Passivity Analysis and Passification of Discrete-Time Hybrid Systems

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Abstract

For discrete-time hybrid systems in piecewise affine or piecewise polynomial form, this note proposes sufficient passivity analysis and synthesis criteria based on the computation of piecewise quadratic or piecewise polynomial storage functions. By exploiting linear matrix inequality techniques and sum of squares decomposition methods, passivity analysis and synthesis of passifying controllers can be carried out through standard semidefinite programming packages, providing a tool particularly important for stability of interconnected heterogenous dynamical systems.

I. INTRODUCTION

Passivity is a widely adopted tool for analyzing the stability of interconnections of dynamical systems [1] and is used in several domains of engineering sciences, such as in the analysis of electrical circuits and of mechanical systems. In particular, passivity is exploited in robotics as a key concept for stability analysis of human/machine interaction (see, e.g., [2]).

Stability analysis of interconnected systems hinges upon the ability of characterizing the passivity properties of each single dynamical system. A solid theory and analytical/numerical criteria are available for linear systems, and theoretical characterizations were developed for smooth nonlinear dynamical systems [1]. Although most of the passivity characterizations were proposed for continuous-time models, a few results were developed for discrete-time models [3].

In many practical applications, some of the system components exhibit a heterogeneous dynamical discrete and continuous nature that cannot be captured by smooth models because of abrupt mode switches. The study of such *hybrid systems*, that has massively emerged in the last few years, was devoted to analyzing the dynamical interaction between continuous and discrete signals in one common framework. Passivity analysis of hybrid models has received very little attention, except for the contributions of [4], [5], [6] and [7], in which notions of passivity for continuous-time hybrid systems are formulated.

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In this note we address the passivity property of discrete-time hybrid systems in the widely exploited piecewise affine (PWA) form and, more generally, in the piecewise polynomial (PWP) form. For PWA systems, in the spirit of [8], [9], quadratic and piecewise quadratic storage functions are computed via the solution of a number of linear matrix inequality (LMI) problems. The proposed method also yields a LMI-based procedure for computing piecewise linear state-feedback controllers ensuring passivity of the closed-loop system. We also propose a method for proving passivity of a PWA/PWP system by means of polynomial or piecewise polynomial storage functions. Such functions are constructed via semidefinite programming by means of the sum of squares (SOS) decomposition of multivariate polynomials [10]. SOS methods for the computation of piecewise polynomial Lyapunov functions have been exploited for analyzing stability of continuous-time hybrid and switched systems [11]. In this note we use a similar idea for passivity analysis in discrete-time, although the approach can be easily generalized to the continuous-time case. Preliminary work leading to the results reported in this note was presented in [12] and in [13].

The paper is organized as follows. After reporting some preliminary definitions and results, and formulating the passivity analysis problem in Section II, in Section III we present an LMI-based passivity test for PWA systems based on the construction of piecewise quadratic storage functions. Section IV describes a passivity test for PWA and PWP systems based on the computation of piecewise polynomial storage functions. An application of the proposed results to a simple model derived from haptics is presented in Section V, and finally some concluding remarks are drawn in Section VI.

II. Notation, Preliminaries, and Problem Formulation

In this note we consider discrete-time time-invariant hybrid systems of the form

$$\begin{cases} x_{k+1} = f_i(x_k, u_k) \\ y_k = h_i(x_k, u_k) \end{cases} \quad \text{if } [x_k^T \quad u_k^T]^T \in \chi_i, \;\; i \in \mathcal{I}, \tag{1}$$

where $x_k \in \mathbb{R}^n$ is the state vector, $u_k \in \mathbb{R}^m$ the control input, $y_k \in \mathbb{R}^p$ the output vector, $k \in \mathcal{T} \triangleq \{0, 1, \ldots\}$ the discrete-time counter, $\mathcal{I} \triangleq \{1, \ldots, n_{\mathcal{I}}\}$ the set of mode-indices, and $f_i : \mathbb{R}^{n+m} \to \mathbb{R}^n$, $h_i : \mathbb{R}^{n+m} \to \mathbb{R}^p$ are suitable vector fields. Let $\{\chi_i\}_{i\in\mathcal{I}}$ be a *partition* of $\mathbb{R}^{n+m}$, namely

$$\chi_i = \left\{ [x^T u^T]^T \in \mathbb{R}^{n+m} : \; g_{i,r}^x(x) \geq 0, \;\; g_{i,t}^u(u) \geq 0, \;\; r = 1, \ldots, r_i, \;\; t = 1, \ldots, t_i \right\}, \tag{2}$$

with $\mathrm{Int}\chi_i \bigcap \mathrm{Int}\chi_j = \emptyset$, $\forall i, j \in \mathcal{I}$, $i \neq j$ (“Int” denotes the interior), $\bigcup_{i\in\mathcal{I}} \chi_i = \mathbb{R}^{n+m}$, $\chi_i \neq \emptyset$ $\forall i \in \mathcal{I}$, and where $g_{i,r}^x : \mathbb{R}^n \to \mathbb{R}$, $g_{i,t}^u : \mathbb{R}^m \to \mathbb{R}$ are the functions defining the shape of the cells of the partition[1]. Also, let us introduce the following sets of indices

