Global H_∞ Controllers for a Class of Nonlinear Systems

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***Abstract*—In this note, the problem of state feedback H_∞ control for a class of nonlinear systems is considered. The class under study is a generalization of the well-known Lur'e systems. The H_∞ problem is addressed via a class of storage functions of the Lur'e–Postnikov type whose integral term is parameterized by a nonlinear scalar function. The related H_∞ controllers consist of a linear term, which is designed for the underlying linearized system, plus a nonlinear term which depends on the nonlinear function. A simple geometrical criterion is provided for the characterization of the set of controllers which ensure a given level of L_2-performance globally. Some guidelines for an effective design of the controller within this set are discussed via two examples.**

***Index Terms*—Control and optimization, nonlinear H_∞ control, nonlinear systems, optimal control, robust control.**

I. Introduction

Although the theory of state feedback nonlinear H_∞ control is now completely understood (see [1] and the references therein), some difficulties still remain for a reliable design of nonlinear H_∞ controllers. These difficulties are mainly related to the lack of efficient numerical procedures for solving the Hamilton–Jacobi–Isaacs (HJI) inequality. Indeed, a typical way is to employ a storage function containing a quadratic term plus some higher order polynomial terms (see, e.g., [2] and [3]). The quadratic term is chosen such that the related linear controller works satisfactorily for the underlying linearized system, while the higher order terms are designed in order to provide a nonlinear controller ensuring a larger domain of validity. Unfortunately, the procedure is in general completely numerical, thus making it difficult to compare the performance provided by different controllers. Moreover, the important issue of comparing the performance of linear and nonlinear controllers still deserves a deeper understanding (see [1, pp. 219–222], [3], and [4]).

In this note, the domain of validity of state feedback H_∞ controllers is investigated for a class of nonlinear systems, which is a generalization of the well-known Lur'e systems [5], [6]. A class of storage functions depending affinely on one scalar memoryless nonlinear function is considered. The related controllers have the same linear part, which is designed for the underlying linearized system, and a nonlinear term which depends on the free nonlinear function. Moreover, each controller guarantees a given level γ of L_2-performance within its domain of validity. A simple geometrical criterion is presented for the design of the nonlinear part of the controller to ensure that the domain of validity is the whole state–space. Some guidelines for the selection of the controller are illustrated via two numerical examples.

The remainder of the note is organized as follows. Section II contains the problem formulation. The main results are given in Section III. Section IV presents two examples. Finally, some concluding comments are drawn in Section V. Preliminary versions of this note are presented in [7] and [8].

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A. Notation

$\mathbb{R}$: real space; $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$; $\mathbb{R}^n$: real n-space; $x = (x_1, \ldots, x_n)^T \in \mathbb{R}^n$: vector of $\mathbb{R}^n$; $\mathbb{R}^{p\times n}$: real $p \times n$-space; $A = [A_{ij}] \in \mathbb{R}^{p\times n}$: matrix of $\mathbb{R}^{p\times n}$; A^T: transpose of A; A^{-1}: inverse of A; $A > 0$ ($A \geq 0$): positive–definite (semidefinite) matrix; I_n: $n \times n$ identity matrix; $\langle \Psi, \Phi \rangle_R = \Psi^T R \Phi$: inner product of Ψ and Φ (with weighting matrix R).

II. Problem Formulation and Preliminary Results

Consider the following class of nonlinear systems:

$$\begin{cases} \dot{x} = Ax - Fn(\xi) + Bu + Ed \\ \xi = G^T x \\ y = Cx \end{cases} \tag{1}$$

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}$ is the control input, $d \in \mathbb{R}$ is the exogenous (unknown) disturbance, $y \in \mathbb{R}^p$ is the system output, $\xi \in \mathbb{R}$ is an auxiliary output variable, $A \in \mathbb{R}^{n\times n}$, $F \in \mathbb{R}^n$, $B \in \mathbb{R}^n$, $E \in \mathbb{R}^n$, $G \in \mathbb{R}^n$, $C \in \mathbb{R}^{p\times n}$, and $n : \mathbb{R} \to \mathbb{R}$ is a smooth memoryless nonlinearity such that

$$n(0) = n'(0) = 0. \tag{2}$$

Condition (2) ensures that (1) admits the origin as an equilibrium point for $u = d = 0$ and its linear part does not depend on the nonlinearity $n(\cdot)$. Note that, when $u = d = 0$ and $C = G^T$, (1) reduces to the systems involved in the Lur'e problem (see [5] and [6]). In the sequel, we find it convenient to rewrite $n(\xi)$ as

$$n(\xi) = k_n(\xi)\xi.$$

Note that (2) implies that the gain $k_n(\xi)$ satisfies $k_n(0) = 0$.

We are interested in investigating the state feedback H_∞ control of the above class of nonlinear systems. The standard way to deal with this problem is to find a storage function associated with the origin of (1), whose definition is recalled as follows.

Definition 1: Let $V : \mathbb{R}^n \to \mathbb{R}$ be a nonnegative smooth function such that $V(0) = 0$ and let γ be a positive scalar. Then, $V(x)$ is said to be a *storage function* if the HJI inequality

$$\begin{aligned} &\frac{\partial V}{\partial x}(x)\left[Ax - Fn(G^T x)\right] \\ &+ \frac{1}{2}\frac{\partial V}{\partial x}(x)\left[\frac{1}{\gamma^2}EE^T - BB^T\right]\frac{\partial^T V}{\partial x}(x) \\ &+ \frac{1}{2}x^T C^T C x \leq 0 \end{aligned} \tag{3}$$

holds in some neighborhood of the origin. Moreover, the set W of all x satisfying (3) is said to be the *domain of validity* of V.

Once a storage function has been found, the following well-known result directly provides a state feedback H_∞ controller [2].

