Synthesis of robust strictly positive real systems with l_2 parametric uncertainty

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Abstract— **The problem of designing filters ensuring strict positive realness of a family of uncertain polynomials over an assigned region of the complex plane is a longly investigated issue in the analysis of absolute stability of nonlinear Lur'e systems and the design of adaptive schemes. This paper addresses the problem of designing a continuous-time rational filter when the uncertain polynomial family is assumed to be an ellipsoid in coefficient space. It is shown that the stability of all the polynomials of such a family is a necessary and sufficient condition for the existence of the filter. More importantly, contrary to the results available for the case of a polyhedral uncertainty set in coefficient space, it turns out that the filter is a proper rational function with degree smaller than twice the degree of the uncertain polynomials. Furthermore, a closed form solution to the filter synthesis problem based on polynomial factorization is derived.**

Keywords—**strict positive realness, uncertain polynomials, robustness, filter design.**

I. Introduction

Recent years have witnessed a growing interest in the study of invariance of the Strict Positive Realness (SPR) property of rational transfer functions with respect to numerator and denominator perturbations. This issue is relevant to the analysis of absolute stability of nonlinear Lur'e systems and the design of adaptive schemes (see, e.g., [1]-[12]). In the latter case, the connection lies in the well-known fact that a sufficient condition for the convergence of several recursive algorithms of adaptive schemes is the SPR of a suitable family of transfer functions (see, e.g., [13]-[15]).

The key issue investigated in many papers is the robust SPR problem. Given a set of polynomials $\mathcal{P}$ and a region Λ of the complex plane, determine if there exists a polynomial (or rational) filter F such that each transfer function P/F, $P \in \mathcal{P}$ is strictly positive real over Λ. For instance, in the context of recursive identification schemes, the set $\mathcal{P}$ can be viewed as a model of the uncertainty about the true plant and Λ is the region of the complex plane where the power spectral density of the regressor is concentrated.

Several useful results are available on the existence and construction of F for different choices of $\mathcal{P}$ and Λ. In [3]-[5] the continuous-time and discrete-time robust SPR problems are considered when $\mathcal{P}$ is a polyhedron in the coefficient space, while in [6]-[8] the set $\mathcal{P}$ is described in terms of root location regions and Λ is some subset of the complement of the unit disk. Also, the robust-shifted SPR problem is investigated in [9],[10].

Despite of the numerous contributions to the robust SPR problem, some important issues remain unsolved. One of the most important concerns the degree of the filter F. In [4] an important result stating necessary and sufficient conditions for the existence of the sought filter F is given, when $\mathcal{P}$ is assumed to be a polyhedron in the coefficient space. Such condition simply requires that the all the polynomials of the set $\mathcal{P}$ are stable. The corresponding filter turns out to be a polynomial in the discrete-time case and in general a rational function in the continuous-time case. In addition, a procedure for constructing the filter F as a series expansion is given. However, this technique does not provide the filter F in closed form, i.e., F may have an arbitrarily high degree. On the other hand, some sufficient conditions have been given to ensure the existence of a polynomial filter [5] and a finite degree rational filter [11] when $\mathcal{P}$ is an interval polynomial. Finally, a finite degree rational filter can also be designed when the set $\mathcal{P}$ contains only two discrete time polynomials [12].

In this paper, we consider the continuous-time robust SPR problem when the set $\mathcal{P}$ is assumed to be an ellipsoid in the coefficient space. This is a natural choice for the set $\mathcal{P}$ in the context of recursive identification schemes [13]-[15]. First, exploiting the results in [4], it is shown that the stability of the polynomials of $\mathcal{P}$ is a necessary and sufficient condition for the existence of the filter F. Successively, a completely different analysis is performed in order to construct a solution of the problem with an a-priori bounded degree. More specifically, it turns out that the filter F is a rational function having a degree less than twice the degree of the polynomials of the set $\mathcal{P}$. Moreover, F can be obtained in closed form via a suitable polynomial factorization problem.

The paper is structured as follows. Section 2 contains the problem formulation and preliminary results. Section 3 presents the main results for the robust SPR problem. Section 4 contains some application examples to illustrate the features of the results. Section 5 reports some concluding comments. The proofs of several results are reported in the appendix section.

Notation.
$\mathbb{C}$: complex plane;
$s \in \mathbb{C}$: complex number;
$\mathrm{Re}[s], \mathrm{Im}[s]$: real and imaginary parts of s;
$\arg[s]$: argument of s;
$P(s)$: real polynomial;

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∂P : degree of $P(s)$;
$[P(s)]_o$: polynomial containing only the odd powers of $P(s)$;
$\mathbb{R}^n$: real n-space;
$v = (v_1, \ldots, v_n)'$: vector of $\mathbb{R}^n$;
$\|v\|_2$: 2-norm of v;
$\mathcal{H}$: set of Hurwitz polynomials;
$\mathcal{RH}_\infty$: set of stable proper real rational transfer functions

Basic definitions.
Let us recall the definitions of positive realness (PR) and strict positive realness (SPR) of a rational function used throughout the paper [16].

Definition 1: A rational function $\Phi(s)$ is positive real if
1. $\Phi(s)$ is real for real s;
2. $\Phi(s)$ is analytic in $\mathrm{Re}[s] > 0$ and the poles on the imaginary axis are simple and such that the associated residue is non-negative;
3. for any real value of ω for which $s = j\omega$ is not a pole of $\Phi(s)$, $\mathrm{Re}\,[\Phi(j\omega)] \geq 0$.

Definition 2: A rational function $\Phi(s)$ is said to be strictly positive real if
1. $\Phi(s), \Phi^{-1}(s) \in \mathcal{RH}_\infty$;
2. $\mathrm{Re}\,[\Phi(j\omega)] > 0 \quad \forall \omega \geq 0$.