$$\mathcal{S}_i = \{j \in \mathcal{I} \ : \ \Pi_x(\chi_i) = \Pi_x(\chi_j)\}\,, \ i \in \mathcal{I},$$

where $\Pi_x(\chi_i)$ denotes the projection of χ_i over the x-space. Since in (2) we have excluded the more general case $\chi_i = \left\{[x^T u^T]^T \in \mathbb{R}^{n+m} : \ g_{i,s}(x,u) \geq 0, \ s = 1, \ldots, s_i\right\}$ in defining the shape of the cells, we have that $\mathrm{Int}\Pi_x(\chi_j) \bigcap \mathrm{Int}\Pi_x(\chi_i) = \emptyset$, $\forall j \notin \mathcal{S}_i$, $\forall i \in \mathcal{I}$. Let $\mathcal{H} \subseteq \mathcal{I}$ denote a subset of indices $i \in \mathcal{I}$ corresponding to a collection of all sets $\Pi_x(\chi_i)$ without duplicates, and for each $h \in \mathcal{H}$ denote

$$\overline{\chi}_h = \Pi_x(\chi_h). \tag{3}$$

Clearly, $\overline{\chi}_h = \Pi_x(\chi_i)$, $\forall i \in \mathcal{S}_h$, and the collection $\{\overline{\chi}_h\}_{h \in \mathcal{H}}$ forms a partition of $\mathbb{R}^n$. Moreover, $\bigcup_{h \in \mathcal{H}} \mathcal{S}_h = \mathcal{I}$ and $\mathcal{S}_h \cap \mathcal{S}_l = \emptyset$, $\forall h, l \in \mathcal{H}$, $h \neq l$. Also, introduce the map $h : \mathcal{I} \rightarrow \mathcal{H}$ defined as

$$h(i) = h \in \mathcal{H} \ \text{ such that } \ i \in \mathcal{S}_h.$$

Each set $\overline{\chi}_h$ has the expression

$$\overline{\chi}_h = \{x \in \mathbb{R}^n : \ g^x_{h,r}(x) \geq 0, \ r = 1, \ldots, r_h\}, \ h \in \mathcal{H}.$$

Our definition of partition in (2) generalizes the definition of [8], [15], [16], where the authors consider the case of hybrid systems defined over a partition of the x-space only.

A. *Piecewise Affine (PWA) case*

If the vector fields f_i, h_i in (1), and $g^x_{i,r}$, $g^u_{i,t}$ in (2) are affine functions, then system (1) is in *piecewise affine* (PWA) form:

$$\begin{cases} x_{k+1} = A_i x_k + B_i u_k + \phi_i \\ y_k = C_i x_k + D_i u_k + \psi_i \end{cases} \text{if } [x_k^T \ \ u_k^T]^T \in \chi_i, \ i \in \mathcal{I}, \tag{4}$$

where A_i, B_i, C_i, D_i, ϕ_i, ψ_i constant matrices/vectors of suitable dimension. $\{\chi_i\}_{i \in \mathcal{I}}$ forms a polyhedral partition of $\mathbb{R}^{n+m}$, i.e.,

$$\chi_i = \left\{[x^T u^T]^T \in \mathbb{R}^{n+m} \ : \ F^x_i x \geq f^x_i, F^u_i u \geq f^u_i\right\}, \tag{5}$$

[1]In case of discontinuities across common boundaries of neighboring regions, to avoid ambiguities of the state-update and/or output mappings of the hybrid system (1), one can define sets χ_i in (2) by using strict and nonstrict inequalities. Alternatively, one can replace strict inequalities $g(\cdot) > 0$ by nonstrict inequalities $g(\cdot) \geq \epsilon$, where $\epsilon > 0$ is an arbitrarily small number (e.g., the machine precision), although systems trajectories would not be defined in the interval $0 < g(\cdot) < \epsilon$ (see [14]).

where F_i^x, f_i^x, F_i^u, f_i^u, $i \in \mathcal{I}$, are constant matrices/vectors.
Clearly, the partition $\{\overline{\chi}_h\}_{h\in\mathcal{H}}$ is defined by

$$\overline{\chi}_h = \{x \in \mathbb{R}^n \,:\, F_h^x x \geq f_h^x\}\,,\ \ h \in \mathcal{H}. \tag{6}$$

Let $\mathcal{I}_0 = \{i \in \mathcal{I} : 0 \in \chi_{h(i)}\}$. We assume that $\phi_i = \psi_i = 0$, $\forall i \in \mathcal{I}_0$, i.e., that the origin is an equilibrium point for the system with zero inputs.