Theorem 1: Consider (1) and let $\gamma > 0$. Suppose there exists a storage function $V(x)$ and let W be its domain of validity. Then, the closed-loop system with the feedback control law

$$u = -B^T \frac{\partial^T V}{\partial x}(x)$$

has L_2-gain from d to $[y \quad u]^T$ less than or equal to γ as long as its trajectories are inside W.

Therefore, the key step in nonlinear H_∞ control consists in finding a suitable storage function. A standard approach is to select a function containing a quadratic term, which is chosen on the basis of the linearization of (1) around the origin, plus some higher order polynomial terms (see, e.g., [2] and [3]). In this note, we consider the following class of functions:

$$\Theta = \left\{V_m(x) : V_m(x) = \frac{1}{2}x^T Px + \int_0^\xi m(\sigma)d\sigma\ , \quad \text{and } m(\cdot) \in \mathcal{M}\right\} \tag{4}$$

where $P \in \mathbb{R}^{n\times n}$ is a symmetric positive–semidefinite matrix and $\mathcal{M}$ is the class of continuous functions $m : \mathbb{R} \to \mathbb{R}$, which can be written as

$$m(\sigma) = k_m(\sigma)\sigma \tag{5}$$

where the gain $k_m(\sigma)$ is such that $k_m(0) = 0$. Note that the quadratic term is the same for each function $V_m(x)$, while the higher order terms depend on the choice of the function $m(\cdot) \in \mathcal{M}$. In particular, $V_m(x)$ reduces to the quadratic term when $m(\sigma) \equiv 0$.

The class Θ has a structure which recalls that of the Lur'e-Postnikov Lyapunov functions [6]. More specifically, once $m(\sigma) = \beta n(\sigma)$, $\beta \in \mathbb{R}$, Θ reduces to the Lur'e–Postnikov Lyapunov functions which have a central role in the classical theory of Lur'e systems [5], [6]. The following result pertains to the class Θ.

Lemma 1: Let $\gamma > 0$ and suppose $P = P^T > 0$ is a solution of the Riccati equation

$$A^T P + PA + P\left[\frac{1}{\gamma^2}EE^T - BB^T\right]P + C^T C = -Q \tag{6}$$

for some $Q = Q^T > 0$. Let κ_n and κ_m be two scalar parameters and define the matrix

$$\bar{Q}(\kappa_n, \kappa_m) = Q + \Upsilon(\kappa_n, \kappa_m)G^T + G\Upsilon^T(\kappa_n, \kappa_m) \tag{7}$$

where

$$\begin{aligned}\Upsilon(\kappa_n, \kappa_m) = {}& \kappa_n PF + \kappa_m\left[-A^T G - \frac{1}{\gamma^2}G^T EPE + G^T BPB\right] \\ & + G^T F\kappa_n\kappa_m G + \frac{1}{2}\left((G^T B)^2 - \frac{1}{\gamma^2}(G^T E)^2\right)\kappa_m^2 G.\end{aligned} \tag{8}$$

Then, for each $m \in \mathcal{M}$, the following statements hold.

1) $V_m(x)$ is a storage function and its domain of validity is given by

$$W_m = \{x \in \mathbb{R}^n : x^T \bar{Q}(k_n(\xi), k_m(\xi))x \geq 0 \quad \xi = G^T x \text{ and } V_m(x) \geq 0\}. \tag{9}$$

2) The state feedback controller

$$u_m(x) = -B^T Px - m(G^T x)B^T G$$

guarantees that the L_2-gain from d to $[y \ \ u]^T$ is less or equal to γ within W_m.

Proof: Statement 1) follows by observing that the HJI inequality (3) reduces to the first inequality in (9) once $V(x) = V_m(x)$ and P is selected according to (6). Statement 2) is a direct consequence of Theorem 1. ■

Lemma 1 provides a class of controllers $\{u_m(x), \ \ m(\cdot) \in \mathcal{M}\}$, ensuring the level γ of L_2-performance within the related domain of validity W_m in (9). Each controller consists of a fixed linear term plus a nonlinear term, which depends on th function $m(\cdot)$. In particular, the linear controller $u_0(x) = -B^T Px$, which is obtained for $m(\sigma) \equiv 0$, has been designed via the solution of (6) to work properly for the linearized system of (1) around the origin. On the other hand, the domain of validity W_0 pertaining to $u_0(x)$ may be not as large as desired and, therefore, a suitable nonlinear controller $u_m(x)$ must be looked for.

Motivated by the previous discussion, we are interested in providing criteria for the selection of the function $m(\cdot)$ in order for the related controller $u_m(x)$ to generate the level γ of L_2-performance globally, i.e., $W_m \equiv \mathbb{R}^n$.

III. Main Results

Let us consider the first inequality in (9). Clearly, such inequality holds for all $x \in \mathbb{R}^n$ if $(k_n(\xi), k_m(\xi))$ belongs for all $\xi \in \mathbb{R}$ to the region in the (κ_n, κ_m) plane where the matrix $\bar{Q}(\kappa_n, \kappa_m)$ in (7) is positive semidefinite. Hence, our first goal is to characterize the geometrical shape of such a region. To this purpose, we need the following auxiliary result.

Lemma 2: Consider the one-parameter family of $n \times n$ matrices

$$\bar{R}(\kappa) = R + \kappa(\Psi\Phi^T + \Phi\Psi^T) \tag{10}$$

where $R \in \mathbb{R}^{n\times n}$, $R = R^T > 0$, $\Phi \in \mathbb{R}^n$, $\Psi \in \mathbb{R}^n$ and $\kappa \in \mathbb{R}$. Then, $\bar{R}(\kappa) \geq 0$ if and only if

$$1 + 2\langle\Psi, \Phi\rangle_{R^{-1}}\ \kappa + \left(\langle\Psi, \Phi\rangle_{R^{-1}}^2 - \langle\Psi, \Psi\rangle_{R^{-1}}\langle\Phi, \Phi\rangle_{R^{-1}}\right)\kappa^2 \geq 0. \tag{11}$$

Proof: See the Appendix ■

Exploiting the previous lemma, we have a first result concerning $\bar{Q}(\kappa_n, \kappa_m)$.

