting point for determining $\Phi(s)$ is the next result (see also [31]), where condition (A.24) is rewritten into an equivalent form no longer dependent on the parameter vector δ.

Lemma A.3 *Let $G(j\omega), R(\omega), I(\omega)$ be defined as in (A.14) and (A.15). Then, the following two statements are equivalent:*

1.
$$\mathrm{Re}\left[\Phi(j\omega)\left(1-\delta' G(j\omega)\right)\right] > 0 \quad \forall\omega \geq 0 \quad \forall\delta : \|\delta\|_2 < \rho; \tag{A.26}$$

2.
$$\begin{array}{ll}(a) & \mathrm{Re}[\Phi(j\omega)] > 0 \\ (b) & \|R(\omega) - \gamma_\Phi(\omega) I(\omega)\|_2^2 < \dfrac{1}{\rho^2}\end{array} \qquad \forall\omega \geq 0 \tag{A.27}$$

where
$$\gamma_\Phi(\omega) := \frac{\mathrm{Im}[\Phi(j\omega)]}{\mathrm{Re}[\Phi(j\omega)]}. \tag{A.28}$$

Proof: see Appendix B.

Note that conditions (A.23) and (A.27a) imply that $\Phi(s)$ must be a strictly positive real rational function. Hence, the *RSPR* problem amounts to determine a strictly positive real $\Phi(s)$ such that the inequality

$$\|R(\omega) - \gamma(\omega) I(\omega)\|_2^2 < \frac{1}{\rho^2} \tag{A.29}$$

is satisfied for $\gamma(\omega) = \gamma_\Phi(\omega)$ for all $\omega \geq 0$.

Therefore, a central issue for the solution of the *RSPR* problem is the characterization of the following set of functions

$$\Gamma := \{\gamma(\omega) \ : \ \gamma(\omega) \text{ is bounded continuous and satisfies (A.29)}\}. \tag{A.30}$$

The function
$$\gamma^*(\omega) = \frac{R'(\omega) I(\omega)}{\|I(\omega)\|_2^2} \tag{A.31}$$

defined for $\omega \in \bar{\Omega}_0$ plays a key role in such a characterization.

Lemma A.4 *Let ρ^* be the parametric stability margin of $\mathcal{P}_\rho$ (see Lemma A.2) and suppose $\rho < \rho^*$. Then, the following statements hold.*

1. *Γ is the set of bounded continuous functions $\gamma(\omega)$ such that*

$$\underline{\gamma}(\omega) < \gamma(\omega) < \overline{\gamma}(\omega) \quad \forall\omega \in \bar{\Omega}_0 \tag{A.32}$$

where

$$\underline{\gamma}(\omega) = \min\left\{\gamma^*(\omega) \pm \frac{\sqrt{\Delta(\omega)}}{\|I(\omega)\|_2^2}\right\}$$
$$\overline{\gamma}(\omega) = \max\left\{\gamma^*(\omega) \pm \frac{\sqrt{\Delta(\omega)}}{\|I(\omega)\|_2^2}\right\} \tag{A.33}$$

being $\gamma^*(\omega)$ *as in (A.31) and*

$$\Delta(\omega) = \left[R'(\omega)I(\omega)\right]^2 - \|I(\omega)\|_2^2\left[\|R(\omega)\|_2^2 - \frac{1}{\rho^2}\right]. \tag{A.34}$$

2. Γ *is nonempty.*

Proof: see Appendix B.

The above Lemma makes it clear how it is possible to solve the $RSPR$ problem. Indeed, it is sufficient to find a strictly positive real transfer function $\Phi(s)$ such that $\gamma_\Phi(\omega)$ belongs to the set Γ. Since $\gamma_\Phi(\omega)$ is bounded continuous when $\Phi(s)$ is strictly positive real, it is enough to satisfy the relation

$$\underline{\gamma}(\omega) < \gamma_\Phi(\omega) < \overline{\gamma}(\omega) \quad \forall \omega \in \bar{\Omega}_0. \tag{A.35}$$

Consider Fig. A.1(a), where the functions $\underline{\gamma}(\omega)$ (solid lower line) and $\overline{\gamma}(\omega)$ (solid upper line) are depicted for a given $\rho = \rho_1$. In this case, it is easily verified that the function

$$\Phi(s) = 1 \tag{A.36}$$

solves the $RSPR$ problem, since $\gamma_\Phi(\omega) = 0$ is between $\underline{\gamma}(\omega)$ and $\overline{\gamma}(\omega)$. Such a solution leads to the filter

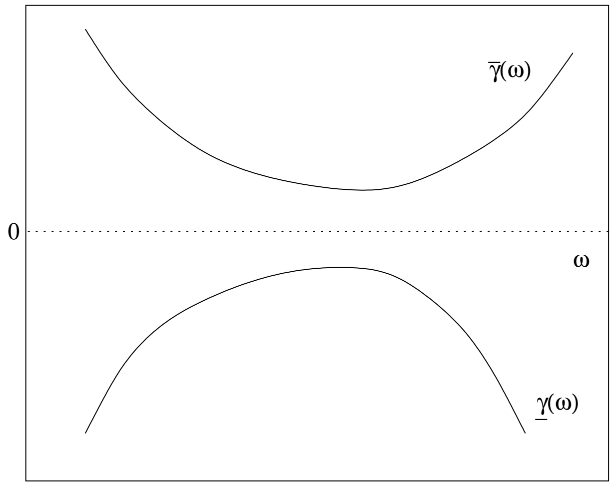
$$F(s) = P_0(s) \tag{A.37}$$

which is the nominal polynomial itself.

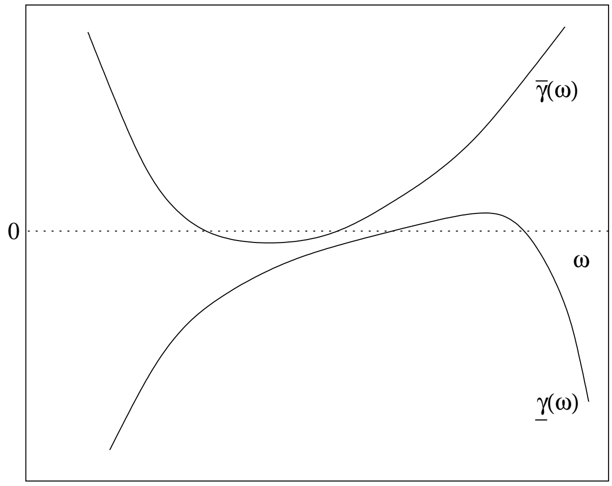
It is clear that such a filter is likely to perform well for small uncertainty, i.e., for values of ρ sufficiently smaller than ρ^*. Indeed, this is the usual way for designing $F(s)$ in several application contexts (see [37],[36]). For larger values of ρ, this is no longer guaranteed as shown in Fig. A.1(b), where $\rho = \rho_2 > \rho_1$ is considered. In this case, the band is narrower and a different solution must be found.

Notice that the following relation holds (see (A.33))

$$\gamma^*(\omega) = \frac{\underline{\gamma}(\omega) + \overline{\gamma}(\omega)}{2} \tag{A.38}$$



(a)



(b)

Figure A.1: (a): $\underline{\gamma}(\omega)$ and $\overline{\gamma}(\omega)$ for $\rho = \rho_1$; (b): $\underline{\gamma}(\omega)$ and $\overline{\gamma}(\omega)$ for $\rho = \rho_2$,$\rho_1 < \rho_2 < \rho^*$.

i.e., the function $\gamma^*(\omega)$ is at each ω the middle point of the band defined by $\underline{\gamma}(\omega)$ and $\overline{\gamma}(\omega)$ for whatever value of ρ less than ρ^*. This observation suggests to look for a strictly positive real rational function $\Phi(s)$ such that $\gamma_\Phi(\omega)$ is as close as possible to $\gamma^*(\omega)$. Since $\gamma^*(\omega)$ does not depend on ρ, such an approach is likely to provide a solution of the $RSPR$ problem for ρ arbitrarily close to ρ^*.

To proceed, we derive some properties of $\gamma^*(\omega)$.

The next Lemma relates the function $\gamma^*(\omega)$ to the polynomial

$$\Pi(s) = \sum_{i=1}^{n} P_0(s)P_i(-s)\left[P_0(-s)P_i(s)\right]_o . \tag{A.39}$$

Lemma A.5 *The following properties hold for the polynomial* $\Pi(s)$*:*

1.

$$\Pi(j\omega) = \left[P_0(j\omega)P_0(-j\omega)\right]^2 \left[I'(\omega)I(\omega) + jR'(\omega)I(\omega)\right] ; \tag{A.40}$$

2.

$$\begin{array}{ll} \mathrm{Re}\left[\Pi(j\omega)\right] \geq 0 & \forall\omega \geq 0 \\ \mathrm{Re}\left[\Pi(j\omega)\right] > 0 & \forall\omega \in \bar{\Omega}_0 \end{array} ; \tag{A.41}$$

3.

$$\gamma^*(\omega) = \frac{\mathrm{Im}\left[\Pi(j\omega)\right]}{\mathrm{Re}\left[\Pi(j\omega)\right]}. \tag{A.42}$$

Proof: see Appendix B.