For ease of notation, by setting $\overline{x} = [x^T\ 1]^T$, $\overline{u} = [u^T\ 1]^T$, $\overline{y} = [y^T\ 0]^T$, we rewrite (4) in the more compact form

$$\begin{cases} \overline{x}_{k+1} = \overline{A}_i \overline{x}_k + \overline{B}_i \overline{u}_k \\ \overline{y}_k = \overline{C}_i \overline{x}_k + \overline{D}_i \overline{u}_k \end{cases} \text{if } [x_k^T \quad u_k^T]^T \in \chi_i,\ i \in \mathcal{I}, \tag{7}$$

and $\chi_i = \left\{[x^T u^T]^T \in \mathbb{R}^{n+m} : \overline{F}_i^x \overline{x} \geq 0, \overline{F}_i^u \overline{u} \geq 0\right\}$, where $\overline{A}_i = \left[\begin{smallmatrix} A_i & \phi_i \\ 0 & 1 \end{smallmatrix}\right]$, $\overline{B}_i = \left[\begin{smallmatrix} B_i & 0 \\ 0 & 0 \end{smallmatrix}\right]$, $\overline{C}_i = \left[\begin{smallmatrix} C_i & \psi_i \\ 0 & 0 \end{smallmatrix}\right]$, $\overline{D}_i = \left[\begin{smallmatrix} D_i & 0 \\ 0 & 0 \end{smallmatrix}\right]$, and $\overline{F}_i^x = [F_i^x \ \ -f_i^x]$, $\overline{F}_i^u = [F_i^u \ \ -f_i^u]$.

Likewise, if $f_i(x,u)$, $h_i(x,u)$, $g_{i,r}^x(x)$, and $g_{i,r}^u(u)$ in (1),(2) are multivariate polynomials in x and u, then the system is termed *piecewise polynomial* (PWP).

B. Discrete-time passivity

In this note, we refer to the standard notion of passivity for discrete-time systems [1], [3].

Definition 1: Consider system (1), and let $m = p$. The system is said to be *passive* if there exists a positive definite function $V : \mathbb{R}^n \to \mathbb{R}$ with $V(0) = 0$ (called the *storage function*) such that along all possible system trajectories (x_k, u_k, y_k), $k \in \mathcal{T}$, the following *dissipation inequality* holds

$$V(x_{k+1}) - V(x_k) - y_k^T u_k \leq 0. \tag{8}$$

Note that even in the linear case, the usual sampled equivalent of a passive continuous-time system, which assumes a zero-order holder on the input and sets y_k as the output value at the k-th sampling instant, in general does not preserve passivity. A passivity-preserving discretization scheme for linear dynamics was proposed in [12].

III. Passivity Analysis of Piecewise Affine Systems

The most common way to investigate passivity of general nonlinear systems is to check the dissipation inequality (8) against storage functions of prescribed structure. In this respect, quadratic functions are the most common choice. Such an approach can be applied straightforwardly to the case of PWA systems of the form (4), (5). Indeed, it is easily shown that passivity of the system is ensured if there exists a common quadratic storage function satisfying the passivity inequality for all the linear subsystems defined by (A_i, B_i, C_i, D_i), $i \in \mathcal{I}$. Moreover, by a standard

Kalman-Yakubovich-Popov (KYP) lemma argument, checking the passivity of each subsystem via a quadratic storage function is known to boil down to an LMI condition [1].

It is apparent that the common quadratic storage function approach is likely to be overly conservative for hybrid systems in PWA form (4),(5) since the switching conditions are completely ignored. By following the line proposed in [8], [9] in the context of stability analysis, in the sequel we illustrate an LMI criterion for passivity analysis based on the computation of piecewise quadratic storage functions. This task is accomplished by specializing the positivity and dissipation inequalities in Definition 1 so as to capture the relevant features of the switching behavior and hence to reduce conservatism.

A. Passivity Analysis via Piecewise Quadratic Storage Functions

For system (4), we consider a piecewise quadratic (PWQ) candidate storage function $V : \mathbb{R}^n \to \mathbb{R}$ defined on the partition $\{\overline{\chi}_h\}_{h\in\mathcal{H}}$ of the state space as

$$V(x) = \overline{x}^T \overline{P}_h \overline{x} \quad \forall x \in \overline{\chi}_h,\ h \in \mathcal{H}, \tag{9}$$

where $\overline{P}_h$ are suitable $(n+1)\times(n+1)$ symmetric matrices. Note that, in order for $V(x)$ to be zero at the origin and positive definite, $\overline{P}_h$ is constrained to have the form $\overline{P}_h = \begin{bmatrix} P_h & 0 \\ 0 & 0 \end{bmatrix}$ for all $h \in \mathcal{H}$ such that $\overline{\chi}_h$ contains the origin, i.e., $\forall h \in \mathcal{I}_0 \cap \mathcal{H}$, where $P_h \in \mathbb{R}^{n\times n}$ is a positive definite symmetric matrix.

According to (8), if matrices $\overline{P}_h$, $h \in \mathcal{H}$ exist such that the dissipation inequality with $V(x)$ as in (9) holds for all system trajectories, then system (4) is passive. If this is the case, then the system will be termed *PWQ passive*.