Lemma 3: The following conditions are equivalent:

$$1)\quad \bar{Q}(\kappa_n, \kappa_m) \geq 0 \tag{12}$$

$$2)\quad \langle G, G\rangle_{Q^{-1}}\langle\Upsilon(\kappa_n, \kappa_m), \Upsilon(\kappa_n, \kappa_m)\rangle_{Q^{-1}} - \left(1 + \langle\Upsilon(\kappa_n, \kappa_m), G\rangle_{Q^{-1}}\right)^2 \leq 0. \tag{13}$$

Proof: It is easily verified that $\bar{Q}(\kappa_n, \kappa_m)$ in (7) has exactly the form in (10) once $R = Q$, $\Psi = \Upsilon(\kappa_n, \kappa_m)$, $\Phi = G$, and $\kappa = 1$. Hence, the proof follows by observing that in this case (11) reduces to (13). ■

Lemma 3 provides a convenient equivalent expression for $\bar{Q}(\kappa_n, \kappa_m) \geq 0$. The next step is to show that (13) defines a very simple geometrical constraint on the parameters κ_n and κ_m. Indeed, exploiting the expression of $\Upsilon(\kappa_n, \kappa_m)$ in (8), it turns out that (13) simplifies to

$$\kappa_n^2(h_1h_2 - h_3^2) + \kappa_m^2(l_1h_2 - l_3^2) + 2\kappa_n\kappa_m(-h_2l_2 - h_3l_3) + 2\kappa_n h_3 + 2\kappa_m l_3 - 1 \leq 0 \tag{14}$$

where

$$\begin{aligned}
h_1 &= \langle PF, PF\rangle_{Q^{-1}} \\
h_2 &= \langle G, G\rangle_{Q^{-1}} \\
h_3 &= -\langle PF, G\rangle_{Q^{-1}} \\
l_1 &= \left\langle -A^T G - \frac{1}{\gamma^2}G^T EPE + G^T BPB, -A^T G - \frac{1}{\gamma^2}G^T EPE + G^T BPB\right\rangle_{Q^{-1}} - (G^T B)^2 + \frac{1}{\gamma^2}(G^T E)^2 \\
l_2 &= G^T F - \langle PF, -A^T G - \frac{1}{\gamma^2}G^T EPE + G^T BPB\rangle_{Q^{-1}} \\
l_3 &= -\langle G, -A^T G - \frac{1}{\gamma^2}G^T EPE + G^T BPB\rangle_{Q^{-1}}
\end{aligned} \tag{15}$$

are quantities which depend on the linear part only of (1).

Clearly, (14) can be written equivalently as

$$[\,\kappa_n \quad \kappa_m \quad 1\,]\, T \begin{bmatrix} \kappa_n \\ \kappa_m \\ 1 \end{bmatrix} \le 0 \tag{16}$$

where

$$T = [T_{ij}] = \begin{bmatrix} h_1 h_2 - h_3^2 & -h_2 l_2 - h_3 l_3 & h_3 \\ -h_2 l_2 - h_3 l_3 & l_1 h_2 - l_3^2 & l_3 \\ h_3 & l_3 & -1 \end{bmatrix}. \tag{17}$$

Consider the region of the (κ_n, κ_m) plane defined as

$$\Omega = \left\{ (\kappa_n, \kappa_m) : \quad [\,\kappa_n \quad \kappa_m \quad 1\,]\, T \begin{bmatrix} \kappa_n \\ \kappa_m \\ 1 \end{bmatrix} \le 0 \right\} \tag{18}$$

and let

$$\Delta = T_{12}^2 - T_{11} T_{22}. \tag{19}$$

The next result follows directly from Lemma 3 and the equivalence of (13) and (16).

Lemma 4: The following conditions are equivalent:

$$1) \quad \bar{Q}(\kappa_n, \kappa_m) \ge 0 \tag{20}$$

$$2) \quad (\kappa_n, \kappa_m) \in \Omega. \tag{21}$$

From (17) and (18), it follows that Ω contains the origin of the (κ_n, κ_m) plane. Moreover, its boundary $\partial\Omega$ is described by a simple geometrical curve, i.e., either an ellipse ($\Delta < 0$), or a parabola ($\Delta = 0$), or a hyperbole ($\Delta > 0$).

Let us now investigate the second inequality in (9), i.e., $V_m(x) \ge 0$. Consider the half plane

$$\Pi = \left\{ (\kappa_n, \kappa_m) : \quad \kappa_m > -\frac{1}{G^T P^{-1} G} \right\} \tag{22}$$

which clearly contains the origin. The following result characterizes the condition $V_m(x) \ge 0$.

Lemma 5: If $(k_n(\xi), k_m(\xi)) \in \Pi$ for all $\xi \in \mathbb{R}$, then $V_m(x) \ge 0$ for all $x \in \mathbb{R}^n$.