The following result relating PR and SPR will be employed in the paper.

Lemma 1: Let $\Phi^*(s) = \dfrac{P_1(s)}{P_2(s)}$ be positive real. Then, for sufficiently small $\varepsilon, \delta > 0$, the function

$$\Phi(s) = \Phi^*(s+\varepsilon)(1+\delta s)^{\partial P_2 - \partial P_1} \tag{1}$$

is strictly positive real.

Proof: See Appendix.

Remark 1: From a well-known property concerning the relative degree of a positive real rational function, it follows that ∂P_1 and ∂P_2 in (1) satisfy the relation:

$$-1 \leq \partial P_2 - \partial P_1 \leq 1.$$

II. Problem formulation and preliminary results

The robust SPR problem in the continuous-time case can be stated as follows [3],[4]. Given a set of polynomials $\mathcal{P}$, determine, if it exists, a polynomial (or in general a rational function) $F(s)$ such that for any $P(s) \in \mathcal{P}$ the function $P(s)/F(s)$ is strictly positive real over the closed right half plane.

A fundamental contribution to the robust SPR problem was given in [4], where $\mathcal{P}$ was assumed to be a polyhedron in the coefficient space. In that paper, it is shown that the robust SPR problem has a solution if and only if the family $\mathcal{P}$ has no roots in the closed right half plane.

In this paper, we address the robust SPR problem for a different set of polynomials, i.e., an ellipsoid in coefficient space centered at a given nominal polynomial.

Definition 3: An ellipsoidal set of polynomials of degree l is the set

$$\mathcal{P}_\rho := \left\{ P(s) = P_0(s) + \sum_{i=1}^{n} q_i P_i(s) : \|q\|_2 \leq \rho \right\}$$

where $P_0(s)$, $P_1(s)$, $\ldots$,$P_n(s)$ are such that $\partial P_0 = l$, $\partial P_i < l$ for all $i = 1, \ldots, n$, $q = (q_1 \ldots q_n)' \in \mathbb{R}^n$ is the parameter vector, and $\rho > 0$.

Recalling Definition 2, we can state the robust SPR ($RSPR$) problem in the following way.

$RSPR$ problem. Given the set $\mathcal{P}_\rho$, determine a transfer function $F(s)$, if it exists, such that the SPR conditions

1.
$$\frac{P(s)}{F(s)}, \frac{F(s)}{P(s)} \in \mathcal{RH}_\infty \tag{2}$$

2.
$$\mathrm{Re}\left[\frac{P(j\omega)}{F(j\omega)}\right] > 0 \ \ \forall \omega \geq 0. \tag{3}$$

hold for all $P(s) \in \mathcal{P}_\rho$.

Remark 2: We recall that condition (3) is equivalent to the phase condition

$$|\arg[P(j\omega)] - \arg[F(j\omega)]| < \pi/2 \quad \forall \omega \geq 0.$$

Clearly, for the solvability of the $RSPR$ problem it is mandatory that condition (2) must hold for $P_0(s)$. Therefore, recalling that the denominator of any real rational rational function in $\mathcal{RH}_\infty$ is necessarily a Hurwitz polynomial, the following requirement on the set $\mathcal{P}_\rho$ can be enforced without loss of generality.

Assumption. The nominal polynomial $P_0(s)$ is Hurwitz.

A preliminary result for the $RSPR$ problem can be obtained quite readily. To this purpose, let ρ^* denote the l_2 parametric stability margin of $\mathcal{P}_\rho$, i.e., the maximal ρ such that $\mathcal{P}_\rho$ cointains all Hurwitz polynomials

$$\rho^* = \sup_{\mathcal{P}_\rho \subset \mathcal{H}} \rho.$$

According to condition (2), it follows that the condition $\rho < \rho^*$ is necessary for the solution of the $RSPR$ problem. Exploiting convexity of $\mathcal{P}_\rho$ and the results in [4], it turns out that such a condition is also sufficient.

Theorem 1: Consider the set $\mathcal{P}_\rho$ of uncertain polynomials and suppose that $\rho < \rho^*$. Then, there exist a non-negative integer M and a Hurwitz polynomial $R(s)$ of degree $l + M$ such that the rational function

$$F(s) = \frac{R(s)}{(s+1)^M} \tag{4}$$

solves the $RSPR$ problem.

Proof: It follows from a straightforward extension of Theorem 3.1 in [4], once the finite set $\{n_i(s)\}$ is replaced by the convex set $\mathcal{P}_\rho$. Indeed, let

$$\overline{\phi}(\omega) =: \sup_{P \in \mathcal{P}_\rho} \arg[P(j\omega)] \ \ ; \ \ \underline{\phi}(\omega) =: \inf_{P \in \mathcal{P}_\rho} \arg[P(j\omega)].$$

Since $\mathcal{P}_\rho$ is a convex degree-invariant set of Hurwitz polynomials, the following condition is true (see [2])

$$\overline{\phi}(\omega) - \underline{\phi}(\omega) < \pi \ \ \forall \omega \geq 0. \tag{5}$$

Now, introducing the function

$$\phi^*(\omega) := \frac{\overline{\phi}(\omega) + \underline{\phi}(\omega)}{2},$$

it can be easily checked that the relation

$$|\arg[P(j\omega)] - \phi^*(\omega)| < \frac{\pi}{2} \quad \forall \omega \geq 0 \tag{6}$$

holds for each polynomial $P(s) \in \mathcal{P}_\rho$.
Thus, from Remark 2 it turns out that the $RSPR$ problem is solved if a function $F^*(s)$ is determined such that ${F^*}^{-1}(s) \in \mathcal{RH}_\infty$ and its phase on the imaginary axis satisfies

$$\arg[F^*(j\omega)] = \phi^*(\omega).$$

Employing a series expansion as in [4], it can be shown that $F^*(s)$ can be arbitrarily approximated via a rational function of the form (4) for suitable $R(s)$ and M. ■

Although quite interesting from a conceptual viewpoint, Theorem 1 only provides a partial solution to the $RSPR$ problem. Indeed, since $F(s)$ is computed via a procedure based on a series expansion, there is no a-priori knowledge of the degree of the filter F. In the next section, we will overcome this drawback by introducing a completely new approach to the $RSPR$ problem, which allows us to construct a rational filter F with a finite known degree via the solution of a suitable factorization problem.