Let us define the set of index pairs

$$\mathcal{S} = \left\{(i,j) : \exists x \in \mathbb{R}^n, u, w \in \mathbb{R}^m : [x^T u^T]^T \in \chi_i,\ [(A_i x + B_i u + \phi_i)^T w^T]^T \in \chi_j,\ i,j \in \mathcal{I}\right\}, \tag{10}$$

i.e., the set of ordered pairs of indices corresponding to all transitions from cell χ_i at any time k to cell χ_j at time $k+1$ which are actually allowed to occur along system trajectories. The set $\mathcal{S}$ can be computed by means of reachability analysis based on linear programming [17]. Moreover, for all $(i,j) \in \mathcal{S}$, let

$$\widetilde{\chi}_i^j = \{[x^T\ u^T]^T \in \chi_i : \exists\, w : [(A_i x + B_i u + \phi_i)^T\ w^T]^T \in \chi_j\}$$

be the subsets of all state-input pairs in cell χ_i which can evolve into cell χ_j in one step. It is easily seen that, since $\chi_j \neq \emptyset$, the set $\widetilde{\chi}_i^j$ is the following polytope of $\mathbb{R}^{n+m}$

$$\widetilde{\chi}_i^j = \{[x^T\ u^T]^T \in \mathbb{R}^{n+m} : \overline{F}_i^x \overline{x} \geq 0,\ \overline{F}_i^u \overline{u} \geq 0,\ \overline{F}_{ij}^x \overline{x} + \overline{F}_{ij}^u \overline{u} \geq 0\},$$

where $\overline{F}_{ij}^x = [F_j^x A_i \ \ F_j^x \phi_i - f_j^x]$ and $\overline{F}_{ij}^u = [F_j^x B_i \ \ 0]$. The following result, in the spirit of [8], [15], [16], gives a sufficient condition for PWQ passivity of the PWA system (4) that can be tested by semidefinite programming.

Theorem 1: Let $\overline{U}_i$, $\overline{V}_i$, $i \in \mathcal{I}$, $\overline{Z}_h$, $h \in \mathcal{H}$, and $\overline{W}_{ij}$, $(i,j) \in \mathcal{S}$, be unknown matrices of suitable dimensions with nonnegative entries and define

$$\begin{array}{lllclll} \overline{G}_{ij} & = & (\overline{F}_i^x)^T \overline{U}_i \overline{F}_i^x + (\overline{F}_{ij}^x)^T \overline{W}_{ij} \overline{F}_{ij}^x & ; & \overline{J}_{ij} & = & (\overline{F}_i^u)^T \overline{V}_i \overline{F}_i^u + (\overline{F}_{ij}^u)^T \overline{W}_{ij} \overline{F}_{ij}^u \\ \overline{H}_{ij} & = & (\overline{F}_{ij}^u)^T \overline{W}_{ij} \overline{F}_{ij}^x & ; & \overline{L}_h & = & (\overline{F}_h^x)^T \overline{Z}_h \overline{F}_h^x. \end{array} \tag{11}$$

Let $\overline{P}_h \in \mathbb{R}^{(n+1)\times(n+1)}$, $h \in \mathcal{H}$, be symmetric matrices. If a selection of matrices $\overline{P}_h$, $\overline{Z}_h$, $h \in \mathcal{H}$, $\overline{U}_i$, $\overline{V}_i$, $i \in \mathcal{I}$, and $\overline{W}_{i,j}$, $(i,j) \in \mathcal{S}$, exists that satisfies the set of LMIs

$$\begin{cases} \overline{P}_h - \overline{L}_h > 0, \ \forall h \in \mathcal{H}, h \notin \mathcal{I}_0 \\ \overline{P}_h = \left[\begin{smallmatrix} P_h & 0 \\ 0 & 0 \end{smallmatrix}\right], \ [I_n \ 0]\,(\overline{P}_h - \overline{L}_h) \left[\begin{smallmatrix} I_n \\ 0 \end{smallmatrix}\right] > 0, \ \forall h \in \mathcal{H} \cap \mathcal{I}_0 \\ \begin{bmatrix} \overline{A}_i^T \overline{P}_{h(j)} \overline{A}_i - \overline{P}_{h(i)} + \overline{G}_{ij} & \overline{A}_i^T \overline{P}_{h(j)} \overline{B}_i - \dfrac{\overline{C}_i^T}{2} + \overline{H}_{ij}^T \\ \overline{B}_i^T \overline{P}_{h(j)} \overline{A}_i - \dfrac{\overline{C}_i}{2} + \overline{H}_{ij} & \overline{B}_i^T \overline{P}_{h(j)} \overline{B}_i - \dfrac{\overline{D}_i + \overline{D}_i^T}{2} + \overline{J}_{ij} \end{bmatrix} \le 0, \ \forall (i,j) \in \mathcal{S}, \end{cases} \tag{12}$$

with $\overline{G}_{ij}, \overline{J}_{ij}, \overline{H}_{ij}, \overline{L}_h$ as in (11), then system (4) is PWQ passive with storage function (9).