Proof: Let $k_m(\xi) = (-1/G^T P^{-1} G) + \epsilon_m(\xi)$ where $\epsilon_m(\xi) > 0$ for all $\xi \in \mathbb{R}$. We have

$$\begin{aligned} V_m(x) &= \frac{1}{2} x^T P x + \int_0^{G^T x} m(\sigma) d\sigma = \frac{1}{2} x^T P x + \int_0^{G^T x} \sigma k_m(\sigma) d\sigma \\ &= \frac{1}{2} x^T P x - \int_0^{G^T x} \frac{\sigma}{G^T P^{-1} G} d\sigma + \int_0^{G^T x} \sigma \epsilon_m(\sigma) d\sigma \\ &\ge \frac{1}{2} x^T P x - \frac{1}{2} \frac{(G^T x)^2}{G^T P^{-1} G} = \frac{1}{2} x^T \left(P - \frac{G G^T}{G^T P^{-1} G} \right) x. \end{aligned}$$

To show that $V_m(x) \ge 0\ \forall x \in \mathbb{R}^n$ it suffices to prove that the matrix $P - GG^T / G^T P^{-1} G$ is positive semidefinite. From a standard determinantal result we have

$$\det \left(P - \kappa \frac{G G^T}{G^T P^{-1} G} \right) = (1 - \kappa) \det(P).$$

Hence, since $P > 0$, the previous expression implies that $P - GG^T / G^T P^{-1} G \ge 0$. ∎

Consider now the region

$$\Lambda = \Omega \cap \Pi \tag{23}$$

which clearly depends on the linear part of (1) only and contains the origin. Such a region plays a key role in the main result of the note.

Theorem 2: Let $\mathcal{C}$ be the curve in the (κ_n, κ_m) plane described parametrically as

$$\mathcal{C} = \begin{cases} \kappa_n = k_n(\xi) \\ \kappa_m = k_m(\xi) \end{cases} \quad \xi \in \mathbb{R} \tag{24}$$

and suppose that

$$\mathcal{C} \subset \Lambda. \tag{25}$$

Then, the domain of validity of the controller

$$u_m(x) = -B^T P x - k_m(G^T x) G^T x \cdot B^T G \tag{26}$$

is the whole state–space, i.e., $W_m \equiv \mathbb{R}^n$.

Proof: Since $\mathcal{C} \subset \Pi$, Lemma 5 guarantees that the second inequality in (9) holds for all $x \in \mathbb{R}^n$. Since $\mathcal{C} \subset \Omega$, Lemma 4 ensures that $\bar{Q}(k_n(G^T x), k_m(G^T x)) \ge 0$ for all $x \in \mathbb{R}^n$ and, therefore, also the first inequality in (9) holds globally. ∎

Theorem 2 provides a simple geometrical criterion for determining controllers which globally ensure the level γ of L_2-performance. Given γ and Q and evaluated P according to Lemma 1, it is enough to compute the region Λ exploiting (17), (18), (22), and (23). The controller (26) is then obtained by looking for a function $k_m(\cdot)$ such that the curve $\mathcal{C}$ in (24) satisfies condition (25).

A necessary and sufficient condition for the existence of such $k_m(\xi)$ is given next. The key to this result is the sector pertaining to the system nonlinearity $n(\cdot)$. It is worth to recall that $n(\xi)$ is said to belong to the sector $(\underline{\kappa}, \overline{\kappa})$ (i.e., $n(\xi) \in \mathrm{sect}(\underline{\kappa}, \overline{\kappa})$) if $\underline{\kappa} \le k_n(\xi) \le \overline{\kappa}$ for all $\xi \in \mathbb{R}$.

Let us define the following quantities which belong to $\overline{\mathbb{R}}$:

$$\underline{\kappa}_n = \inf_{(\kappa_n, \kappa_m) \in \Lambda} \kappa_n \quad \overline{\kappa}_n = \sup_{(\kappa_n, \kappa_m) \in \Lambda} \kappa_n. \tag{27}$$

Theorem 3: There exists $k_m(\xi)$ such that (24)–(25) hold if and only if

$$n(\xi) \in \mathrm{sect}(\underline{\kappa}_n, \overline{\kappa}_n). \tag{28}$$

Proof: Since Λ is a connected region, from the definition of $\underline{\kappa}_n$ and $\overline{\kappa}_n$ in (27) it follows that (28) is equivalent to pointwise existence of some function $k_m(\xi)$ such that $(k_n(\xi), k_m(\xi)) \in \Lambda$ for all $\xi \in \mathbb{R}$. Since the region Λ also contains the origin of the (κ_n, κ_m) plane, such a function can be chosen such that $k_m(0) = 0$, thus concluding the proof. ∎

It can be shown that $\underline{\kappa}_n$ and $\overline{\kappa}_n$ can be computed explicitly by exploiting the property that $\partial\Omega \cap \partial\Pi = \varnothing$ [9]. This property ensures that to compute $\underline{\kappa}_n$ and $\overline{\kappa}_n$ it is sufficient to consider only the constraint (16), which defines Ω. The explicit expressions of $\underline{\kappa}_n$ and $\overline{\kappa}_n$ are summarized in the following table:

	$\Delta > 0$, $T_{22} < 0$	$\Delta > 0$, $T_{22} > 0$, $T_{12} < 0$	$\Delta > 0$, $T_{22} > 0$, $T_{12} > 0$	$\Delta < 0$
$\underline{\kappa}_n$	$-\infty$	κ^+	$-\infty$	κ^+
$\overline{\kappa}_n$	$+\infty$	$+\infty$	κ^-	κ^-

(29)

where Δ is as in (19) and

$$\kappa^{\pm} = \frac{-(T_{12} l_3 - T_{22} h_3) \pm \sqrt{(T_{12} l_3 - T_{22} h_3)^2 - (l_3^2 + T_{22}) \Delta}}{\Delta}. \tag{30}$$

Note that in the ellipse case ($\Delta < 0$) the sector of the system nonlinearity $n(\cdot)$ is not allowed to be unbounded. Due to space limitations,