The following Lemma states the existence of a transfer function $\Phi^*(s)$ such that

$$\gamma_{\Phi^*}(\omega) = \frac{\mathrm{Im}[\Phi^*(j\omega)]}{\mathrm{Re}[\Phi^*(j\omega)]} = \gamma^*(\omega). \tag{A.43}$$

Lemma A.6 *Let* $\Pi_1(s)$ *and* $\Pi_2(s)$ *be any two polynomials such that*

$$\Pi_1(s)\Pi_2(-s) = \Pi(s) \tag{A.44}$$

with $\Pi(s)$ *as in (A.39), and define*

$$\Phi^*(s) = \frac{\Pi_1(s)}{\Pi_2(s)}. \tag{A.45}$$

Then,

$$\left\{ \begin{array}{l} \gamma_{\Phi^*}(\omega) = \gamma^*(\omega) \\ \mathrm{Re}[\Phi^*(j\omega)] > 0 \end{array} \right. \quad \forall\omega \in \bar{\Omega}_0. \tag{A.46}$$

Proof: see Appendix B.

Lemma A.6 suggests the following idea for providing a solution to the $RSPR$ problem: determine a positive real rational function $\Phi^*(s)$ of the form (A.45) and perform a small perturbation of its coefficients in order to obtain an SPR transfer function (see Lemma A.1).

For ease of illustration, we first develop the case $\Omega_0 = \{0\}$, which indeed represents the generic situation (see Remark 2.1). The general case, which requires the same basic steps but some additional technicalities, will be dealt with later.

The following property is a straightforward consequence of the fact that $\Pi(s)$ is zero on the imaginary axis only for $s = 0$ (see (A.40)).

Lemma A.7 *Suppose $\Omega_0 = \{0\}$. Then, $\Pi(s)$ can be factorized as follows:*

$$\Pi(s) = As^r \bar{\Pi}_1(s)\bar{\Pi}_2(-s) \tag{A.47}$$

where A is a real constant, $r \geq 1$ is an integer and $\bar{\Pi}_1(s)$ and $\bar{\Pi}_2(s)$ are uniquely determined monic Hurwitz polynomials. Moreover, $\bar{\Pi}_1(s)$ contains $P_0(s)$ as a factor.

Let us introduce the functions

$$\Phi_e^*(s) = \frac{\bar{\Pi}_1(s)}{\bar{\Pi}_2(s)} \tag{A.48}$$

defined for even r, and

$$\Phi_o^*(s) = \frac{\bar{\Pi}_1(s)}{\bar{\Pi}_2(s)}\; s^{\mathrm{sgn}A\;(-1)^{(r-1)/2}} \tag{A.49}$$

defined for odd r.

We are now ready to give the main result which relies on the fact that $\Phi_e^*(s)$ and $\Phi_o^*(s)$ turn out to be positive real.

Theorem A.2 *Given the set $\mathcal{P}_\rho$, let ρ^* be the parametric stability margin of $\mathcal{P}_\rho$ and suppose the following conditions hold:*

1. *$\rho < \rho^*$;*

2. *$\Omega_0 = \{0\}$.*

Let $\Phi_e^*(s)$ and $\Phi_o^*(s)$ be defined as in (A.48) and (A.49). Then, for sufficiently small positive ε and τ, the rational function

$$\Phi(s) = \begin{cases} \Phi_e^*(s)(1+\tau s)^{\partial\bar{\Pi}_2 - \partial\bar{\Pi}_1} & \text{for even } r \\ \Phi_o^*(s)\left(\dfrac{s+\varepsilon}{s}\right)^{\operatorname{sgn}A\ (-1)^{(r-1)/2}} \cdot \\ \cdot(1+\tau s)^{\partial\bar{\Pi}_2 - \partial\bar{\Pi}_1 - \operatorname{sgn}A\ (-1)^{(r-1)/2}} & \text{for odd } r \end{cases} \tag{A.50}$$

satisfies (A.23) and (A.24) for all $\omega \geq 0$, i.e., the filter

$$F(s) = \frac{P_0(s)}{\Phi(s)} \tag{A.51}$$

solves the robust SPR problem for $\mathcal{P}_\rho$.

Proof: First, it can be easily verified that $\Phi(s)$ in (A.50) satisfies (A.23) by construction.

Lemma A.3 states that condition (A.24) is equivalent to condition (A.27)-(A.28). Thus, we have to prove that $\Phi(s)$ is strictly positive real and such that the inequality

$$\|R(\omega) - \gamma(\omega)I(\omega)\|_2^2 < \frac{1}{\rho^2} \tag{A.52}$$

holds for $\gamma(\omega) = \gamma_\Phi(\omega)$ for all $\omega \geq 0$.

Suppose r is even. As $\bar{\Pi}_1(s)$ and $\bar{\Pi}_2(s)$ are monic Hurwitz polynomials, and (A.41) holds for sufficiently small non-zero ω, it turns out that $Aj^r > 0$. Hence, from (A.47) it follows that $\Pi(s)$ can be rewritten as

$$\begin{aligned} \Pi(s) &= |A|\ s^{r/2}\ (-s)^{r/2}\ \bar{\Pi}_1(s)\bar{\Pi}_2(-s) = \\ &= \left[|A|^{1/2}\ s^{r/2}\ \bar{\Pi}_1(s)\right]\left[|A|^{1/2}\ (-s)^{r/2}\ \bar{\Pi}_2(-s)\right]. \end{aligned} \tag{A.53}$$

Then, by Lemma A.6, the rational function $\Phi_e^*(s)$ satisfies

$$\begin{cases} \gamma_{\Phi_e^*}(\omega) = \gamma^*(\omega) \\ \operatorname{Re}[\Phi_e^*(j\omega)] > 0 \end{cases} \quad \forall\omega > 0. \tag{A.54}$$

Furthermore, since $\bar{\Pi}_1(s)$ and $\bar{\Pi}_2(s)$ are monic Hurwitz polynomials, it follows that $\operatorname{Re}[\Phi_e^*(0)] > 0$, and therefore we have

$$\operatorname{Re}[\Phi_e^*(j\omega)] > 0 \quad \forall\omega \geq 0. \tag{A.55}$$

Since $\bar{\Pi}_2(s)$ is Hurwitz, we conclude that $\Phi_e^*(s)$ is a positive real rational function.

Now, consider the function

$$\Phi(s) = \Phi_e^*(s)(1+\tau s)^{\partial\bar{\Pi}_2 - \partial\bar{\Pi}_1}. \tag{A.56}$$

According to Definition A.2, by (A.55) $\Phi(s)$ is strictly positive real for sufficiently small positive τ.

It remains to show that, if $\rho < \rho^*$, $\Phi(s)$ satisfies (A.52) for suitable τ. Now, exploiting Lemmas A.4, A.5, and A.6 and the fact that $\gamma_{\Phi_e^*}(0) = 0$, it turns out that $\Phi_e^*(s)$ satisfies (A.52) for $\gamma(\omega) = \gamma_{\Phi_e^*}(\omega)$ for any $\rho < \rho^*$ and any $\omega \geq 0$. Moreover, since $\gamma_\Phi(\omega)$ is continuous with respect to τ, it turns out that the left term of inequality (A.52) for $\gamma(\omega) = \gamma_\Phi(\omega)$ depends continuously on τ. Hence, observing that $\gamma_\Phi(0) = 0$, we can conclude that for sufficiently small positive τ, condition (A.52) is also satisfied by $\gamma_\Phi(\omega)$ for all $\omega \geq 0$.

Now suppose r is odd. Again from (A.47) and taking (A.41) into account, $\Pi(s)$ can be expressed as

$$\begin{aligned} \Pi(s) &= A\ s\ s^{r-1}\ (-1)^{(r-1)/2}\ (-1)^{(r-1)/2}\ \bar{\Pi}_1(s)\bar{\Pi}_2(-s) = \\ &= A\ s\ (-1)^{(r-1)/2}\ s^{(r-1)/2}\ (-s)^{(r-1)/2}\ \bar{\Pi}_1(s)\bar{\Pi}_2(-s) = \\ &= s\ \mathrm{sgn}\ A\ (-1)^{(r-1)/2}\ \left[|A|^{1/2}\ s^{(r-1)/2}\ \bar{\Pi}_1(s)\right] \cdot \\ &\quad \cdot \left[|A|^{1/2}\ (-s)^{(r-1)/2}\bar{\Pi}_2(-s)\right]. \end{aligned} \tag{A.57}$$

By Lemma A.6, $\Phi_o^*(s)$ satisfies

$$\begin{cases} \gamma_{\Phi_o^*}(\omega) = \gamma^*(\omega) \\ \mathrm{Re}[\Phi_o^*(j\omega)] > 0 \end{cases} \quad \forall \omega > 0. \tag{A.58}$$

Obviously, $\Phi_o^*(s)$ is analytic for $\mathrm{Re}[s] > 0$.

In order to prove that $\Phi_o^*(s)$ is positive real, it suffices to show that both $\Phi_o^*(s)$ and its inverse $\Phi_o^{*-1}(s)$ have real positive residues in $s = 0$, when $s = 0$ is actually a (simple) pole of either transfer function.