Proof: By the first LMI in (12) and the fourth of (11) it turns out that $\overline{x}^T \overline{L}_h \overline{x} \ge 0, \forall x \in \overline{\chi}_h, \ h \in \mathcal{H}$, and hence $V(x)$ in (9)is positive definite. Moreover, along any trajectory such that $[x_k^T \ u_k^T]^T \in \chi_i$ and $[x_{k+1}^T \ u_{k+1}^T]' \in \chi_j$ for some $(i,j) \in \mathcal{S}$, it holds that

$$\begin{aligned} & V(x_{k+1}) - V(x_k) - y_k^T u_k = \\ & [\overline{x}_k^T \ \overline{u}_k^T] \begin{bmatrix} \overline{A}_i^T \overline{P}_{h(j)} \overline{A}_i - \overline{P}_{h(i)} + \overline{G}_{ij} & \overline{A}_i^T \overline{P}_{h(j)} \overline{B}_i - \dfrac{\overline{C}_i^T}{2} + \overline{H}_{ij}^T \\ \overline{B}_i^T \overline{P}_{h(j)} \overline{A}_i - \dfrac{\overline{C}_i}{2} + \overline{H}_{ij} & \overline{B}_i^T \overline{P}_{h(j)} \overline{B}_i - \dfrac{\overline{D}_i + \overline{D}_i^T}{2} + \overline{J}_{ij} \end{bmatrix} \begin{bmatrix} \overline{x}_k \\ \overline{u}_k \end{bmatrix} \\ & -[\overline{x}_k^T \overline{G}_{ij} \overline{x}_k + 2\overline{u}_k^T \overline{H}_{ij} \overline{x}_k + \overline{u}_k^T \overline{J}_{ij} \overline{u}_k]. \end{aligned}$$

Therefore, by the last LMI in (12) and (11), we have that $V(x_{k+1}) - V(x_k) - y_k^T u_k \le 0$ along any system trajectory and hence the system is passive according to Definition 1. ■

Note that the condition $\overline{P}_h = \left[\begin{smallmatrix} P_h & 0 \\ 0 & 0 \end{smallmatrix}\right]$ in (12) implies linear equality constraints on $\overline{U}_i$, $\overline{W}_{ij}$, $\overline{V}_i$, $\overline{Z}_h$ for $i, h \in \mathcal{I}_0$.

A simpler but more conservative version of Theorem 1 can be obtained by removing the unknowns $\overline{U}_i$, $\overline{V}_i$, $\overline{Z}_h$, $\overline{W}_{ij}$ and the terms $\overline{G}_{ij}$, $\overline{J}_{ij}$, $\overline{H}_{ij}$, $\overline{L}_h$ from the LMI problem (12). This amounts to ignoring the switching conditions defined by the sets χ_i in (7) for the PWA dynamics.

B. Passivity Enforcement via Piecewise Linear State Feedback

We now consider the problem of synthesizing a piecewise linear state feedback control law for PWA systems in order to make the resulting closed-loop system passive. More specifically, we look for a piecewise linear function $k^{pl}:\mathbb{R}^n \to \mathbb{R}^m$ such that system (4) with state feedback

$$u_k = k^{pl}(x_k) + v_k \tag{13}$$

is PWQ passive, i.e., there exists a PWQ storage function $V(x)$ such that the dissipation inequality $V(x_{k+1}) - V(x_k) \leq y_k^T v_k$ holds for any system trajectory (x_k, v_k, y_k), $k \in \mathcal{T}$. The approach proposed here extends the one used in [18] in the context of stabilization. In order to meet space limitations and to avoid introducing excessive technicalities, we only address the problem for system (4) with zero affine terms, i.e., $\phi_i = \psi_i = 0$, $\forall i \in \mathcal{I}$.

It is apparent that the partition $\{\chi_i\}_{i\in\mathcal{I}}$ cannot be exploited to define the piecewise linear feedback in (13), since the partition itself depends on the control input u. Hence, it is natural to look for a piecewise linear feedback defined on the polyhedral partition $\{\bar{\chi}_h\}_{h\in\mathcal{H}}$ of the state space defined by (6), i.e., a control law of the form

$$u_k = -K_h x_k + v_k,\ x_k \in \overline{\chi}_h,\ h \in \mathcal{H}. \tag{14}$$

Based on the PWQ storage function

$$V(x) = x^T P_h x,\ \forall x \in \chi_h,\ h \in \mathcal{H}, \tag{15}$$

we want to provide a criterion for synthesizing feedback gains K_h, $h \in \mathcal{H}$, such that the closed-loop system

$$\begin{cases} x_{k+1} = A_i^{cl} x_k + B_i v_k \\ y_k = C_i^{cl} x_k + D_i v_k \end{cases} \quad \text{if } \begin{bmatrix} x_k \\ K_{h(i)} x_k + v_k \end{bmatrix} \in \chi_i,\ i \in \mathcal{I},$$

with input v_k and output y_k, is passive, where $A_i^{cl} = A_i - B_i K_{h(i)}$, $C_i^{cl} = C_i - D_i K_{h(i)}$.