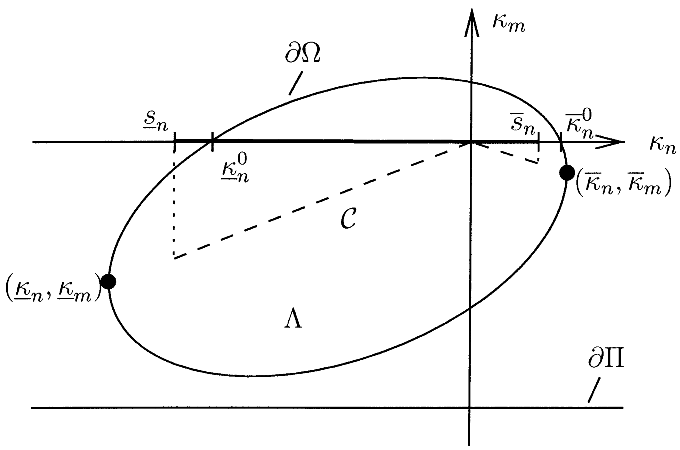

Fig. 1. Graphical interpretation of Theorems 2–4 and Corollary 1. The two quantities $\underline{\boldsymbol{s}}_n$ and $\overline{\boldsymbol{s}}_n$ are defined as $\underline{\boldsymbol{s}}_n = \mathbf{inf}\ \boldsymbol{k}_n(\boldsymbol{\xi}), \overline{\boldsymbol{s}}_n = \mathbf{sup}\ \boldsymbol{k}_n(\boldsymbol{\xi})$.

the parabola case ($\Delta = 0$) and the hyperbole case ($\Delta > 0$) when $T_{22} = 0$ or $T_{12} = 0$ have been omitted and can be found in [9].

Let us define the following quantities:

$$\underline{\kappa}_n^0 = \inf_{(\kappa_n, 0)\in\Lambda} \kappa_n; \quad \overline{\kappa}_n^0 = \sup_{(\kappa_n, 0)\in\Lambda} \kappa_n \tag{31}$$

which can be easily computed from (16) and are such that $\underline{\kappa}_n^0 \geq \underline{\kappa}_n$ and $\overline{\kappa}_n^0 \leq \overline{\kappa}_n$. The next result allows for a direct comparison of linear versus nonlinear controllers.

Corollary 1: The domain of validity W_0 of the linear controller $u_0(x) = -B^T P x$ is the whole state–space if

$$n(\xi) \in \mathrm{sect}(\underline{\kappa}_n^0, \overline{\kappa}_n^0). \tag{32}$$

Proof: It directly follows from Theorem 2 once the curve $\mathcal{C}$ in (24) is replaced by the straight line in the (κ_n, κ_m) plane parametrized by $\{(k_n(\xi), 0), \ \xi \in \mathbb{R}\}$, and observing that the region Λ is convex. ■

Fig. 1 provides a graphical interpretation of Theorem 2, Theorem 3, and Corollary 1 for $\Delta < 0$. Note that in this case, condition (32) fails since $\underline{s}_n = \inf k_n(\xi) < \underline{k}_n^0$ and, therefore, Corollary 1 cannot be applied. On the contrary, (28) is satisfied and, hence, Theorems 2 and 3 hold.

Clearly, any curve $\mathcal{C}$, which lies entirely in Λ and crosses the origin, implicitly provides a solution for $k_m(\cdot)$. The next result provides a closed form expression of $k_m(\cdot)$ for a piecewise linear curve.

Theorem 4: Let condition (28) be satisfied and compute $k_m(\xi)$ as

$$k_m(\xi) = \begin{cases} \underline{\theta} k_n(\xi) & \forall \xi : k_n(\xi) < 0 \\ \overline{\theta} k_n(\xi) & \forall \xi : k_n(\xi) \geq 0 \end{cases} \tag{33}$$

where $\underline{\theta}$ and $\overline{\theta}$ are given by (34)

	$\Delta > 0$, $T_{22} < 0$	$\Delta > 0$, $T_{22} > 0$, $T_{12} < 0$	$\Delta > 0$, $T_{22} > 0$, $T_{12} > 0$	$\Delta < 0$
$\underline{\theta}$	$\frac{(-T_{12}+\sqrt{\Delta})}{T_{22}}$	$\frac{\underline{\kappa}_m}{\underline{\kappa}_n}$	$\frac{-T_{12}}{T_{22}}$	$\frac{\underline{\kappa}_m}{\underline{\kappa}_n}$
$\overline{\theta}$	$\frac{(-T_{12}-\sqrt{\Delta})}{T_{22}}$	$\frac{-T_{12}}{T_{22}}$	$\frac{\overline{\kappa}_m}{\overline{\kappa}_n}$	$\frac{\overline{\kappa}_m}{\overline{\kappa}_n}$

being $\underline{\kappa}_m$ and $\overline{\kappa}_m$ defined as

$$\underline{\kappa}_m = -\frac{T_{12}\underline{\kappa}_n + l_3}{T_{22}} \ ; \ \overline{\kappa}_m = -\frac{T_{12}\overline{\kappa}_n + l_3}{T_{22}}. \tag{35}$$

Then, the related controller (26) ensures the level γ of L_2-performance globally.

Proof: Since the region Λ is convex and contains the origin, there exists a function $k_m(\cdot)$ such that $\mathcal{C} = \{(k_n(\xi), k_m(\xi)) : \xi \in \mathbb{R}\}$ is a piecewise linear curve in Λ containing the origin. Indeed, exploiting table (29), it is not difficult to verify that (33)–(34) yield one such function (see Fig. 1). ■

It is worth noting that Theorem 4 provides just one possible choice of the free function $k_m(\cdot)$. Indeed, this degree of freedom can be exploited to meet other performance indices (e.g., the transient behavior) of the closed loop, as it is shown in the example section.