- If $\mathrm{sgn}\ A\ (-1)^{(r-1)/2} = -1$ we have

$$\mathrm{Res}[\Phi_o^*(s), 0] = \frac{\bar{\Pi}_1(0)}{\bar{\Pi}_2(0)} > 0 \tag{A.59}$$

  since $\bar{\Pi}_1(s)$ and $\bar{\Pi}_2(s)$ are monic and Hurwitz;

- If $\mathrm{sgn}\ A\ (-1)^{(r-1)/2} = 1$

$$\mathrm{Res}[\Phi_o^{*-1}(s), 0] = \frac{\bar{\Pi}_2(0)}{\bar{\Pi}_1(0)} > 0. \tag{A.60}$$

Hence, $\Phi_o^*(s)$ is positive real. Introducing the rational function

$$\Phi(s) = \Phi_o^*(s) \left(\frac{s+\varepsilon}{s} \right)^{\mathrm{sgn}A \ (-1)^{(r-1)/2}} \cdot \\ \cdot (1+\tau s)^{\partial \bar{\Pi}_2 - \partial \bar{\Pi}_1 - \mathrm{sgn}A \ (-1)^{(r-1)/2}} \tag{A.61}$$

by the positive real character of $\Phi_o^*(s)$, taking into account (A.58) and the fact that $\mathrm{Re}[\Phi(0)] > 0$, it turns out that $\Phi(s)$ is strictly positive real for sufficiently small positive ε, τ.

Now, Lemmas A.4, A.5, and A.6 ensure that $\gamma(\omega) = \gamma_{\Phi_o^*}(\omega)$ satisfies condition (A.52) for any $\rho < \rho^*$ and $\omega > 0$. Moreover, since $\gamma_\Phi(\omega)$ is continuous with respect to τ and ϵ, it turns out that the left term of inequality (A.52) for $\gamma(\omega) = \gamma_\Phi(\omega)$ depends continuously on τ and ϵ. Hence, observing that $\gamma_\Phi(0) = 0$, it follows that for sufficiently small positive ε and τ, (A.52) holds for $\gamma(\omega) = \gamma_\Phi(\omega)$ for all $\omega \geq 0$. $\diamondsuit$

Remark A.2 The parameters ε and τ in the expression (A.50) of the solution $\Phi(s)$ are introduced in order to obtain a strictly positive real rational function such that $\gamma_\Phi(\omega)$ belongs to the set Γ. Indeed, in the limiting case $\varepsilon = 0$ and $\tau = 0$, $\Phi(s)$ reduces to $\Phi_e^*(s)$ for even r and $\Phi_o^*(s)$ for odd r. These two functions are in general guaranteed to be positive real only. On the other hand, as ρ approaches ρ^*, the band defined by $\underline{\gamma}(\omega)$ and $\overline{\gamma}(\omega)$ becomes narrower as depicted in Fig. A.1, and therefore $\gamma_\Phi(\omega)$ has to be chosen sufficiently close to $\gamma^*(\omega)$. Since $\gamma^*(\omega) = \gamma_{\Phi_e^*}(\omega)$ for even r and $\gamma^*(\omega) = \gamma_{\Phi_o^*}(\omega)$ for odd r, it turns out that the closer ρ is to ρ^*, the smaller ε and τ have to be chosen. Moreover, some general guidelines for the selection of ε and τ can be derived from the frequency properties of $\Phi_e^*(s)$ and $\Phi_o^*(s)$. For example, since ε defines a low frequency pole or zero of $\Phi(s)$, it should be chosen at least one decade smaller than all the singularities of $\Phi_o^*(s)$. Similarly, τ should be chosen such that $1/\tau$ is at least one decade larger than all the singularities of $\Phi_e^*(s)$ and $\Phi_o^*(s)$ (see also Example 1).

Remark A.3 Since $\Phi_e^*(s)$ and $\Phi_o^*(s)$ are positive real, Property A.2 implies that

$$\begin{array}{ll} -1 \leq \partial \bar{\Pi}_2 - \partial \bar{\Pi}_1 \leq 1 & r \text{ even} \\ -1 \leq \partial \bar{\Pi}_2 - \partial \bar{\Pi}_1 - \mathrm{sgn}A \ (-1)^{(r-1)/2} \leq 1 & r \text{ odd.} \end{array} \tag{A.62}$$

Exploiting the factorization in Lemma A.7, we can determine an upper bound on the degree of the solution filter $F(s)$ in (A.51), which involves the degree l of the set $\mathcal{P}_\rho$.

Let

$$F(s) = \frac{N_F(s)}{D_F(s)}, \tag{A.63}$$

the following result holds.

Corollary A.1 *Let the assumptions in Theorem A.2 be fulfilled. Then,*

$$\begin{aligned} \partial D_F &\le l-2 \quad \text{for even } r \\ \partial D_F &\le l-1 \quad \text{for odd } r\ . \end{aligned} \tag{A.64}$$

Proof: Let

$$\mu = \partial\bar{\Pi}_2 - \partial\bar{\Pi}_1 \tag{A.65}$$

$$\sigma = \begin{cases} \text{sgn}A\ (-1)^{(r-1)/2} & \text{for odd } r \\ 0 & \text{for even } r \end{cases} \tag{A.66}$$

$$e = \mu - \sigma. \tag{A.67}$$

Note that e is the relative degree of either $\Phi_e^*(s)$ or $\Phi_o^*(s)$ (see (A.48), (A.49), and (A.50)). Since these functions are positive real, we have

$$e \in \{-1, 0, 1\}. \tag{A.68}$$

By (A.47) and the assumptions in Definition A.3, we get

$$\partial\Pi = r + \partial\bar{\Pi}_1 + \partial\bar{\Pi}_2 \le 2(2l-1) \tag{A.69}$$

and hence from (A.65)

$$\partial\bar{\Pi}_1 \le 2l - 1 - \frac{r+\mu}{2}. \tag{A.70}$$

From (A.51) and (A.63), it follows that

$$\frac{N_F(s)}{D_F(s)} = \frac{P_0(s)\bar{\Pi}_2(s)}{\bar{\Pi}_1(s)(s+\varepsilon)^{\sigma}(1+\tau s)^{\mu-\sigma}}. \tag{A.71}$$

Since from Lemma A.7 we know that $\bar{\Pi}_1(s)$ contains $P_0(s)$ as a factor, assuming the worst case $\sigma \ge 0$, $\mu - \sigma = e \ge 0$, we get

$$\partial D_F \le \partial\bar{\Pi}_1 - l + \mu. \tag{A.72}$$

Taking into account (A.70), we have

$$\partial D_F \le l - 1 - \frac{r-\mu}{2} = l - 1 - \frac{r-e-\sigma}{2} \tag{A.73}$$

where the equality follows from (A.67). Finally, note that if r is odd we have $r \ge 1$, otherwise $r \ge 2$. By substituting the minimum value of r and the maximum values of e and σ in either case, we obtain (A.64). ♢

Remark A.4 The above approach to the solution of the *RSPR* problem provides an upper bound for the degree of the filter $F(s)$ that was lacking in [1] (see Theorem A.1).

Now, we move to the general case in which the set Ω_0 contains other frequencies in addition to $\omega = 0$, i.e. it has the general form (2.66). The following result parallels Lemma A.7.

Lemma A.8 *Suppose* $\Omega_0 = \{0, \omega_1, \ldots, \omega_k\}$. *Then, the polynomial* $\Pi(s)$ *in (A.39) is factorizable as*

$$\Pi(s) = As^{r_0} \prod_{i=1}^{k} \left(s^2 + \omega_i^2\right)^{r_i} \bar{\Pi}_1(s)\bar{\Pi}_2(-s), \tag{A.74}$$

where A *is a real number,* r_i *are suitable non-negative integers,* $\bar{\Pi}_1(s)$ *and* $\bar{\Pi}_2(s)$ *are uniquely determined monic Hurwitz polynomials. Moreover,* $\bar{\Pi}_1(s)$ *contains* $P_0(s)$ *as a factor.*

Let

$$\tilde{\Pi}_i(s) = \frac{\Pi(s)}{(s^2 + \omega_i^2)^{r_i}}, \quad i = 1 \ldots k, \tag{A.75}$$

and introduce the rational function

$$\Phi^*(s) = \frac{\bar{\Pi}_1(s)}{\bar{\Pi}_2(s)} s^{N_0} \prod_{i=1}^{k} (s^2 + \omega_i^2)^{N_i}, \tag{A.76}$$

where $\bar{\Pi}_1(s)$ and $\bar{\Pi}_2(s)$ are as in Lemma A.8,

$$N_0 = \begin{cases} 0 & \text{if } r_0 \text{ is even} \\ \operatorname{sgn} A \ (-1)^{(r_0-1)/2} & \text{if } r_0 \text{ is odd} \end{cases} \tag{A.77}$$

$$N_i = \begin{cases} 0 & \text{if } r_i \text{ is even} \\ -1 & \text{if } r_i \text{ is odd and } \operatorname{Im}\left[\tilde{\Pi}_i(j\omega_i)\right] > 0 \\ 1 & \text{if } r_i \text{ is odd and } \operatorname{Im}\left[\tilde{\Pi}_i(j\omega_i)\right] < 0 \end{cases} . \tag{A.78}$$

We have the following general result based on the fact that the function $\Phi^*(s)$ in (A.76) is shown to be positive real.

Theorem A.3 *Given the set* $\mathcal{P}_\rho$, *let* ρ^* *be the parametric stability margin of* $\mathcal{P}_\rho$ *and suppose the following conditions hold*

1. $\rho < \rho^*$;

2. $\Omega_0 = \{0, \omega_1, \ldots, \omega_k\}$.