The following closed-loop passivity condition can be stated.

Lemma 1: Consider system (4) with $\phi_i = \psi_i = 0$, $\forall i \in \mathcal{I}$. If there exist matrices P_h, $h \in \mathcal{H}$, and K_h, $h \in \mathcal{H}$ such that the inequalities

$$\begin{cases} P_h = P_h^T > 0,\ \forall h \in \mathcal{H} \\ \begin{bmatrix} (A_i^{cl})^T P_l A_i^{cl} - P_{h(i)} & (A_i^{cl})^T P_l B_i - \dfrac{(C_i^{cl})^T}{2} \\ B_i^T P_l A_i^{cl} - \dfrac{C_i^{cl}}{2} & B_i^T P_l B_i - \dfrac{D_i + D_i^T}{2} \end{bmatrix} \leq 0,\ \forall\, i \in \mathcal{I},\ \forall l \in \mathcal{H} \end{cases} \tag{16}$$

hold, then the system with piecewise linear feedback (14) is PWQ passive .

Proof: It suffices to note that the second inequality in (16) implies that $V(x_{k+1}) - V(x_k) - y_k^T v_k \leq 0$ along all possible system trajectories. Indeed, the feedback gain $K_{h(i)}$ is applied for all $x_k \in \chi_i$ and independent of the cell $\overline{\chi}_l$ the vector x_{k+1} belongs to. Moreover, all possible transitions are covered. ■

The PWQ passivity condition provided by Lemma 1 is not computationally appealing since the inequalities in (16) are bilinear in K_h and P_h, and hence the synthesis problem cannot be approached by means of convex optimizations techniques. Nevertheless, such inequalities can be exploited to derive a LMI sufficient condition for computing the passifying piecewise linear feedback (14). This is accomplished through a standard Schur complement argument as the following result shows.

Theorem 2: Consider system (4) and let $\phi_i = \psi_i = 0$, $\forall i \in \mathcal{I}$. If there exist matrices Q_h, R_h, Y_h, $h \in \mathcal{H}$ such that the set of LMIs

$$\begin{cases} Q_h = Q_h^T > 0, \ \forall h \in \mathcal{H} \\ \begin{bmatrix} R_{h(i)} + R_{h(i)}^T - Q_{h(i)} & \frac{1}{2}(R_{h(i)}^T C_i^T - Y_{h(i)}^T D_i^T) & R_{h(i)}^T A_i^T - Y_{h(i)}^T B_i^T \\ \frac{1}{2}(C_i R_{h(i)} - D_i Y_{h(i)}) & \frac{D_i + D_i^T}{2} & B_i^T \\ A_i R_{h(i)} - B_i Y_{h(i)} & B_i & Q_l \end{bmatrix} \geq 0, \ \forall i \in \mathcal{I}, \ \forall l \in \mathcal{H} \end{cases} \tag{17}$$

holds, then the system with piecewise linear state feedback (14) with

$$K_h = Y_h R_h^{-1}, \ h \in \mathcal{H},$$

is PWQ passive with respect to the storage function (15), with $P_h = Q_h^{-1}$.

Proof: Since $Q_{h(i)} > 0$ and $R_{h(i)} + R_{h(i)}^T \geq Q_{h(i)}$ by (17), then $R_{h(i)}$ is nonsingular and moreover it is easy to see that $R_{h(i)}^T Q_{h(i)}^{-1} R_{h(i)} \geq R_{h(i)} + R_{h(i)}^T - Q_{h(i)} \geq 0$. Hence (17) implies

$$\begin{cases} Q_h = Q_h^T > 0, \ \forall h \in \mathcal{H} \\ \begin{bmatrix} R_{h(i)}^T Q_{h(i)}^{-1} R_{h(i)} & \frac{1}{2}(R_{h(i)}^T C_i^T - Y_{h(i)}^T D_i^T) & R_{h(i)}^T A_i^T - Y_{h(i)}^T B_i^T \\ \frac{1}{2}(C_i R_{h(i)} - D_i Y_{h(i)}) & \frac{D_i + D_i^T}{2} & B_i^T \\ A_i R_{h(i)} - B_i Y_{h(i)} & B_i & Q_l \end{bmatrix} \geq 0, \ \forall i \in \mathcal{I}, \ \forall l \in \mathcal{H} \end{cases} \tag{18}$$

By left-multiplying (18) by $\begin{bmatrix} R_{h(i)}^{-T} & 0 \\ 0 & I \end{bmatrix}$ and right-multiplying by $\begin{bmatrix} R_{h(i)}^{-1} & 0 \\ 0 & I \end{bmatrix}$ we obtain

$$\begin{cases} Q_h = Q_h^T > 0, \ \forall h \in \mathcal{H} \\ \begin{bmatrix} Q_{h(i)}^{-1} & \frac{{C_i^{cl}}^T}{2} & (A_i^{cl})^T \\ \frac{C_i^{cl}}{2} & \frac{D_i + D_i^T}{2} & B_i^T \\ A_i^{cl} & B_i & Q_l \end{bmatrix} \geq 0, \ \forall i \in \mathcal{I}, \ \forall l \in \mathcal{H}, \end{cases}$$

which is equivalent to (16) by a Schur complement argument, where $Q_{h(i)} = P_{h(i)}^{-1}$. The result then follows by Lemma 1. ∎