III. Main results

To solve the robust SPR problem we first need the expression of the l_2 stability margin of $\mathcal{P}_\rho$. To this purpose, let

$$G(s) := \left[-\frac{P_1(s)}{P_0(s)} \cdots -\frac{P_n(s)}{P_0(s)} \right]' \tag{7}$$

and introduce the two functions

$$R(\omega) := \mathrm{Re}[G(j\omega)], \quad I(\omega) := \mathrm{Im}[G(j\omega)]. \tag{8}$$

Consider the two complementary sets of frequencies

$$\Omega_0 = \{\omega \geq 0 \ : \ I(\omega) = 0\}, \tag{9}$$
$$\overline{\Omega}_0 = \{\omega \geq 0 \ : \ I(\omega) \neq 0\}. \tag{10}$$

Since the set Ω_0 plays an important role in the characterization of the form of the filter F, we briefly discuss its structure. It is easily verified that Ω_0 contains at most a finite number k of frequencies in addition to $\omega = 0$, i.e.,

$$\Omega_0 = \{0, \omega_1, \ldots, \omega_k\}. \tag{11}$$

Furthermore, since any frequency $\omega_i \in \Omega_0$, $i = 1, \ldots, k$ must be a common root of n polynomials in ω of degree less than $2l$, we observe that the existence of such frequencies is not generic, especially for large n. Therefore, the case $\Omega_0 = \{0\}$ can be considered as the generic case.
We have the following well-known result [2].

Lemma 2: Let

$$\rho_0 = \min_{\omega \in \Omega_0} \frac{1}{\|R(\omega)\|_2} \tag{12}$$

$$\bar{\rho} = \inf_{\omega \in \bar{\Omega}_0} \rho_{\bar{\Omega}_0}(\omega) \tag{13}$$

where

$$\rho_{\bar{\Omega}_0}(\omega) =$$

$$= \begin{cases} \dfrac{\|I(\omega)\|_2}{\left[\|I(\omega)\|_2^2 \, \|R(\omega)\|_2^2 - (R'(\omega)I(\omega))^2 \right]^{1/2}} & \text{if } \omega \in \bar{\Omega}_s \\ +\infty & \text{if } \omega \notin \bar{\Omega}_s \end{cases} \tag{14}$$

being

$$\bar{\Omega}_s = \left\{ \omega \in \bar{\Omega}_0 \ : \ \|I(\omega)\|_2^2 \, \|R(\omega)\|_2^2 - (R'(\omega)I(\omega))^2 \neq 0 \right\}. \tag{15}$$

Then, the l_2 parametric stability margin of $\mathcal{P}_\rho$ is given by

$$\rho^* = \begin{cases} \rho_0 & \text{if } n = 1 \\ \min\{\rho_0, \bar{\rho}\} & \text{if } n > 1 \end{cases}. \tag{16}$$

Proof: see Appendix.

It is straightforward to check that the $RSPR$ problem can be restated equivalently as follows. Determine a function $\Phi(s)$ such that

$$\Phi(s), \Phi^{-1}(s) \in \mathcal{RH}_\infty \tag{17}$$

and

$$\mathrm{Re}\left[\Phi(j\omega)\left(1 - q'G(j\omega)\right)\right] > 0 \quad \forall \omega \geq 0 \quad \forall q : \|q\|_2 \leq \rho. \tag{18}$$

Obviously, once $\Phi(s)$ has been determined, $F(s)$ is readily obtained via the relation

$$F(s) = \frac{P_0(s)}{\Phi(s)}.$$

The starting point for determining $\Phi(s)$ is the next result (see also [17]), where condition (18) is rewritten into an equivalent form no longer dependent on the parameter vector q.

Lemma 3: Let $G(j\omega), R(\omega), I(\omega)$ be defined as in (7) and (8). Then, the following two statements are equivalent:
1.

$$\mathrm{Re}\left[\Phi(j\omega)\left(1 - q'G(j\omega)\right)\right] > 0 \quad \forall \omega \geq 0 \quad \forall q : \|q\|_2 \leq \rho; \tag{19}$$

2.

$$\begin{array}{ll} (a) & \mathrm{Re}[\Phi(j\omega)] > 0 \\ (b) & \|R(\omega) - \gamma_\Phi(\omega) I(\omega)\|_2^2 < \dfrac{1}{\rho^2} \end{array} \quad \forall \omega \geq 0 \tag{20}$$

where

$$\gamma_\Phi(\omega) := \frac{\mathrm{Im}[\Phi(j\omega)]}{\mathrm{Re}[\Phi(j\omega)]}. \tag{21}$$

Proof: see Appendix.

Note that conditions (17) and (20a) imply that $\Phi(s)$ must be a strictly positive real rational function. Hence, the $RSPR$ problem amounts to determine a strictly positive real $\Phi(s)$ such that the inequality

$$\|R(\omega) - \gamma(\omega) I(\omega)\|_2^2 < \frac{1}{\rho^2} \tag{22}$$

is satisfied for $\gamma(\omega) = \gamma_\Phi(\omega)$ for all $\omega \geq 0$.