Let $\Phi^*(s)$, N_0, N_i *be as in (A.76), (A.77), (A.78), respectively. Then, for sufficiently small positive* ε *and* τ, *the rational function*

$$\Phi(s) = \Phi^*(s+\varepsilon)(1+\tau s)^{\partial\bar{\Pi}_2 - \partial\bar{\Pi}_1 - N_0 - 2\sum_{i=1}^{k} N_i} \tag{A.79}$$

satisfies (A.23) and (A.24) all $\omega \geq 0$, *i.e. the filter*

$$F(s) = \frac{P_0(s)}{\Phi(s)} \tag{A.80}$$

solves the robust SPR problem for $\mathcal{P}_\rho$.

Proof: see Appendix B.

A result concerning the degree of the filter $F(s)$ can be given also for this general case.

Corollary A.2 *Let the assumptions in Theorem A.3 be fulfilled. Then,*

$$\partial D_F \leq 2l - 1. \tag{A.81}$$

Proof: see Appendix B.

Remark A.5 The upper bound in this case is larger than in the case $\Omega_0 = \{0\}$ (see Corollary A.1). The increase in the upper bound is due to the fact that the numerator of $\Phi^*(s+\varepsilon)$ contains $\bar{\Pi}_1(s+\varepsilon)$ in place of $\bar{\Pi}_1(s)$ and therefore $P_0(s)$ cannot be canceled in (A.80), as it was done in Corollary A.1.

Under a very mild additional assumption on $\Phi^*(s)$ in (A.76), a simplified form for the solution $\Phi(s)$ can be given.

Theorem A.4 *Given the set* $\mathcal{P}_\rho$, *let* ρ^* *be its parametric stability margin and suppose the following conditions hold:*

1. $\rho < \rho^*$;

2. $\Omega_0 = \{0, \omega_1, \ldots, \omega_k\}$;

3. There exists no $i \in \{1, \ldots, k\}$ *such that* r_i *is even and* $\mathrm{Re}\left[\tilde{\Pi}_i(j\omega_i)\right] = 0$.

where $\tilde{\Pi}_i(s)$ *is defined as in (A.75).*

Let $\Phi^*(s)$, N_0, N_i *be as in (A.76), (A.77), (A.78), respectively. Then, the robust SPR problem is solved by the filter*

$$F(s) = \frac{P_0(s)}{\Phi(s)} \tag{A.82}$$

where $\Phi(s)$ *is the function*

$$\begin{aligned}\Phi(s) = \Phi^*(s)\left(\frac{s+\varepsilon}{s}\right)^{N_0}\prod_{i=1}^{k}\left(\frac{s^2+2\zeta_i\omega_i s+\omega_i^2}{s^2+\omega_i^2}\right)^{N_i}\cdot \\ \cdot(1+\tau s)^{\partial\bar{\Pi}_2-\partial\bar{\Pi}_1-N_0-2\sum_{i=1}^{k}N_i}\end{aligned} \tag{A.83}$$

for sufficiently small non-negative ε, τ *and* ζ_i, $i = 1, \dots, k$.

Proof: see Appendix B.

Note that in this case we can recover the stronger condition of Corollary A.1 concerning the degree of D_F, since according to (A.83) and (A.76), $P_0(s)$ is a factor of the numerator of $\Phi(s)$ (recall from Lemma A.8 that $P_0(s)$ is a factor of $\bar{\Pi}_1(s)$). Indeed, we have the following result.

Corollary A.3 *Let the assumptions in Theorem A.4 be satisfied. Then,*

$$\partial D_F \leq l - 1. \tag{A.84}$$

A.4 Examples

In this section we develop some numerical examples to illustrate the features of the proposed results. One specific goal is to show that the filter $F(s) = P_0(s)$, that is quite often used in several application contexts (see [37],[36]), is not an appropriate choice especially when ρ is close to ρ^*.

Example A.1 Let

$$\mathcal{P}_\rho = \left\{P(s;\delta) = (s+1)^3 + \delta_1 s + \delta_2 \quad : \quad \|\delta\|_2 < \rho\right\}.$$

The function $G(s)$ is given by

$$G(s) = -\begin{bmatrix} \dfrac{s}{(s+1)^3} & \dfrac{1}{(s+1)^3}\end{bmatrix}'$$

and, moreover,

$$\Omega_0 = \{0\}.$$

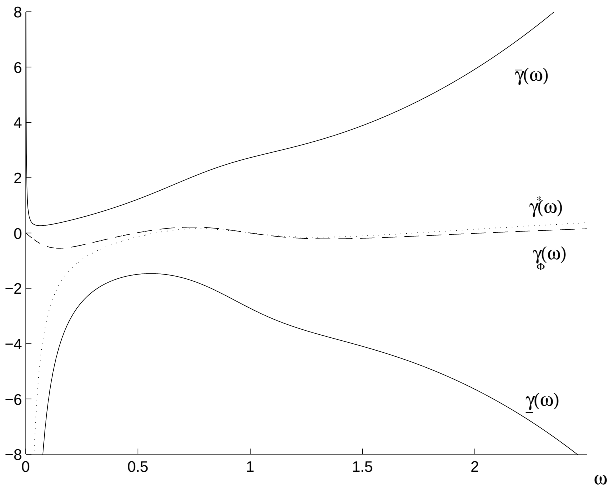


Figure A.2: Example 1: $\underline{\gamma}(\omega)$ and $\overline{\gamma}(\omega)$ (solid), $\gamma^*(\omega)$ (dotted), $\gamma_\Phi(\omega)$ (dashed).

According to Lemma A.2, the l_2 stability margin of $\mathcal{P}_\rho$ is given by $\rho^* = \rho_0 = 1$. The associated $\Pi(s)$ can be factorized as in Lemma A.7 yielding

$$\Pi(s) = 3(-s)(s+1)^4(s^2 - 2/3s + 1)$$

and, therefore,

$$A = 3; \qquad r = 1; \qquad \overline{\Pi}_1(s) = (s+1)^4; \qquad \overline{\Pi}_2(s) = (s^2 - 2/3s + 1).$$

From the application of Theorem A.2, we get that, for sufficiently small positive ε and τ, the rational function

$$\Phi(s) = \frac{(s+1)^4}{(s+\varepsilon)(s^2 + 2/3s + 1)(1 + \tau s)}$$

solves the *RSPR* problem for $\rho < 1$. The corresponding filter $F(s)$ is

$$F(s) = \frac{(s+\varepsilon)(s^2 + 2/3s + 1)(1 + \tau s)}{s+1}.$$

Note that the transfer function

$$\Phi_o^*(s) = \frac{(s+1)^4}{s(s^2 + 2/3s + 1)}$$

is positive real, but not strictly positive real.

Concerning the selection of ε and τ, we can proceed as discussed in Remark A.2. Since

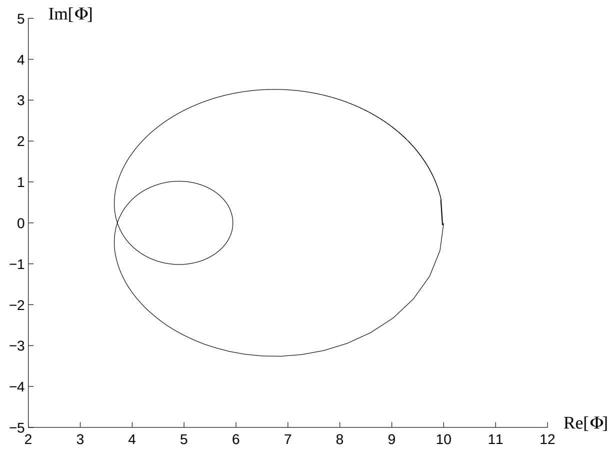


Figure A.3: Example 1: Nyquist plot of $\Phi(s)$

all the singularities of $\Phi_o^*(s)$ are at $\omega = 1$, we can select $\varepsilon = 0.1$ and $\tau = 0.1$. This choice allows for the solution of the $RSPR$ problem for ρ very close to one. As an example, Fig. A.2 shows the functions $\underline{\gamma}(\omega)$ and $\overline{\gamma}(\omega)$ (solid line) calculated for $\rho = 0.97$ along with $\gamma^*(\omega)$ (dotted line) and $\gamma_\Phi(\omega)$ (dashed line). In Fig. A.3 the Nyquist plot of such $\Phi(s)$ is depicted, thus showing its SPR.

Note that in this case the filter $F(s) = P_0(s) = (s+1)^3$ is a solution of the $RSPR$ problem, since $\gamma(\omega) = 0$ lies within the band defined by $\overline{\gamma}(\omega)$ and $\underline{\gamma}(\omega)$ for all $\omega \geq 0$.