IV. EXAMPLES

Example 1: Consider the system of the form (1) defined by the matrices

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 3 \end{bmatrix}$$
$$B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$
$$C = [0.2 \quad -2]$$
$$E = F = B$$
$$G = C^T$$

and the nonlinearity

$$n(\xi) = 6(\tanh \xi - \xi). \tag{36}$$

Note that $n(\xi) \in \mathrm{sect}(-6, 0)$.

Let us consider the H_∞ control problem for $\gamma = 1.3$. It turns out that for $Q = I_2$ the Riccati equation (6) is solved for

$$P = \begin{bmatrix} 17.63 & 0.47 \\ 0.47 & 15.63 \end{bmatrix}$$

which yields the quadratic term of the storage function $V_m(x)$ in (4). From (15), it turns out that the region Ω in (18) is defined by the matrix

$$T = \begin{bmatrix} 16.59 & -17.04 & 31.16 \\ -17.04 & 31.43 & -13.45 \\ 31.16 & -13.45 & -1 \end{bmatrix}$$

and since $\Delta = -231.17$, $\partial\Omega$ is an ellipse [see Fig. 2(a)]. Moreover, the half-plane Π in (22) is defined by $\kappa_m > -3.85$ and, hence, $\Lambda \equiv \Omega$. Moreover, it turns out that both $\underline{\kappa}_n$ and $\overline{\kappa}_n$ are finite and, from (29), we get

$$\underline{\kappa}_n = -6.63 \ \overline{\kappa}_n = 0.14.$$

Since (28) holds, Theorem 4 can be applied. From (35), we compute

$$\underline{\kappa}_m = -3.17 \ \overline{\kappa}_m = 0.50$$

and finally, from (33) and (34)

$$k_m(\xi) = 0.48 k_n(\xi) = \frac{2.88(\tanh \xi - \xi)}{\xi}. \tag{37}$$

Hence, the nonlinear feedback controller (26) with $k_m(\xi)$ in (37) globally solves the considered nonlinear H_∞ control problem.

Clearly, other choices of $k_m(\cdot)$ can be exploited in order to improve the performance of the closed loop. For instance, let us choose $k_m(\xi) = \hat{k}_m(\xi)$ where $\hat{k}_m(\xi)$ is given as

$$\hat{k}_m(\xi) = 0.1056 k_n^2(\xi) + 1.2500 k_n(\xi) \qquad \forall \xi \in \mathbb{R}. \tag{38}$$

The corresponding curve $\hat{\mathcal{C}}$ is entirely contained in Λ [see Fig. 2(a)] and, therefore, the controller $\hat{u}_m(x)$ defined via (38) ensures the same

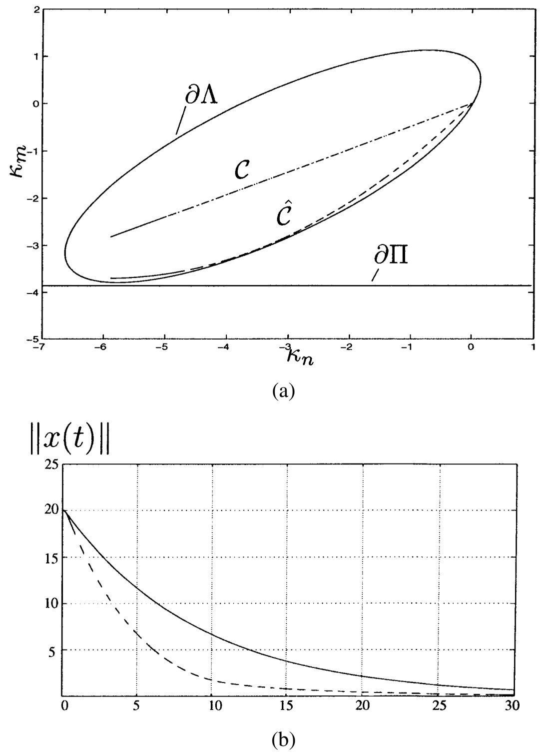

Fig. 2. Example 1. (a) Representation of $\boldsymbol{\Lambda}$ (solid ellipse), curves $\boldsymbol{\mathcal{C}}$ (dash-dot) and $\hat{\boldsymbol{\mathcal{C}}}$ (dashed). (b) Responses ($\|\boldsymbol{x(t)}\|$) to initial state $\boldsymbol{x(0)} = [\mathbf{20}\ \ \mathbf{0}]'$ corresponding to $\boldsymbol{u}_m(\boldsymbol{x})$ (solid) and $\hat{\boldsymbol{u}}_m(\boldsymbol{x})$ (dashed).

level of L_2-performance of the control law $u_m(x)$ related to (37). On the other hand, since $|\hat{k}_m(\xi)| \geq |k_m(\xi)|$ it is expected that (38) provides higher feedback gains than (37) and consequently a higher convergence rate of the corresponding closed-loop system. This is indeed confirmed by numerical simulations. As an example, the closed loop responses from initial condition $x(0) = [20\ \ 0]'$ of $\hat{u}_m(x)$ and $u_m(x)$ are depicted in Fig. 2(b).

Finally, we note that (32) does not hold. Hence, Corollary 1 does not guarantee the global validity of the linear controller $u_0(x) = -B^T Px$ defined by $m(\sigma) \equiv 0$. In this respect, it can be shown that the exact domain of validity, which can be computed numerically via (9), is not the whole state space. Therefore, in this case, the application of Corollary 1 does not show any conservatism.

Example 2: Consider a system of the form (1) where

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 4 \end{bmatrix} \quad B = E = F = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad C = G^T = [-2 \quad -5]$$

and

$$n(\xi) = \xi^3.$$

Note that $n(\xi) \in \mathrm{sect}(0, +\infty)$.