Therefore, a central issue for the solution of the $RSPR$ problem is the characterization of the following set of functions

$$\Gamma := \{\gamma(\omega) \ : \ \gamma(\omega) \text{ is bounded continuous and satisfies (22)}\}.$$

The function

$$\gamma^*(\omega) = \frac{R'(\omega) I(\omega)}{\|I(\omega)\|_2^2} \tag{23}$$

defined for $\omega \in \bar{\Omega}_0$ plays a key role in such a characterization.

Lemma 4: Let ρ^* be the parametric stability margin of $\mathcal{P}_\rho$ (see Lemma 2) and suppose $\rho < \rho^*$. Then, the following statements hold.

1. Γ is the set of bounded continuous functions $\gamma(\omega)$ such that

$$\underline{\gamma}(\omega) < \gamma(\omega) < \overline{\gamma}(\omega) \quad \forall \omega \in \bar{\Omega}_0 \tag{24}$$

where

$$\begin{aligned} \underline{\gamma}(\omega) &= \min \left\{ \gamma^*(\omega) \pm \frac{\sqrt{\Delta(\omega)}}{\|I(\omega)\|_2^2} \right\} \\ \overline{\gamma}(\omega) &= \max \left\{ \gamma^*(\omega) \pm \frac{\sqrt{\Delta(\omega)}}{\|I(\omega)\|_2^2} \right\} \end{aligned} \tag{25}$$

being $\gamma^*(\omega)$ as in (23) and

$$\Delta(\omega) = [R'(\omega) I(\omega)]^2 - \|I(\omega)\|_2^2 \left[\|R(\omega)\|_2^2 - \frac{1}{\rho^2} \right]. \tag{26}$$

2. Γ is nonempty.

Proof: see Appendix.

The above Lemma makes it clear how it is possible to solve the $RSPR$ problem. Indeed, it is sufficient to find a strictly positive real transfer function $\Phi(s)$ such that $\gamma_\Phi(\omega)$ belongs to the set Γ. Since $\gamma_\Phi(\omega) = \frac{\mathrm{Im}[\Phi(j\omega)]}{\mathrm{Re}[\Phi(j\omega)]}$ is bounded continuous when $\Phi(s)$ is strictly positive real, it is enough to satisfy the relation

$$\underline{\gamma}(\omega) < \gamma_\Phi(\omega) < \overline{\gamma}(\omega) \quad \forall \omega \in \bar{\Omega}_0.$$

Consider Fig. 1(a), where the functions $\underline{\gamma}(\omega)$ (solid lower line) and $\overline{\gamma}(\omega)$ (solid upper line) are depicted for a given $\rho = \rho_1$. In this case, it is easily verified that the function

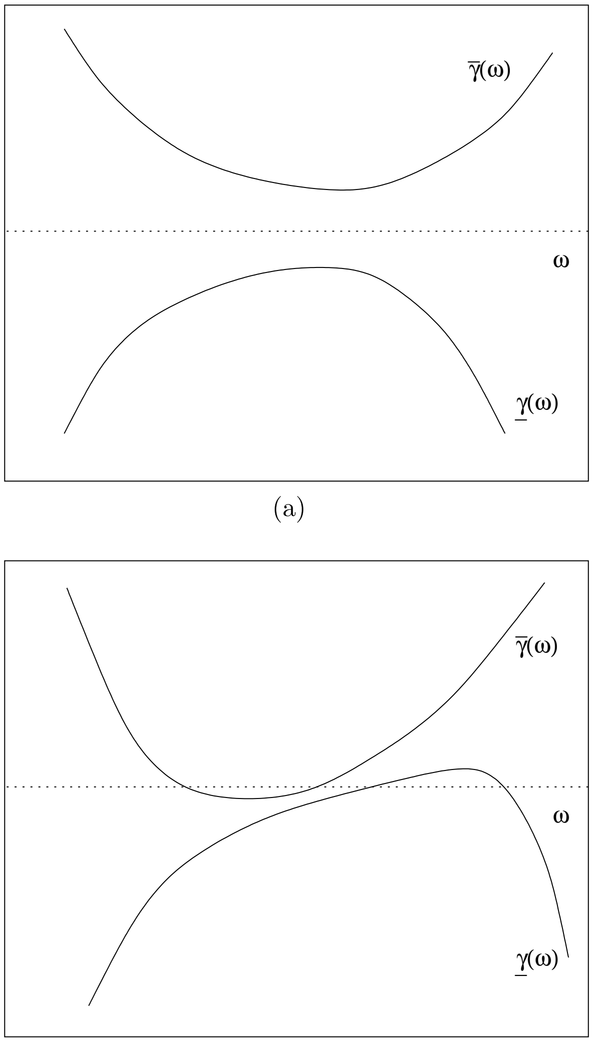
$$\Phi(s) = 1$$

solves the $RSPR$ problem, since $\gamma_\Phi(\omega) = 0$ is between $\underline{\gamma}(\omega)$ and $\overline{\gamma}(\omega)$. Such a solution leads to the filter

$$F(s) = P_0(s)$$

that is the nominal polynomial itself.

It is clear that such a filter is likely to perform well for small uncertainty, i.e., for values of ρ sufficiently smaller than ρ^*. Indeed, this is the usual way for designing $F(s)$ in several application contexts (see [6],[7]). For larger values of ρ, this is no longer guaranteed as shown in Fig. 1(b), where $\rho = \rho_2 > \rho_1$ is considered. In this case, the band is narrower and a different solution must be found.