Example A.2 Let

$$\mathcal{P}_\rho = \left\{ P(s;\delta) = (s+1)^3 + \delta_1 s^2 + \delta_2 s \quad : \quad \|\delta\|_2 < \rho \right\}.$$

We get

$$\rho^* = \bar{\rho} = \sqrt{7}$$

$$\Omega_0 = \{0\}$$

and, according to Lemma A.7,

$$\Pi(s) = -s^2(s+1)^4(s^2 - 0.78s + 3.54)(s^2 - 0.22s + 0.28).$$

The application of Theorem A.2 leads to the rational function

$$\Phi(s) = \frac{(s+1)^4}{(s^2 + 0.78s + 3.54)(s^2 + 0.22s + 0.28)},$$

which solves the $RSPR$ problem for $\rho < \sqrt{7}$. The corresponding filter $F(s)$ is given by

$$F(s) = \frac{(s^2 + 0.78s + 3.54)(s^2 + 0.22s + 0.28)}{s+1}.$$

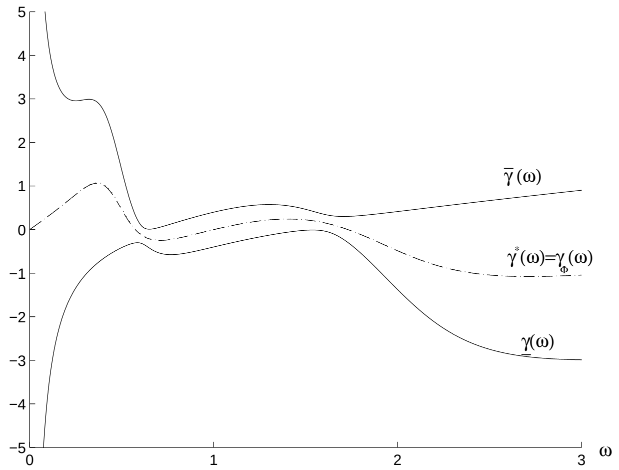


Figure A.4: Example 2: $\underline{\gamma}(\omega)$ and $\overline{\gamma}(\omega)$ (solid), $\gamma^*(\omega)$ (dotted), $\gamma_\Phi(\omega)$ (dashed).

Note that in this case $\Phi(s) = \Phi_e^*(s)$, i.e. $\Phi_e^*(s)$ turns out to be strictly positive real.

The function $\gamma_\Phi(\omega) = \gamma^*(\omega)$ for $\rho = 2.63$ is shown in Fig. A.4. Note that the filter $F(s) = P_0(s)$ does not solve the $RSPR$ problem.

Example A.3 Consider

$$\mathcal{P}_\rho = \left\{ P(s;\delta) = (s+1)^3 + q_1 s^2 + q_2 \quad : \quad \|q\|_2 < \rho \right\}.$$

We have

$$\Omega_0 = \{0, \sqrt{3}\}$$

$$\rho^* = \rho_0 = 1$$

and, according to Lemma A.8,

$$\Pi(s) = (-s)(s^2+3)(s+1)^3(s^2+\sqrt{2}s+1)(s^2-\sqrt{2}s+1).$$

Note that the two imaginary roots of $\Pi(s)$ are simple. Hence, the solution of the $RSPR$ problem can be obtained by applying Theorem A.4. For sufficiently small positive ε and ζ, the rational function

$$\Phi(s) = \frac{(s+1)^3}{(s+\varepsilon)(s^2+2\sqrt{3}\zeta s+3)}$$

solves the $RSPR$ problem for $\rho < 1$ and the corresponding filter $F(s)$ turns out to be a polynomial

$$F(s) = (s+\varepsilon)(s^2+2\sqrt{3}\zeta s+3).$$

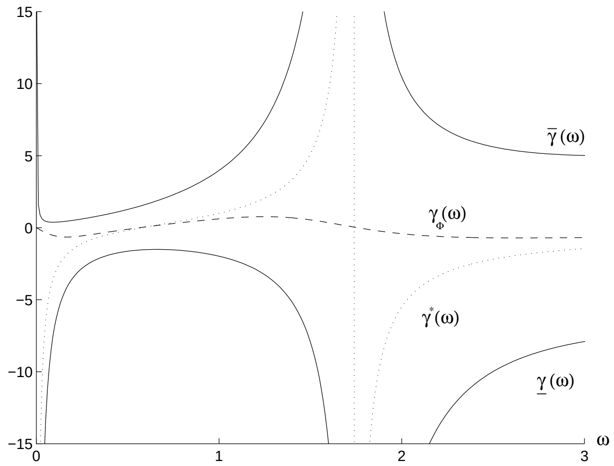


Figure A.5: Example 3: $\underline{\gamma}(\omega)$ and $\overline{\gamma}(\omega)$ (solid), $\gamma^*(\omega)$ (dotted), $\gamma_\Phi(\omega)$ (dashed).

The diagram of Figure A.5 is obtained for $\rho = 0.97$, $\varepsilon = 0.1$, and $\zeta = 0.2$. Note that $\underline{\gamma}(\omega)$, $\overline{\gamma}(\omega)$, and $\gamma^*(\omega)$ are unbounded for $\omega = \sqrt{3}$, while the nominal filter $F(s) = P_0(s)$ is a valid solution.

Example A.4 Let

$$\mathcal{P}_\rho = \big\{P(s;\delta) = s^4 + 3s^3 + 5.5s^2 + 4.5s + 5.5+$$
$$+\delta_1(s^2+s+3) + \delta_2(s^3+s-1) \;:\; \|\delta\|_2 < \rho\big\}\,.$$

We have

$$\rho^* = \rho_0 = 1.0607$$
$$\Omega_0 = \{0, \sqrt{2}\}$$

and, according to Lemma A.8,

$$\Pi(s) = (-s)(s^2+2)^2(s^4+3s^3+5.5s^2+4.5s+5.5)\cdot$$
$$\cdot(s^5+3.5s^3+3s^2+0.5s+8.5) =$$
$$= (-s)(s^2+2)^2(s^4+3s^3+5.5s^2+4.5s+5.5)\cdot$$
$$\cdot(s+1.3569)(s^2-0.1306s+3.2591)(s^2-1.2263s+1.9220).$$

In this case assumption 3 of Theorem A.4 does not hold, since the roots at $s = \pm j\sqrt{2}$ are double and $\mathrm{Re}[\Pi(s)/(s+2)^2]\big|_{s=j\sqrt{2}} = 0$.

Thus, we have to apply Theorem A.3. First, according to (A.76), we compute the positive real rational function

$$\Phi^*(s) = \frac{(s^4+3s^3+5.5s^2+4.5s+5.5)(s+1.3569)}{s(s^2+0.1306s+3.2591)(s^2+1.2263s+1.9220)}$$

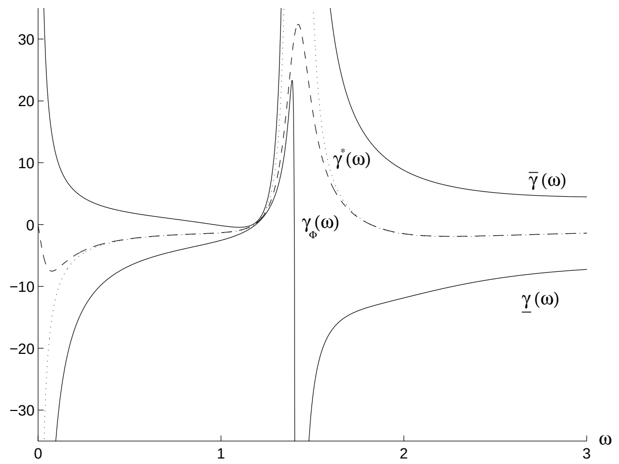


Figure A.6: Example 4: $\underline{\gamma}(\omega)$ and $\overline{\gamma}(\omega)$ (solid), $\gamma^*(\omega)$ (dotted), $\gamma_\Phi(\omega)$ (dashed).

according to (A.76). Then, for sufficiently small ε, the rational function

$$\Phi(s) = \Phi^*(s+\varepsilon)$$

solves the $RSPR$ problem for $\rho < 1.0607$.

The plot in Figure A.6 is calculated for $\rho = 1$ and $\varepsilon = 0.005$. Note that in this case, $F(s) = P_0(s)$ is not a solution of the $RSPR$ problem.

Example A.5 In this example we show that the filter $F(s) = P_0(s)$ is not in general a solution of the $RSPR$ problem, especially for values of ρ close to ρ^*. This has been already pointed out in Example 4, where however the considered problem led to a peculiar form of $\Pi(s)$ that forced to use Theorem A.3. Indeed, consider the set

$$\mathcal{P}_\rho = \big\{P(s;\delta) = s^4 + 3s^3 + 5.5s^2 + 4.5s + 5.5+ \\ +\delta_1(s^2+s+3) + \delta_2(s^3+s-0.5) \ : \ \|\delta\|_2 < \rho\big\}\,.$$

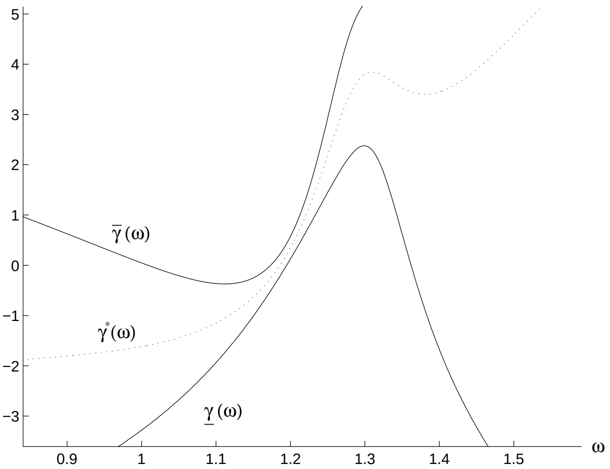
which is a slight modification of the one of the previous example (only $P_2(0)$ has been changed).