IV. Passivity Analysis for Piecewise Polynomial Systems

In this section we consider the problem of assessing the passivity property of hybrid systems in piecewise polynomial (PWP) form, i.e., of systems of the form (1), (2) when it is assumed that the vector fields $f_i(x, u)$, $h_i(x, u)$, $g_{i,r}^x(x)$ and $g_{i,r}^u(u)$ are multivariate polynomials in x and u. Our approach is based on the computation of piecewise polynomial storage functions by exploiting the SOS decomposition of multivariate polynomials. It is well-known that the SOS decomposition provides a satisfactory relaxation for proving polynomial positivity via semidefinite programming [10].

In the sequel, piecewise polynomial storage functions $V(x)$ will be considered and passivity conditions will be formulated in terms of inequalities based on (8). To employ the convex programming techniques mentioned above, such inequalities must be interpreted in the SOS sense.

Remark 1: The PWA models analyzed in the previous section clearly fall into the wider class of PWP systems. It is worth to note that looking for higher order piecewise polynomial storage functions may generally yield significantly less conservative passivity tests than those based on piecewise quadratic storage functions, as it will be pointed out in Section V. Unfortunately, contrary to PWQ methods, SOS tests which employ superquadratic storage functions do not easily extend to compute feedback control laws that ensure passivity of the closed-loop system. Moreover, such tests may be more sensitive to numerical errors.

Consider a PWP system of the form (1), (2) and the following piecewise polynomial candidate storage function $V : \mathbb{R}^n \to \mathbb{R}$ defined on the partition $\{\overline{\chi}_h\}_{h\in\mathcal{H}}$ in (3):

$$V(x) = V_h(x),\ \forall x \in \overline{\chi}_h,\ \ h \in \mathcal{H}, \tag{19}$$

where $V_h(x)$, $h \in \mathcal{H}$ are polynomials. Let us define the set of index pairs

$$\mathcal{S} = \left\{(i,j) :\ \exists x \in \mathbb{R}^n, u, w \in \mathbb{R}^m :\ [x^T\ u^T]^T \in \chi_i, \quad [f_i(x,u)^T\ w^T]^T \in \chi_j,\ i,j \in \mathcal{I}\right\}\ .$$

Clearly, this set plays the same role as the set $\mathcal{S}$ in (10) for the PWA case.

A PWP passivity test can be devised by proceeding in the same fashion as the PWQ test in Section III.A.

For all $(i,j) \in \mathcal{S}$, let us introduce the set $\tilde{\chi}_i^j \subseteq \chi_i$ defined as

$$\tilde{\chi}_i^j = \left\{[x^T\ u^T]^T \in \chi_i : \exists w : [f_i(x,u)^T\ w^T]^T \in \chi_j\right\},$$

i.e., the subset of state-input pairs in cell χ_i at time k which are allowed to evolve into cell χ_j at time $k+1$. Each $\tilde{\chi}_i^j$ is given by

$$\tilde{\chi}_i^j = \left\{\begin{array}{l} [x^T\ u^T]^T \in \mathbb{R}^n\ \ : \\ g_{i,r}^x(x) \geq 0,\ g_{i,t}^u(u) \geq 0,\ g_{j,s}^x(f_i(x,u)) \geq 0,\ r = 1,\dots r_i,\ t = 1,\dots t_i,\ s = 1,\dots r_j \end{array}\right\}.$$

The following result provides the sought PWP passivity test. The detailed proof is omitted due to space limitation but can be easily conducted in the same fashion as that of Theorem 1 by enforcing both positivity of $V(x)$ in (19) and the dissipation inequality taking into account the switching behavior described by $\mathcal{S}$ and $\tilde{\chi}_i^j$.

Theorem 3: Consider the PWP system (1). If there exist polynomials $V_h(x)$, $h \in \mathcal{H}$, $a_{h,r}^x(x) \geq 0$, $r = 1, \ldots, r_h$, $h \in \mathcal{H}$, $b_{i,j,r}(x,u) \geq 0$, $r = 1, \ldots, r_i$, $c_{i,j,t}(x,u) \geq 0$, $t = 1, \ldots, t_i$, and $d_{i,j,s}(x,u) \geq 0$, $s = 1, \ldots, r_j$, $(i,j) \in \mathcal{S}$ such that $V_h(0) = 0$ and

$$\begin{cases} V_h(x) - \sum_{r=1}^{r_h} a_{h,r}^x(x) g_{h,r}^x(x) > 0, \ \forall x \neq 0, \ \forall h \in \mathcal{H} \\ V_{h(j)}(f_i(x,u)) - V_{h(i)}(x) - h_i^T(x,u)u + \sum_{r=1}^{r_i} b_{i,j,r}(x,u) g_{i,r}^x(x) + \sum_{t=1}^{t_i} c_{i,j,t}(x,u) g_{i,t}^u(u) \\ \quad + \sum_{s=1}^{r_j} d_{i,j,s}(x,u) g_{j,s}^x(f_i(x,u)) \leq 0, \ \forall (x,u), \quad \forall (i,j) \in \mathcal{S} \end{cases}$$

then the system is passive with storage function (19).