(b)

Fig. 1. (a): $\underline{\gamma}(\omega)$ and $\overline{\gamma}(\omega)$ for $\rho = \rho_1$; (b): $\underline{\gamma}(\omega)$ and $\overline{\gamma}(\omega)$ for $\rho = \rho_2$ ($\rho_1 < \rho_2 < \rho^*$).

Notice that the following relation holds (see (25))

$$\gamma^*(\omega) = \frac{\underline{\gamma}(\omega) + \overline{\gamma}(\omega)}{2}$$

i.e., the function $\gamma^*(\omega)$ is at each ω the middle point of the band defined by $\underline{\gamma}(\omega)$ and $\overline{\gamma}(\omega)$ for whatever value of ρ less than ρ^*. This observation suggests to look for a strictly positive real rational function $\Phi(s)$ such that $\gamma_\Phi(\omega)$ is as close as possible to $\gamma^*(\omega)$. Since $\gamma^*(\omega)$ does not depend on ρ, such an approach is likely to provide a solution of the $RSPR$ problem for ρ arbitrarily close to ρ^*.

To proceed, we derive some properties of $\gamma^*(\omega)$.

The next Lemma relates the function $\gamma^*(\omega)$ to the polynomial

$$\Pi(s)=\sum_{i=1}^{n}P_0(s)P_i(-s)\left[P_0(-s)P_i(s)\right]_o. \qquad (27)$$

Lemma 5: The following properties hold for the polynomial $\Pi(s)$:
1.

$$\Pi(j\omega)=\left[P_0(j\omega)P_0(-j\omega)\right]^2\left[I'(\omega)I(\omega)+jR'(\omega)I(\omega)\right]; \qquad (28)$$

2.

$$\begin{array}{ll}\mathrm{Re}\left[\Pi(j\omega)\right]\geq 0 & \forall\omega\geq 0\\ \mathrm{Re}\left[\Pi(j\omega)\right]>0 & \forall\omega\in\bar{\Omega}_0\end{array}; \qquad (29)$$

3.

$$\gamma^*(\omega)=\frac{\mathrm{Im}\left[\Pi(j\omega)\right]}{\mathrm{Re}\left[\Pi(j\omega)\right]}.$$

Proof: see Appendix.

The following Lemma states the existence of a transfer function $\Phi^*(s)$ such that

$$\gamma_{\Phi^*}(\omega)=\frac{\mathrm{Im}[\Phi^*(j\omega)]}{\mathrm{Re}[\Phi^*(j\omega)]}=\gamma^*(\omega).$$

Lemma 6: Let $\Pi_1(s)$ and $\Pi_2(s)$ be any two polynomials such that

$$\Pi_1(s)\Pi_2(-s)=\Pi(s) \qquad (30)$$

with $\Pi(s)$ as in (27), and define

$$\Phi^*(s)=\frac{\Pi_1(s)}{\Pi_2(s)}. \qquad (31)$$

Then,

$$\left\{\begin{array}{l}\gamma_{\Phi^*}(\omega)=\gamma^*(\omega)\\ \mathrm{Re}[\Phi^*(j\omega)]>0\end{array}\right. \quad \forall\omega\in\bar{\Omega}_0. \qquad (32)$$

Proof: see Appendix.

Lemma 6 suggests the following idea for providing a solution to the $RSPR$ problem: determine a positive real rational function $\Phi^*(s)$ of the form (31) and perform a small perturbation of its coefficients in order to obtain an SPR transfer function (see Lemma 1).
For ease of illustration, we first develop the case $\Omega_0=\{0\}$, which indeed represents the generic situation (see the discussion after (11)). The general case, which requires the same basic steps but some additional technicalities, will be dealt with later.

The following property is a straightforward consequence of the fact that $\Pi(s)$ is zero on the imaginary axis only for $s=0$ (see (28)).

Lemma 7: Suppose $\Omega_0=\{0\}$. Then, $\Pi(s)$ can be factorized as follows:

$$\Pi(s)=As^r\bar{\Pi}_1(s)\bar{\Pi}_2(-s) \qquad (33)$$

where A is a real constant, $r\geq 1$ is an integer and $\bar{\Pi}_1(s)$ and $\bar{\Pi}_2(s)$ are uniquely determined monic Hurwitz polynomials. Moreover, $\bar{\Pi}_1(s)$ contains $P_0(s)$ as a factor.

Let us introduce the functions

$$\Phi_e^*(s)=\frac{\bar{\Pi}_1(s)}{\bar{\Pi}_2(s)} \qquad (34)$$

defined for even r, and

$$\Phi_o^*(s)=\frac{\bar{\Pi}_1(s)}{\bar{\Pi}_2(s)}\; s^{\mathrm{sgn}A\ (-1)^{(r-1)/2}} \qquad (35)$$

defined for odd r.
We are now ready to give the main result which relies on the fact that $\Phi_e^*(s)$ and $\Phi_o^*(s)$ turn out to be positive real.

Theorem 2: Given the set $\mathcal{P}_\rho$, let ρ^* be the parametric stability margin of $\mathcal{P}_\rho$ and suppose the following conditions hold:
1. $\rho<\rho^*$;
2. $\Omega_0=\{0\}$.
Let $\Phi_e^*(s)$ and $\Phi_o^*(s)$ be defined as in (34) and (35). Then, for sufficiently small positive ε and δ, the rational function

$$\Phi(s)=\left\{\begin{array}{ll}\Phi_e^*(s)(1+\delta s)^{\partial\bar{\Pi}_2-\partial\bar{\Pi}_1} & \text{for even } r\\ \Phi_o^*(s)\left(\dfrac{s+\varepsilon}{s}\right)^{\mathrm{sgn}A\ (-1)^{(r-1)/2}}\cdot & \text{for odd } r\\ \cdot(1+\delta s)^{\partial\bar{\Pi}_2-\partial\bar{\Pi}_1-\mathrm{sgn}A\ (-1)^{(r-1)/2}} & \end{array}\right. \qquad (36)$$

satisfies (17) and (18) for all $\omega\geq 0$, i.e. the filter

$$F(s)=\frac{P_0(s)}{\Phi(s)} \qquad (37)$$

solves the robust SPR problem for $\mathcal{P}_\rho$.