In this case, we have $\Omega_0 = \{0\}$ and $\rho^* = \bar{\rho} \approx 0.99$ and we can apply Theorem A.2. We get

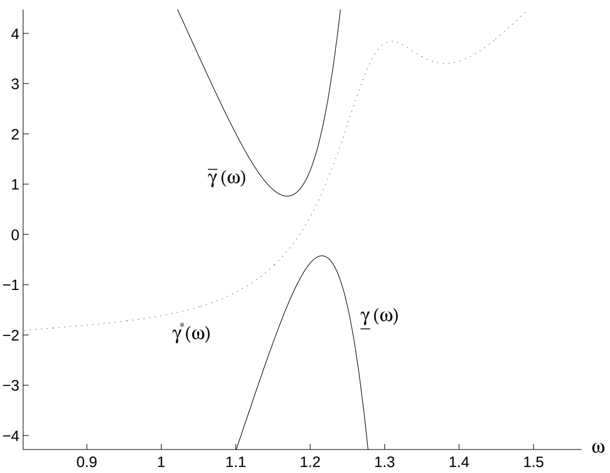
$$\Phi^*(s) = \frac{(s^4+3s^3+5.5s^2+4.5s+5.5)(s+1.32)(s^2+0.26s+1.81)}{s(s^2+0.11s+4.02)(s^2+1.27s+1.64)(s^2+0.20s+1.76)}$$

and

$$F(s) = (s+\varepsilon)\cdot\frac{(s^2+0.11s+4.02)(s^2+1.27s+1.64)(s^2+0.20s+1.76)}{(s+1.32)(s^2+0.26s+1.81)}.$$



(a)



(b)

Figure A.7: Example 5: $\underline{\gamma}(\omega)$ and $\overline{\gamma}(\omega)$ (solid), $\gamma^*(\omega)$ (dotted), $\gamma_\Phi(\omega)$ (dashed); (a) $\rho = 0.89$; (b) $\rho = 0.3$.

The plots of $\overline{\gamma}(\omega)$, $\underline{\gamma}(\omega)$ and $\gamma^*(\omega)$ are depicted in Fig. A.7 (a) for $\rho = 0.89$, making it clear that the filter $F(s) = P_0(s)$ is not working. Indeed, it turns out the solution $\Phi(s) = 1$ can be used as long as $\rho < 0.32$. As an example, the case $\rho = 0.3$ is reported in Fig. A.7 (b).

Appendix B

Proof of some results

B.1 Chapter 2

Proof of Theorem 2.1: Let

$$E = \left\{ \lambda \in (a, b] \ : \ P(s; \lambda') \text{ has all is roots in } \mathcal{S} \ \ \forall \lambda' \in (a, \lambda) \right\}. \tag{B.1}$$

Since $P(s; a)$ has all its roots in $\mathcal{S}$, then there exists $\varepsilon > 0$ such that

$$P(s; \lambda') \text{ has all is roots in } \mathcal{S} \ \ \forall \lambda' \in [a, \lambda + \varepsilon) \cap I. \tag{B.2}$$

Hence, E is nonempty for $a + \varepsilon/2 \in E$. Moreover, it is easy to see that E is an interval and that if

$$\bar{\lambda} = \sup_{\lambda \in E} \lambda \tag{B.3}$$

then $E = (a, \bar{\lambda}]$.

Now, $P(s; \bar{\lambda})$ cannot have all its roots in $\mathcal{S}$, since there would exist $\varepsilon > 0$ such that $\bar{\lambda} + \varepsilon < b$ and $P(s; \lambda')$ has all roots in $\mathcal{S}$ for all $\lambda' \in (\bar{\lambda} - \varepsilon, \bar{\lambda} + \varepsilon) \cap I$. It would turn out that $\bar{\lambda} + \varepsilon/2 \in E$ and this contradicts the definition of $\bar{\lambda}$.

On the other hand, $P(s; \bar{\lambda})$ cannot have roots in $\mathcal{U}_0$, since $\mathcal{U}_0$ is an open set and in that case there would exist $\varepsilon > 0$ such that $P(s; \lambda')$ has roots in $\mathcal{U}_0$ for all $\lambda' \in (\bar{\lambda} - \varepsilon, \bar{\lambda} + \varepsilon) \cap I$, and this contradicts the fact that $\bar{\lambda} - \varepsilon \in E$ for sufficiently small ε.

Hence, we conclude that $P(s; \bar{\lambda})$ has all its roots in $\mathcal{S} \cup \partial\mathcal{S}$ and at least one root in $\partial\mathcal{S}$.$\diamondsuit$

Proof of Lemma 2.1: Recall that Δ_ω is a vector space of dimension $l - 1$. Then, the orthogonal complement $\Delta_\omega^\perp$ has dimension 2. Let $\{Q_1(s), Q_2(s)\}$ be an orthonormal basis of$\Delta_\omega^\perp$. By definition of orthogonal projection we have that

$$P_0(s) - \pi_{P_0|\Delta_\omega}(s) = \frac{< P_0(s), Q_1(s) >}{\|Q_1(s)\|^2} Q_1(s) + \frac{< P_0(s), Q_2(s) >}{\|Q_2(s)\|^2} Q_2(s) \tag{B.4}$$

while the distance d_ω can be expressed as

$$d_\omega^2 = \|P_0(s) - \pi_{P_0|\Delta_\omega}(s)\|^2 \tag{B.5}$$

and since $\{Q_1(s), Q_2(s)\}$ is an orthonormal basis

$$d_\omega^2 = \frac{< P_0(s), Q_1(s) >^2}{\|Q_1(s)\|^2} + \frac{< P_0(s), Q_2(s) >^2}{\|Q_2(s)\|^2} \tag{B.6}$$

Now let $l = 2k$. An orthonormal basis for $\Delta_\omega^\perp$ is given by

$$\begin{aligned} Q_1(s) &= 1 - \omega^2 s^2 + \omega^4 s^4 + \cdots + (-1)^k \omega^{2k} s^{2k} \\ Q_2(s) &= s - \omega^2 s^3 + \omega^4 s^5 + \cdots + (-1)^{k-1} \omega^{2(k-1)} s^{2k-1}. \end{aligned} \tag{B.7}$$

This basis satisfies

$$\begin{aligned} P_e(s) &= < Q_1(s), P_0(s) > \\ P_o(s) &= < Q_2(s), P_0(s) > \end{aligned} \tag{B.8}$$

and moreover

$$\begin{aligned} \|Q_1(s)\|_2^2 &= 1 + \omega^4 + \cdots + \omega^{4k} \\ \|Q_2(s)\|_2^2 &= 1 + \omega^4 + \cdots + \omega^{4(k-1)}. \end{aligned} \tag{B.9}$$

Hence, the first of (2.26) follows by (B.6).

If $l = 2k + 1$ we can choose

$$\begin{aligned} Q_1(s) &= 1 - \omega^2 s^2 + \omega^4 s^4 + \cdots + (-1)^k \omega^{2k} s^{2k} \\ Q_2(s) &= s - \omega^2 s^3 + \omega^4 s^5 + \cdots + (-1)^{k-1} \omega^{2(k-1)} s^{2k-1} + (-1)^k \omega^{2k} s^{2k+1}. \end{aligned} \tag{B.10}$$

Therefore

$$\|Q_1(s)\|_2^2 = \|Q_2(s)\|_2^2 = 1 + \omega^4 + \cdots + \omega^{4k} \tag{B.11}$$

and since properties (B.8) still hold, from (B.6) we get the second of (2.26). $\diamondsuit$

B.2 Chapter 4

Proof of Lemma 4.2: Rewriting (4.42) as

$$\begin{cases} \mathrm{Re}[\Phi(j\omega)] > 0 \\ \delta' \dfrac{\mathrm{Re}[\Phi(j\omega) G_\vartheta(j\omega)]}{\mathrm{Re}[\Phi(j\omega)]} < 1 \end{cases} \quad \forall \omega \geq 0 \;\; \forall \delta : \|\delta\|_2 < \rho \tag{B.12}$$

by the properties of the 2-norm one obtains the following equivalent form

$$\begin{cases} \mathrm{Re}[\Phi(j\omega)] > 0 \\ \mathrm{Re}^2[\Phi(j\omega)] - \mathrm{Re}[\Phi(j\omega)\ \rho G'_\vartheta(j\omega)] \cdot \mathrm{Re}[\Phi(j\omega)\ \rho G_\vartheta(j\omega)] > 0 \end{cases} \quad \forall\omega \geq 0 \tag{B.13}$$

and finally, by employing the properties of the Schur complement,

$$\mathrm{Re}\left[\Phi(j\omega)\begin{bmatrix} I & \rho G_\vartheta(j\omega) \\ \rho G'_\vartheta(j\omega) & 1 \end{bmatrix}\right] > 0 \quad \forall\omega \geq 0. \tag{B.14}$$

♢

B.3 Appendix A

Proof of Lemma A.1: It is easily checked that $\Phi(s)$ satisfies condition 1 of Definition A.2 for any $\varepsilon, \tau > 0$. To prove that also condition 2 holds, we proceed as follows.
It can be shown (see [21, pp. 63–65]) that $\Phi^*(s)$ being positive real implies $\mathrm{Re}[\Phi^*(j\omega + \varepsilon)] > 0$ for all $\omega \geq 0$ for some small $\varepsilon > 0$. Therefore, if $\partial P_1 = \partial P_2$ the proof is already concluded.
On the contrary, suppose that $\partial P_2 - \partial P_1 = 1$ and let

$$\begin{aligned} \Phi(s) &= \Phi^*(s+\varepsilon)(1+\tau s) \\ R_\varepsilon(\omega) &= \mathrm{Re}[\Phi^*(j\omega+\varepsilon)] \\ I_\varepsilon(\omega) &= \mathrm{Im}[\Phi^*(j\omega+\varepsilon)]. \end{aligned} \tag{B.15}$$