Let us consider the H_∞ control problem for $\gamma = 1.05$. Solving (6) for $Q = I_2$ yields

$$P = \begin{bmatrix} 89.53 & 2.26 \\ 2.26 & 89.71 \end{bmatrix}.$$

From (15) and (17) we get $\Delta = 1.48 \cdot 10^6$. Hence, in this case $\partial\Omega$ is a hyperbole. The half-plane Π is defined by $\kappa_m > -3.14$ and $\partial\Lambda$ can be shown to reduce to the upper hyperbole branch. From (29), we get

$$\underline{\kappa}_n = -0.06 \quad \overline{\kappa}_n = +\infty$$

and, therefore, (28) of Theorem 3 holds.

Since $\Delta > 0, T_{22} > 0, T_{12} < 0$ and $k_n(\xi)$ is positive definite, from Theorem 4 we get

$$k_m(\xi) = 85.82 k_n(\xi) = 85.82\xi^2.$$

Concerning the linear contrroller $u_0(x)$, it is clear that Corollary 1 cannot be applied since $\underline{\kappa}_n^0 = -3.26 \cdot 10^{-2}$ and $\overline{\kappa}_n^0 = 8.41 \cdot 10^{-4}$. Indeed, it turns out by numerical computation that also in this case the domain of validity of $u_0(x)$ is not the whole state space.

V. Conclusion

A family of state feedback H_∞ controllers for a class of nonlinear systems is singled out. These controllers, which have the same linear part while the nonlinear part depends on a free scalar memoryless nonlinear function, ensure the same level of L_2-performance on the corresponding domain of validity. A geometrical criterion for selecting the free nonlinear function in order to provide controllers possessing the whole state–space as domain of validity is given. Such a criterion allows us to show that nonlinear controllers provide in general larger domains of validity than linear ones and it makes it possible to take into account also other performance indexes in the controller design.

Appendix

Proof of Lemma 2

Since $\bar{R}(0) > 0$ and $\bar{R}(\kappa)$ is continuous with respect to κ, positive definiteness of $\bar{R}(\kappa)$ can be investigated by examining its determinant. In this respect, observe that the following matrix equivalence holds:

$$\left[\begin{array}{c|c} \bar{R}(\kappa) & \kappa[\Psi \quad \Phi] \\ \hline \begin{bmatrix} 0 \\ 0 \end{bmatrix} & I_2 \end{array}\right] = \left[\begin{array}{c|c} R & \kappa[\Psi \quad \Phi] \\ \hline \begin{bmatrix} -\Phi^T \\ -\Psi^T \end{bmatrix} & I_2 \end{array}\right] \left[\begin{array}{c|c} I_n & [0 \quad 0] \\ \hline \begin{bmatrix} \Phi^T \\ \Psi^T \end{bmatrix} & I_2 \end{array}\right]$$

and, therefore, it turns out that

$$\begin{aligned} \det(\bar{R}(\kappa)) &= \det \left[\begin{array}{c|c} R & \kappa[\Psi \quad \Phi] \\ \hline \begin{bmatrix} -\Phi^T \\ -\Psi^T \end{bmatrix} & I_2 \end{array}\right] \\ &= \det\left(I_2 + \kappa \begin{bmatrix} \Phi^T \\ \Psi^T \end{bmatrix} R^{-1} [\Psi \quad \Phi]\right) \det(R) \\ &= \det \begin{bmatrix} 1 + \kappa\langle\Psi, \Phi\rangle_{R^{-1}} & \kappa\langle\Phi, \Phi\rangle_{R^{-1}} \\ \kappa\langle\Psi, \Psi\rangle_{R^{-1}} & 1 + \kappa\langle\Psi, \Phi\rangle_{R^{-1}} \end{bmatrix} \det(R). \end{aligned} \tag{39}$$

Since $R > 0$, the structure of the last expression in (39) implies that $\bar{R}(\kappa)$ is positive semidefinite if and only if (11) holds.

References

[1] A. J. van der Schaft, *$\boldsymbol{L}_2$-Gain and Passivity Techniques in Nonlinear Control*. London, U.K.: Springer-Verlag, 1999.

[2] ——, “$\boldsymbol{L}_2$-gain analysis of nonlinear systems and nonlinear state feedback $\boldsymbol{H}_\infty$ control,” *IEEE Trans. Automat. Contr.*, vol. 37, pp. 770–784, May 1992.

[3] A. Isidori and W. Kang, “$\boldsymbol{H}_\infty$ control via measurement feedback for general nonlinear systems,” *IEEE Trans. Automat. Contr.*, vol. 40, pp. 466–472, Mar. 1995.

[4] M. J. Yazdanpanah, K. Khorasani, and R. V. Patel, “On the domain of validity of nonlinear $\boldsymbol{H}_\infty$ control,” in *Proc. 36th Conf. Decision Control*, San Diego, CA, 1997, pp. 654–655.

[5] M. Vidyasagar, *Nonlinear System Analysis*, 2nd ed. Upper Saddle River, NJ: Prentice-Hall, 1992.

[6] K. S. Narendra and J. H. Taylor, *Feedback Systems: Input-Output Properties*. San Diego, CA: Academic, 1973.

[7] G. Bianchini, R. Genesio, A. Parenti, and A. Tesi, "On the estimation of the validity domain for a class of nonlinear H_∞ controllers," in *Proc. 38th Conf. Decision Control*, Phoenix, AZ, 1999, pp. 3776–3781.
[8] ——, "On H-infinity control of a class of nonlinear systems," in *Proc. Euro. Control Conf. 2001*, Porto, Portugal, 2001, pp. 693–696.
[9] ——, "Global H_∞ controllers for a class of nonlinear systems," Dipartimento di Ingegneria dell'Informazione, Universita' di Siena, Siena, Italy, Tech. Rep. 08/03, 2003.