Remark 2: The set of allowed transitions $\mathcal{S}$ is in general quite difficult to compute for an arbitrary PWP system. Clearly, the above result can be applied successfully in the case of PWA systems. Otherwise, a more conservative version of Theorem 3 is readily obtained by replacing $\mathcal{S}$ with the Cartesian product $\mathcal{I} \times \mathcal{I}$ and taking $a_{h,r}^x(x) = b_{i,j,r}(x,u) = c_{i,j,t}(x,u) = d_{i,j,s}(x,u) = 0$.

V. Application Example

In this section we apply the proposed passivity criteria to stability analysis of a simple model derived from haptics. Indeed, the stability of a haptic loop (human operator, haptic device and virtual environment) can be assessed provided that passivity of the dynamics relating the applied force and the velocity of the end effector can be ensured [2].

Consider the simple haptic interaction model in Fig. 1. The idea of modelling the interaction between a haptic device and a virtual environment with a sampled-data equivalent of a spring-mass-damper system with Coulomb friction is quite standard [2]. Clearly, the haptic device

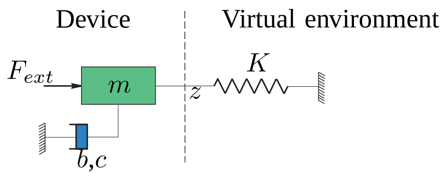

Fig. 1. Spring-mass-damper haptic model with Coulomb friction

dynamics is continuous-time, while the virtual environment is a computer simulated model and hence a pure discrete-time system. It is well-known that the coupling of a discrete-time system with a continuous-time system can lead to loss of passivity. A possible approach to analyzing passivity of the overall model, where the input u is the external force F_{ext} applied to the haptic device and the output y is the velocity $\dot{z}$ of the end effector, is to derive a discrete-time multirate model in which different sampling times nT_s and T_s ($n > 1$) are used for the simulated and the haptic device dynamics, respectively [19]. Of course, this still involves a certain degree of approximation since the intersample behavior of the "fast" subsystem is neglected. It is worth noting that the problem of stability analysis and controller design for haptic systems can indeed be addressed in a purely discrete-time setting by exploiting the framework proposed in [20], [21]. In this respect, we believe that an extension of the above framework to haptic systems involving hybrid components is viable by exploiting the results presented here but this goes beyond the scope of this note and is the subject of current research.

The dynamics of the haptic device (fast system) is modeled in discrete time by sampling every T_s seconds the dynamics of a spring-mass system (with mass m and damping b) through a passivity-preserving discretization scheme as described in [12]. The forces exciting the mass-spring system are the applied force F_{ext} (acting as the input to the overall system), the Coulomb friction $F_c = c \cdot \text{sign}\,(\dot{x})$ acting on the end effector, and the spring force (with stiffness K) from the virtual environment (slow system), which is only allowed to change every nT_s seconds. The physical parameters of the system are $m = 0.01$, $b = 0.9$, $c = 0.1$, $K = 1$, $T_s = 10^{-4}$, $n = 10$ (international units).

Let x_1, x_2 be the position and velocity, respectively, of the end-effector. Let x_3 be the value of x_1 at multiples of the slow sampling time nT_s, and let x_4 be the state of an auxiliary counter that is reset every n steps. The overall system can be described as the fourth-order PWA model (4), defined over four polyhedral cells ($\mathcal{I} = \{1, 2, 3, 4\}$), reported in [22]. The set $\mathcal{S}$ of admissible switches is $\mathcal{S} = \mathcal{I} \times \mathcal{I} \setminus \{(1,1), (1,3), (3,1), (3,3)\}$.

By applying the PWQ criterion of Theorem 1, a piecewise quadratic storage function is found that proves the discrete-time passivity of the system. When damping and Coulomb friction parameters are decreased to $b = 0.01$ and $c = 0.001$, respectively, while Theorem 1 fails to prove PWQ passivity, Theorem 3 provides a valid piecewise quartic storage function. The expressions of both functions are reported in [22].

VI. Conclusion

This note has proposed sufficient passivity analysis criteria for discrete-time hybrid systems in piecewise affine or piecewise polynomial form, and a tool for the synthesis of passifying state-feedback piecewise linear control laws for piecewise affine systems. The proposed approach appears particularly encouraging in the analysis and design of (possibly heterogeneous) interconnected systems, such as those modeling human-machine interaction.

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