We have

$$\mathrm{Re}[\Phi(j\omega)] = R_\varepsilon(\omega) - \tau\omega I_\varepsilon(\omega). \tag{B.16}$$

Then, $\mathrm{Re}[\Phi(j\omega)] > 0 \ \ \forall\omega \geq 0$ if and only if

$$\frac{1}{\tau} > \sup_{\omega\geq 0} \frac{\omega I_\varepsilon(\omega)}{R_\varepsilon(\omega)}. \tag{B.17}$$

Hence, there exists $\tau > 0$ such that $\Phi(s)$ is SPR if and only if

$$\sup_{\omega\geq 0} \frac{\omega I_\varepsilon(\omega)}{R_\varepsilon(\omega)} < +\infty. \tag{B.18}$$

Now, since $\Phi^*(s+\varepsilon)$ is by construction a minimum phase (i.e., all its poles and zeros have negative real part) relative degree one rational function, it turns out that $I_\varepsilon(\omega)$

is bounded for any finite $\omega \geq 0$ and negative for $\omega \to +\infty$. Taking into account that $R_\varepsilon(\omega) > 0$ for all $\omega \geq 0$, we get that (B.18) holds.
A similar argument applies for $\partial P_2 - \partial P_1 = -1$. $\diamondsuit$

Proof of Lemma A.3: Taking $\delta = 0$ in (A.26) yields (A.27a). Therefore, (A.26) can be rewritten equivalently as

$$\begin{array}{ll} (a) & \mathrm{Re}[\Phi(j\omega)] > 0 \\ (b) & \delta' \left[R(\omega) - \gamma_\Phi(\omega) I(\omega)\right] < 1 \end{array} \quad \forall \omega \geq 0 \quad \forall \delta : \|\delta\|_2 < \rho \; . \tag{B.19}$$

By a standard property of the 2-norm, (B.19b) holds for all q such that $\|q\|_2 < \rho$ if and only if (A.27b) holds. $\diamondsuit$

Proof of Lemma A.4:

1. Let us rewrite (A.29) as

$$\|I(\omega)\|_2^2 \gamma^2(\omega) - 2R'(\omega)I(\omega)\gamma(\omega) + \|R(\omega)\|_2^2 - \frac{1}{\rho^2} < 0. \tag{B.20}$$

   Note that, for each fixed ω, the left hand side term of (B.20) is a second order polynomial with respect to $\gamma(\omega)$. Moreover, since $\rho < \rho^*$, Lemma A.2 ensures that $\rho < \overline{\rho}$ (see (A.22)) and it is therefore straightforward to verify (see (A.19)-(A.21)) that, for all $\omega \in \overline{\Omega}_0$, inequality (B.20) holds for any $\gamma(\omega)$ satisfying

$$\underline{\gamma}(\omega) < \gamma(\omega) < \overline{\gamma}(\omega) \tag{B.21}$$

   where

$$\begin{aligned} \underline{\gamma}(\omega) &= \min\left\{\gamma^*(\omega) \pm \frac{\sqrt{\Delta(\omega)}}{\|I(\omega)\|_2^2}\right\} \\ \overline{\gamma}(\omega) &= \max\left\{\gamma^*(\omega) \pm \frac{\sqrt{\Delta(\omega)}}{\|I(\omega)\|_2^2}\right\} \end{aligned} \tag{B.22}$$

   and

$$\Delta(\omega) = \left[R'(\omega)I(\omega)\right]^2 - \|I(\omega)\|_2^2 \left[\|R(\omega)\|_2^2 - \frac{1}{\rho^2}\right]. \tag{B.23}$$

   Finally, again from Lemma A.2 (see (A.18) and (A.22)), it turns out that, for all $\omega \in \Omega_0$, inequality (B.20) holds for $\gamma(\omega)$ being any real value.

2. Under the assumption $\rho < \rho_0$, it can be easily verified that for each $\omega_0 \in \Omega_0$ there exists a neighbourhood $\mathcal{N}(\omega_0)$ of ω_0 such that, for all $\omega \in \mathcal{N}(\omega_0) \setminus \{\omega_0\}$, $\underline{\gamma}(\omega)$ and

$\overline{\gamma}(\omega)$ are continuous functions of opposite sign. Moreover, as stated above, $\gamma(\omega_0)$ can be any real value. Thus, any bounded $\gamma(\omega)$ satisfying (B.21) for $\omega \in \bar{\Omega}_0$ can be extended to a continuous solution of (A.29) for all $\omega \geq 0$. Hence, Γ is nonempty. $\diamondsuit$

Proof of Lemma A.5: Exploiting (A.14), we rewrite (A.39) as

$$\Pi(s) = P_0(s)[P_0(-s)G'(-s)][P_0(-s)P_0(s)G(s)]_o.$$

Thus, $\Pi(j\omega)$ can be calculated as

$$\begin{aligned} \Pi(j\omega) &= P_0(j\omega)P_0(-j\omega)G'(-j\omega)\cdot \\ &\cdot j\mathrm{Im}\{P_0(-j\omega)P_0(j\omega)G(j\omega)\} = \\ &= [P_0(j\omega)P_0(-j\omega)]^2[R'(\omega) - jI'(\omega)]\cdot jI(\omega) = \\ &= [P_0(j\omega)P_0(-j\omega)]^2[I'(\omega)I(\omega) + jR'(\omega)I(\omega)]. \end{aligned} \tag{B.24}$$

This proves property 1. Property 2 directly follows from (B.24) and the fact that $I(\omega) \neq 0$ for $\omega \in \bar{\Omega}_0$, while property 3 derives from (B.24) and (A.31). $\diamondsuit$

Proof of Lemma A.6: From (A.44) and(A.45) we get

$$\begin{aligned} \gamma_{\Phi^*}(\omega) = \frac{\mathrm{Im}\,[\Pi_1(j\omega)\Pi_2(-j\omega)]}{\mathrm{Re}\,[\Pi_1(j\omega)\Pi_2(-j\omega)]} = \frac{\mathrm{Im}\,[\Pi(j\omega)]}{\mathrm{Re}\,[\Pi(j\omega)]} = \\ = \gamma^*(\omega) \quad \forall \omega \in \bar{\Omega}_0. \end{aligned} \tag{B.25}$$

Moreover, we have

$$\mathrm{Re}[\Phi^*(j\omega)] = \frac{\mathrm{Re}[\Pi(j\omega)]}{|\Pi_2(j\omega)|^2}. \tag{B.26}$$

Hence, $\mathrm{Re}[\Phi^*(j\omega)] > 0 \;\; \forall \omega \in \bar{\Omega}_0$ follows from (A.41) in Lemma A.5. $\diamondsuit$

Proof of Theorem A.3: First, it can be easily verified that $\Phi(s)$ in (A.79) satisfies (A.23) by construction. Lemma A.3 states that condition (A.24) is equivalent to condition (A.27)-(A.28). Thus, we have to prove that $\Phi(s)$ satisfies this condition. We start with the following consideration: if $\Phi^*(s)$ in (A.76) satisfies (A.27) for all $\omega \in \bar{\Omega}_0$, then, for sufficiently small non-negative ε, (A.27) also holds for $\Phi^*(s+\varepsilon)$ and $\omega \in \bar{\Omega}_0$. Moreover, assumption 1 implies that $\rho < \rho_0$, and therefore from (A.18) it follows that $\Phi^*(s+\varepsilon)$ satisfies (A.27b) for $\omega \in \Omega_0$, too. Hence, all we have to show amounts to:

i) $\Phi^*(s)$ satisfies (A.27) for all $\omega \in \bar{\Omega}_0$;

ii) for sufficiently small $\varepsilon, \tau > 0$, $\Phi(s)$ is a SPR function.

Let us rewrite the non-negative integers r_i in Lemma A.8 as $r_i = 2p_i + q_i$, where p_i is a non-negative integer and $q_i \in \{0, 1\}$. Accordingly, (A.74) has the form

$$\Pi(s) = As^{r_0} \prod_{i=1}^{k} \left\{ \left(s^2 + \omega_i^2\right)^{2p_i} \left(s^2 + \omega_i^2\right)^{q_i} \right\} \bar{\Pi}_1(s)\bar{\Pi}_2(s). \tag{B.27}$$

As in Theorem A.2, the next step is to factorize $\Pi(s)$ in a suitable way. Taking into account (A.41), $\Pi(s)$ can be expressed as

$$\begin{aligned} \Pi(s) = C_0 \; s^{q_0} \prod_{i=1}^{k} \left(s^2 + \omega_i^2\right)^{q_i} \cdot \\ \cdot \left[|A|^{1/2} \; \bar{\Pi}_1(s) \; s^{p_0} \prod_{i=1}^{k} \left(s^2 + \omega_i^2\right)^{p_i} \right] \cdot \\ \cdot \left[|A|^{1/2} \; \bar{\Pi}_2(-s) \; (-s)^{p_0} \prod_{i=1}^{k} \left(s^2 + \omega_i^2\right)^{p_i} \right] \end{aligned} \tag{B.28}$$

where $C_0 = 1$ if $q_0 = 0$ and $C_0 = \text{sgn}A \; (-1)^{p_0}$ if $q_0 = 1$.

By applying Lemmas A.4, A.5, A.6, it can be verified that the rational function

$$\Phi^*(s) = \frac{\bar{\Pi}_1(s)}{\bar{\Pi}_2(s)} s^{N_0} \prod_{i=1}^{k} (s^2 + \omega_i^2)^{N_i} \tag{B.29}$$

satisfies (A.27) for all $\omega \in \bar{\Omega}_0$. This completes the proof of point i).