Proof: First, it can be easily verified that $\Phi(s)$ in (36) satisfies (17) by construction.
Lemma 3 states that condition (18) is equivalent to condition (20)-(21). Thus, we have to prove that $\Phi(s)$ is strictly positive real and such that the inequality

$$\|R(\omega)-\gamma(\omega)I(\omega)\|_2^2<\frac{1}{\rho^2} \qquad (38)$$

holds for $\gamma(\omega)=\gamma_\Phi(\omega)$ for all $\omega\geq 0$.
Suppose r is even. As $\bar{\Pi}_1(s)$ and $\bar{\Pi}_2(s)$ are monic Hurwitz polynomials, and (29) holds for sufficiently small non-zero ω, it turns out that $Aj^r>0$. Hence, from (33) it follows that $\Pi(s)$ can be rewritten as

$$\begin{array}{c}\Pi(s)=|A|\ s^{r/2}\ (-s)^{r/2}\ \bar{\Pi}_1(s)\bar{\Pi}_2(-s)=\\ =\left[|A|^{1/2}\ s^{r/2}\ \bar{\Pi}_1(s)\right]\left[|A|^{1/2}\ (-s)^{r/2}\ \bar{\Pi}_2(-s)\right].\end{array}$$

Then, by Lemma 6, the rational function $\Phi_e^*(s)$ satisfies

$$\left\{\begin{array}{l}\gamma_{\Phi_e^*}(\omega)=\gamma^*(\omega)\\ \mathrm{Re}[\Phi_e^*(j\omega)]>0\end{array}\right. \quad \forall\omega>0. \qquad (39)$$

Furthermore, since $\bar{\Pi}_1(s)$ and $\bar{\Pi}_2(s)$ are monic Hurwitz polynomials, it follows that $\mathrm{Re}[\Phi_e^*(0)]>0$, and therefore we have

$$\mathrm{Re}[\Phi_e^*(j\omega)]>0 \quad \forall\omega\geq 0. \qquad (40)$$

Since $\bar{\Pi}_2(s)$ is Hurwitz, we conclude that $\Phi_e^*(s)$ is a positive real rational function.
Now, consider the function

$$\Phi(s) = \Phi_e^*(s)(1+\delta s)^{\partial\bar{\Pi}_2 - \partial\bar{\Pi}_1}.$$

According to Definition 2, by (40) $\Phi(s)$ is strictly positive real for sufficiently small positive δ.
It remains to show that, if $\rho < \rho^*$, $\Phi(s)$ satisfies (38) for suitable δ. Now, exploiting Lemmas 4, 5, and 6 and the fact that $\gamma_{\Phi_e^*}(0) = 0$, it turns out that $\Phi_e^*(s)$ satisfies (38) for $\gamma(\omega) = \gamma_{\Phi_e^*}(\omega)$ for any $\rho < \rho^*$ and any $\omega \geq 0$. Moreover, since $\gamma_\Phi(\omega)$ is continuous with respect to δ, it turns out that the left term of inequality (38) for $\gamma(\omega) = \gamma_\Phi(\omega)$ depends continuously on δ. Hence, observing that $\gamma_\Phi(0) = 0$, we can conclude that for sufficiently small positive δ, condition (38) is also satisfied by $\gamma_\Phi(\omega)$ for all $\omega \geq 0$.
Now suppose r is odd. Again from (33) and taking (29) into account, $\Pi(s)$ can be expressed as

$$\begin{aligned}\Pi(s) &= A\ s\ s^{r-1}\ (-1)^{(r-1)/2}\ (-1)^{(r-1)/2}\ \bar{\Pi}_1(s)\bar{\Pi}_2(-s) = \\ &= A\ s\ (-1)^{(r-1)/2}\ s^{(r-1)/2}\ (-s)^{(r-1)/2}\ \bar{\Pi}_1(s)\bar{\Pi}_2(-s) = \\ &= s\ \mathrm{sgn}\ A\ (-1)^{(r-1)/2}\ \left[|A|^{1/2}\ s^{(r-1)/2}\ \bar{\Pi}_1(s)\right]\cdot \\ &\quad \cdot\left[|A|^{1/2}\ (-s)^{(r-1)/2}\bar{\Pi}_2(-s)\right].\end{aligned}$$

By Lemma 6, $\Phi_o^*(s)$ satisfies

$$\begin{cases} \gamma_{\Phi_o^*}(\omega) = \gamma^*(\omega) \\ \mathrm{Re}[\Phi_o^*(j\omega)] > 0 \end{cases} \quad \forall\omega > 0. \tag{41}$$

Obviously, $\Phi_o^*(s)$ is analytic for $\mathrm{Re}[s] > 0$.
In order to prove that $\Phi_o^*(s)$ is positive real, it suffices to show that both $\Phi_o^*(s)$ and its inverse $\Phi_o^{*-1}(s)$ have real positive residues in $s = 0$, when $s = 0$ is actually a (simple) pole of either transfer function.