A New Method for Singular Value Loop Shaping in Design of Multiple-Channel Controllers

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***Abstract*—In this note, simple symmetric interval bounds on the singular values of a matrix based on its Gershgorin disks are proposed. This allows the Gershgorin theorem to be used not only to provide information about the location of the eigenvalues of a matrix but also its singular values. This is utilized for the proposition of a new design technique for singular value loop shaping based on the diagonal dominance methodology for design of linear multivariable plants. In return, this allows multiple-channel simply structured controllers to be designed with a view to robustness and to meet constraints and specifications on the behavior of its singular values. A design example is given demonstrating the effectiveness of this approach.**

***Index Terms*—Diagonal dominance, linear multivariable controller design, loop-shaping, singular value decomposition (SVD) analysis.**

I. Introduction

For a matrix $A = [a_{ij}] \in \mathbf{C}^{m\times m}$, the radius of its column Gershgorin disks $C_j(A)$ also referred to as the *deleted absolute column sum* and row gershgorin disks $R_i(A)$ also referred to as the *deleted absolute row sum* are defined, respectively, as

$$C_j(A) = \sum_{\substack{i=1\\ i\neq j}}^{m} |\ a_{ij}\ | \tag{1}$$

$$R_i(A) = \sum_{\substack{j=1\\ i\neq j}}^{m} |\ a_{ij}\ |. \tag{2}$$

Gershgorin's theorem [1] states that the eigenvalues of A lie inside the region defined by these disks centered on the diagonal entries of A

$$G_R(A) \equiv \bigcup_{i=1}^{m} \{s \in \mathbf{C} : |\ s - a_{ii}\ | \leq R_i(A)\} \tag{3}$$

$$G_C(A) \equiv \bigcup_{j=1}^{m} \{s \in \mathbf{C} : |\ s - a_{jj}\ | \leq C_j(A)\}. \tag{4}$$

Note that both $G_R(A)$ and $G_C(A)$ must include the eigenvalues, hence their intersection $G_\mu(A) = G_R(A) \cap G_C(A)$ is the only subset the eigenvalues can truly exist in. $G_\mu(A)$ is referred to as a *minimal Gershgorin set* [2] and other minimal sets may be obtained by considering the intersection of all the Gershgorin sets corresponding to *similar* operators to A (e.g., $\check{A} = S^{-1}AS$).

Rosenborck [3] used Gershgorin's theorem to propose the first frequency-based linear multivariable controller design technique based on the concept of diagonal dominance. This is a design technique that converts a linear multivariable design problem into several single-loop design problems which can then be solved using any number of available single-loop design techniques. In the case of column dominance, for a plant with transfer function matrix $G(s) = [g_{ij}(s)] \in \mathbf{C}^{m\times m}$, this involves finding a pre-compensator matrix $K = [k_{ij}(s)] \in \mathbf{R}^{m\times m}$, such that the resulting open-loop system with transfer function matrix $Q(s) = G(s)K$ satisfies the inequality

$$|\ q_{ii}(s)\ | \geq \sum_{\substack{j=1\\ j\neq i}}^{m} |\ q_{ij}(s)\ |, \qquad (i = 1, \ldots, m) \tag{5}$$

where, here, "$\geq$" denotes "at least equal to, but as much greater than as possible." Should the inequality become an equality the system will just satisfy the diagonal dominance criteria. If such a K can be found, $Q(s)$ may be replaced by $\tilde{Q}(s) = \mathrm{diag}\{Q(s)\} = [q_{ii}(s)] \in \mathbf{C}^{m\times m}$. Next, a diagonal controller matrix $D(s) = [d_{ii}(s)] \in \mathbf{C}^{m\times m}$ can be found such that $\tilde{q}_{ii}(s)d_{ii}(s)$ is as close as possible to $m_i(s)(1 - m_i(s))$, where $M(s) = \mathrm{diag}\{m_i(s)\}$ is the desired transfer function matrix of the closed-loop system, whose actual overall transfer function matrix is $T(s)$.

Note that, since $\tilde{q}_{ij}(s)d_{ij}(s) = 0$ $(\forall\, i \neq j)$, the design of $D(s)$ can be broken down into m single-loop design problems; the transfer function matrix of the corresponding multivariable controller is $C(s) = KD(s)$. If the precompensator matrix K satisfies (5), then this inequality is also satisfied for $Q(s)D(s)$ since $D(s)$ postmultiplies each column of $Q(s)$ by the same gain at each frequency. There are many established techniques of finding the matrix K such as the pseudodiagonalization algorithm of Hawkins [4] and the ALIGN algorithm [5]. More recently, techniques were developed for finding dynamic precompensators [6], [7] which although are typically of first or second order, are able to achieve much greater levels of diagonal dominance.

A major handicap with diagonal dominance is that the controller design only focuses on very few properties of the system [8] and issues such as robustness, disturbance rejection, etc. are implied from the process and not inherently addressed by it. This makes the motivations for this work very clear. By showing that the Gershgorin disks can be used to bound the singular values as well, it allows the designer to not only use the Gershgorin disks to asses the system's interaction, but also guaranteed bounds on the whereabouts of its singular values. In turn, the design process can be made to cater for cases where, for example, there are specifications on the H_∞ norm of the system or the behavior of its singular values.

Needless to say, there are other techniques, which can produce high performance robust controllers, which may satisfy some given constraint on the behavior of the singular values of the system. The most well known of these techniques is the H_∞ mixed sensitivity approach. However, the majority of these techniques are synthesis based, where the designer has ultimately little control over many of the features of the produced controllers, such as structure, complexity, and even sometimes realizability. For example, it is commonly acknowledged that in a H_∞ design study, even if one overcomes the initial nontrivial problem of designing the weighting functions [9], still the secondary problem remains of a controller which is very complex and of a high order. The technique outlined here is on the other hand a "design" technique,

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