In order to prove that $\Phi(s)$ is SPR for sufficiently small ε and $\tau > 0$, we employ Lemma A.1. Hence, it suffices to check the positive real character of $\Phi^*(s)$.

We note that $\Phi^*(s)$ is analytic for $\text{Re}[s] > 0$ and that $\text{Re}\left[\Phi^*(j\omega)\right] \geq 0$ for all ω such that $\Phi^*(s)$ is analytic in $s = j\omega$. Since $\Phi^*(s)$ is positive real if and only if $\Phi^{*-1}(s)$ is, all we have to prove is that both $\Phi^*(s)$ and $\Phi^{*-1}(s)$ possess real positive residues in their respective finite imaginary poles, which are all simple by construction.

To this purpose, we first introduce a useful result concerning the rational functions $\tilde{\Pi}_i(s)$, $i = 1, \ldots, k$, defined in (A.75). Note that for any $i \in \{1, \ldots, k\}$, the following equality holds

$$\Pi(j\omega) = (\omega_i^2 - \omega^2)^{r_i} \left\{ \text{Re}\left[\tilde{\Pi}_i(j\omega)\right] + j\text{Im}\left[\tilde{\Pi}_i(j\omega)\right] \right\} \; . \tag{B.30}$$

Since Lemma A.5 ensures that $\text{Re}\left[\Pi(j\omega)\right] \geq 0$ for all $\omega \geq 0$, it can be shown that $\tilde{\Pi}_i(s)$ must satisfy the following condition

$$r_i \text{ odd} \quad \Longrightarrow \quad \text{Re}\left[\tilde{\Pi}_i(j\omega_i)\right] = 0 \;\; \text{and} \;\; \text{Im}\left[\tilde{\Pi}_i(j\omega_i)\right] \neq 0. \tag{B.31}$$

Let us consider the singularities of $\Phi^*(s)$ on the imaginary axis and their corresponding residues.

From (A.76) and (A.77), $s = 0$ is a singularity of $\Phi^*(s)$ if r_0 is odd and $\text{sgn}A\ (-1)^{(r_0-1)/2} = -1$. The corresponding residue is given by

$$\text{Res}[\Phi^*(s), 0] = \frac{\bar{\Pi}_1(0)}{\bar{\Pi}_2(0)} \prod_{i=1}^{k} \omega_i^{2N_i} \tag{B.32}$$

and it is positive since $\bar{\Pi}_1(s)$ and $\bar{\Pi}_2(s)$ are Hurwitz polynomials.

From (A.76) and (A.78), $s = \pm j\omega_h$ is a singularity if r_h is odd and $\text{Im}\left[\tilde{\Pi}_h(j\omega_h)\right] > 0$. The corresponding residue is given by

$$2\text{Res}[\Phi^*(s), j\omega_h] = \frac{1}{j\omega_h} \frac{\bar{\Pi}_1(j\omega_h)}{\bar{\Pi}_2(j\omega_h)} (j\omega_h)^{N_0} \prod_{h \neq i=1}^{k} \left(\omega_i^2 - \omega_h^2\right)^{N_i}, \tag{B.33}$$

and it can be rewritten as

$$\begin{aligned} 2\text{Res}[\Phi^*(s), j\omega_h] &= \frac{1}{j\omega_h} \frac{\tilde{\Pi}_h(j\omega_h)}{R_h(j\omega_h)R_h(-j\omega_h)} = \\ &= \frac{1}{\omega_h} \frac{\text{Im}\left[\tilde{\Pi}_h(j\omega_h)\right]}{R_h(j\omega_h)R_h(-j\omega_h)}. \end{aligned} \tag{B.34}$$

by introducing the non-zero quantity $R_h(j\omega_h)$, whose complete expression is omitted for brevity, and using (A.75) in the first equality, and exploiting condition (B.31) in the last equality. Thus, the residue is positive since $\text{Im}\left[\tilde{\Pi}_h(j\omega_h)\right] > 0$.

A similar analysis can be performed for $\Phi^{*-1}(s)$. It turns out that the residues are all positive as summarized below:

- $s = 0$ is a singularity if r_0 is odd and $\text{sgn}A\ (-1)^{(r_0-1)/2} = 1$. Its residue satisfies

$$\text{Res}[\Phi^{*-1}(s), 0] = \frac{\bar{\Pi}_2(0)}{\bar{\Pi}_1(0)} \prod_{i=1}^{k} \omega_i^{-2N_i} = \frac{\bar{\Pi}_2(0)}{\bar{\Pi}_1(0)} \prod_{i=1}^{k} \omega_i^{-2N_i} > 0 \tag{B.35}$$

- $s = \pm j\omega_h$ is a singularity if r_h is odd and $\text{Im}\left[\tilde{\Pi}_h(j\omega_h)\right] < 0$. Its residue satisfies

$$\begin{aligned} &2\text{Res}[\Phi^{*-1}(s), j\omega_h] = \\ &= \frac{1}{j\omega_h} \frac{\bar{\Pi}_2(j\omega_h)}{\bar{\Pi}_1(j\omega_h)} (j\omega_h)^{-N_0} \prod_{h \neq i=1}^{k} \left(\omega_i^2 - \omega_h^2\right)^{-N_i} = \\ &= \frac{1}{j\omega_h} \frac{R_h(j\omega_h)R_h(-j\omega_h)}{\tilde{\Pi}_h(j\omega_h)} = -\frac{1}{\omega_h} \frac{R_h(j\omega_h)R_h(-j\omega_h)}{\text{Im}\left[\tilde{\Pi}_h(j\omega_h)\right]} > 0. \end{aligned} \tag{B.36}$$

♢

Proof of Corollary A.2: Proceeding the same way as in Corollary A.1 and observing that in general $P_0(s)$ and $\bar{\Pi}_1(s+\varepsilon)$ have no common factors, the following limitation on ∂D_F can be obtained

$$\partial D_F \leq 2l - 1 + \frac{e}{2} + \frac{N_0}{2} + \sum_{i=1}^{k} N_i - \frac{r_0}{2} - \sum_{i=1}^{k} r_i, \tag{B.37}$$

where e denotes the pole-zero excess in $\Phi^*(s)$. Since $\Phi^*(s)$ is positive real and assuming suitable worst case bounds on other parameters one obtains

$$\partial D_F \leq 2l - 1 + \frac{1}{2} + \frac{1}{2} + k - \frac{1}{2} - k, \tag{B.38}$$

which in turn proves (A.81). ♢

Proof of Theorem A.4: By looking at equations (A.79) and (A.83), it is clear that the two rational functions $\Phi(s)$ in Theorems A.3 and A.4 are generated by perturbing the same $\Phi^*(s)$ of (A.76) in two slightly different ways.

Therefore, from the proof of Theorem A.3, it is clear that we have only to show that $\Phi(s)$ in (A.83) is strictly positive real. Observe that $\Phi(s)$ satisfies (A.23) by construction and, again from the proof of Theorem A.3, it turns out that $\mathrm{Re}[\Phi^*(j\omega)] > 0$ for all $\omega \in \bar{\Omega}_0$. Hence, it remains to prove that

$$\mathrm{Re}[\Phi(j\omega_i)] > 0 \tag{B.39}$$

for all $\omega_i \in \Omega_0$ and some sufficiently small ε, τ, and ζ_j, $j = 1, \ldots, k$.

It can be verified that $\Phi(s)$ can be rewritten as

$$\Phi(s) = \Psi_i(s)\,(1 + \Delta\Psi_i(s; \epsilon, \tau, \zeta_1, \ldots, \zeta_{i-1}, \zeta_{i+1}, \ldots, \zeta_k)) \tag{B.40}$$

where

$$\Psi_i(s) = (s^2 + 2\zeta_i\omega_i s + \omega_i^2)^{N_i}\tilde{\Pi}_i(s), \tag{B.41}$$

being $\tilde{\Pi}_i(s)$ given in (A.75), and

$$\Delta\Psi_i(s; \epsilon, \tau, \zeta_1, \ldots, \zeta_{i-1}, \zeta_{i+1}, \ldots, \zeta_k) \tag{B.42}$$

is a rational function, whose value at $s = j\omega_i$ is continuous with respect to the parameters ε, τ, ζ_j, $j = 1, \ldots, k$, $j \neq i$, and such that

$$\Delta\Psi_i(j\omega_i; 0, 0, 0, \ldots, 0, 0, \ldots, 0) = 0. \tag{B.43}$$

Hence, it suffices to prove that the function $\Psi_i(s)$ satisfies

$$\mathrm{Re}\left[\Psi_i(j\omega_i)\right] > 0 \qquad \text{for some} \quad \zeta_i > 0. \tag{B.44}$$

Suppose r_i is odd. We get

$$\mathrm{Re}\left[\Psi_i(j\omega_i)\right] = 2\zeta_i\omega_i^2 j^{N_i+1}\mathrm{Im}\left[\tilde{\Pi}_i(j\omega_i)\right] \tag{B.45}$$

and therefore (B.44) follows from (A.78).
If r_i is even, from (B.41), (A.78) and (A.41) it turns out that

$$\mathrm{Re}\left[\Psi_i(j\omega_i)\right] = \mathrm{Re}\left[\tilde{\Pi}_i(j\omega_i)\right] \geq 0. \tag{B.46}$$

Hence, (B.44) follows from Assumption 3. $\diamondsuit$

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