- If $\mathrm{sgn}\ A\ (-1)^{(r-1)/2} = -1$ we have

$$\mathrm{Res}[\Phi_o^*(s), 0] = \frac{\bar{\Pi}_1(0)}{\bar{\Pi}_2(0)} > 0$$

since $\bar{\Pi}_1(s)$ and $\bar{\Pi}_2(s)$ are monic and Hurwitz;

- If $\mathrm{sgn}\ A\ (-1)^{(r-1)/2} = 1$

$$\mathrm{Res}[\Phi_o^{*-1}(s), 0] = \frac{\bar{\Pi}_2(0)}{\bar{\Pi}_1(0)} > 0.$$

Hence, $\Phi_o^*(s)$ is positive real. Introducing the rational function

$$\begin{aligned}\Phi(s) = \Phi_o^*(s)\left(\frac{s+\varepsilon}{s}\right)^{\mathrm{sgn}A\ (-1)^{(r-1)/2}}\cdot \\ \cdot(1+\delta s)^{\partial\bar{\Pi}_2 - \partial\bar{\Pi}_1 - \mathrm{sgn}A\ (-1)^{(r-1)/2}}\end{aligned}$$

by the positive real character of $\Phi_o^*(s)$, taking into account (41) and the fact that $\mathrm{Re}[\Phi(0)] > 0$, it turns out that $\Phi(s)$ is strictly positive real for sufficiently small positive ε, δ.
Now, Lemmas 4, 5, and 6 ensure that $\gamma(\omega) = \gamma_{\Phi_o^*}(\omega)$ satisfies condition (38) for any $\rho < \rho^*$ and $\omega > 0$. Moreover, since $\gamma_\Phi(\omega)$ is continuous with respect to δ and ϵ, it turns out that the left term of inequality (38) for $\gamma(\omega) = \gamma_\Phi(\omega)$ depends continuously on δ and ϵ. Hence, observing that $\gamma_\Phi(0) = 0$, it follows that for sufficiently small positive ε and δ, (38) holds for $\gamma(\omega) = \gamma_\Phi(\omega)$ for all $\omega \geq 0$. ∎

Remark 3: The parameters ε and δ in the expression (36) of the solution $\Phi(s)$ are introduced in order to obtain a strictly positive real rational function such that $\gamma_\Phi(\omega)$ belongs to the set Γ. Indeed, in the limiting case $\varepsilon = 0$ and $\delta = 0$, $\Phi(s)$ reduces to $\Phi_e^*(s)$ for even r and $\Phi_o^*(s)$ for odd r. These two functions are in general guaranteed to be positive real only. On the other hand, as ρ approaches ρ^*, the band defined by $\underline{\gamma}(\omega)$ and $\overline{\gamma}(\omega)$ becomes narrower as depicted in Fig. 1, and therefore $\gamma_\Phi(\omega)$ has to be chosen sufficiently close to $\gamma^*(\omega)$. Since $\gamma^*(\omega) = \gamma_{\Phi_e^*}(\omega)$ for even r and $\gamma^*(\omega) = \gamma_{\Phi_o^*}(\omega)$ for odd r, it turns out that the closer ρ is to ρ^*, the smaller ε and δ have to be chosen. Moreover, some general guidelines for the selection of ε and δ can be derived from the frequency properties of $\Phi_e^*(s)$ and $\Phi_o^*(s)$. For example, since ε defines a low frequency pole or zero of $\Phi(s)$, it should be chosen at least one decade smaller than all the singularities of $\Phi_o^*(s)$. Similarly, δ should be chosen such that $1/\delta$ is at least one decade larger than all the singularities of $\Phi_e^*(s)$ and $\Phi_o^*(s)$ (see also Example 1).

Remark 4: Since $\Phi_e^*(s)$ and $\Phi_o^*(s)$ are positive real, Remark 1 implies that

$$\begin{array}{ll} -1 \leq \partial\bar{\Pi}_2 - \partial\bar{\Pi}_1 \leq 1 & r \text{ even} \\ -1 \leq \partial\bar{\Pi}_2 - \partial\bar{\Pi}_1 - \mathrm{sgn}A\ (-1)^{(r-1)/2} \leq 1 & r \text{ odd.} \end{array}$$

Exploiting the factorization in Lemma 7, we can determine an upper bound on the degree of the solution filter $F(s)$ in (37), which involves the degree l of the set $\mathcal{P}_\rho$. Let

$$F(s) = \frac{N_F(s)}{D_F(s)}, \tag{42}$$

the following result holds.

Corollary 1: Let the assumptions in Theorem 2 be fulfilled. Then,

$$\begin{array}{ll} \partial D_F \leq l-2 & \text{for even } r \\ \partial D_F \leq l-1 & \text{for odd } r\ . \end{array} \tag{43}$$

Proof: Let

$$\mu = \partial\bar{\Pi}_2 - \partial\bar{\Pi}_1 \tag{44}$$

$$\sigma = \begin{cases} \mathrm{sgn}A\ (-1)^{(r-1)/2} & \text{for odd } r \\ 0 & \text{for even } r \end{cases} \tag{45}$$

$$e = \mu - \sigma. \tag{46}$$

Note that e is the relative degree of either $\Phi_e^*(s)$ or $\Phi_o^*(s)$ (see (34), (35), and (36)). Since these functions are positive real, we have

$$e \in \{-1, 0, 1\}.$$

By (33) and the assumptions in Definition 3, we get

$$\partial\Pi = r + \partial\bar{\Pi}_1 + \partial\bar{\Pi}_2 \leq 2(2l-1)$$

and hence from (44)

$$\partial\bar{\Pi}_1 \leq 2l - 1 - \frac{r+\mu}{2}. \tag{47}$$

From (37) and (42), it follows that

$$\frac{N_F(s)}{D_F(s)} = \frac{P_0(s)\bar{\Pi}_2(s)}{\bar{\Pi}_1(s)(s+\varepsilon)^\sigma(1+\delta s)^{\mu-\sigma}}.$$

Since from Lemma 7 we know that $\bar{\Pi}_1(s)$ contains $P_0(s)$ as a factor, assuming the worst case $\sigma \geq 0$, $\mu - \sigma = e \geq 0$, we get

$$\partial D_F \leq \partial \bar{\Pi}_1 - l + \mu.$$

Taking into account (47), we have

$$\partial D_F \leq l - 1 - \frac{r-\mu}{2} = l - 1 - \frac{r-e-\sigma}{2}$$

where the equality follows from (46). Finally, note that if r is odd we have $r \geq 1$, otherwise $r \geq 2$. By substituting the minimum value of r and the maximum values of e and σ in either case, we obtain (43). ■

Remark 5: The above approach to the solution of the $RSPR$ problem provides an upper bound for the degree of the filter $F(s)$ that was lacking in [4] (see Theorem 1).

Now, we move to the general case in which the set Ω_0 contains other frequencies in addition to $\omega = 0$, i.e. it has the general form (11). The following result parallels Lemma 7.

Lemma 8: Suppose $\Omega_0 = \{0, \omega_1, \ldots, \omega_k\}$. Then, the polynomial $\Pi(s)$ in (27) is factorizable as

$$\Pi(s) = A s^{r_0} \prod_{i=1}^{k} \left(s^2 + \omega_i^2\right)^{r_i} \bar{\Pi}_1(s) \bar{\Pi}_2(-s), \tag{48}$$

where A is a real number, r_i are suitable non-negative integers, $\bar{\Pi}_1(s)$ and $\bar{\Pi}_2(s)$ are uniquely determined monic Hurwitz polynomials. Moreover, $\bar{\Pi}_1(s)$ contains $P_0(s)$ as a factor.

Let

$$\tilde{\Pi}_i(s) = \frac{\Pi(s)}{(s^2 + \omega_i^2)^{r_i}}, \quad i = 1 \ldots k, \tag{49}$$

and introduce the rational function

$$\Phi^*(s) = \frac{\bar{\Pi}_1(s)}{\bar{\Pi}_2(s)} s^{N_0} \prod_{i=1}^{k} (s^2 + \omega_i^2)^{N_i}, \tag{50}$$

where $\bar{\Pi}_1(s)$ and $\bar{\Pi}_2(s)$ are as in Lemma 8,

$$N_0 = \begin{cases} 0 & \text{if } r_0 \text{ is even} \\ \text{sgn}A \ (-1)^{(r_0-1)/2} & \text{if } r_0 \text{ is odd} \end{cases} \tag{51}$$

$$N_i = \begin{cases} 0 & \text{if } r_i \text{ is even} \\ -1 & \text{if } r_i \text{ is odd and Im}\left[\tilde{\Pi}_i(j\omega_i)\right] > 0 \\ 1 & \text{if } r_i \text{ is odd and Im}\left[\tilde{\Pi}_i(j\omega_i)\right] < 0 \end{cases}. \tag{52}$$

We have the following general result based on the fact that the function $\Phi^*(s)$ in (50) is shown to be positive real.

Theorem 3: Given the set $\mathcal{P}_\rho$, let ρ^* be the parametric stability margin of $\mathcal{P}_\rho$ and suppose the following conditions hold

1. $\rho < \rho^*$;
2. $\Omega_0 = \{0, \omega_1, \ldots, \omega_k\}$.

Let $\Phi^*(s)$, N_0, N_i be as in (50), (51), (52), respectively. Then, for sufficiently small positive ε and δ, the rational function

$$\Phi(s) = \Phi^*(s+\varepsilon)(1+\delta s)^{\partial \bar{\Pi}_2 - \partial \bar{\Pi}_1 - N_0 - 2\sum_{i=1}^{k} N_i} \tag{53}$$

satisfies (17) and (18) all $\omega \geq 0$, i.e. the filter

$$F(s) = \frac{P_0(s)}{\Phi(s)} \tag{54}$$

solves the robust SPR problem for $\mathcal{P}_\rho$.

Proof: see Appendix.

A result concerning the degree of the filter $F(s)$ can be given also for this general case.

Corollary 2: Let the assumptions in Theorem 3 be fulfilled. Then,

$$\partial D_F \leq 2l - 1. \tag{55}$$

Proof: see Appendix.

Remark 6: The upper bound in this case is larger than in the case $\Omega_0 = \{0\}$ (see Corollary 1). The increase in the upper bound is due to the fact that the numerator of $\Phi^*(s+\varepsilon)$ contains $\bar{\Pi}_1(s+\varepsilon)$ in place of $\bar{\Pi}_1(s)$ and therefore $P_0(s)$ cannot be canceled in (54), as it was done in Corollary 1.

Under a very mild additional assumption on $\Phi^*(s)$ in (50), a simplified form for the solution $\Phi(s)$ can be given.

Theorem 4: Given the set $\mathcal{P}_\rho$, let ρ^* be its parametric stability margin and suppose the following conditions hold:

1. $\rho < \rho^*$;
2. $\Omega_0 = \{0, \omega_1, \ldots, \omega_k\}$;
3. There exists no $i \in \{1, \ldots, k\}$ such that r_i is even and Re $\left[\tilde{\Pi}_i(j\omega_i)\right] = 0$.

where $\tilde{\Pi}_i(s)$ is defined as in (49).

Let $\Phi^*(s)$, N_0, N_i be as in (50), (51), (52), respectively. Then, the robust SPR problem is solved by the filter

$$F(s) = \frac{P_0(s)}{\Phi(s)}$$

where $\Phi(s)$ is the function

$$\Phi(s) = \Phi^*(s) \left(\frac{s+\varepsilon}{s}\right)^{N_0} \prod_{i=1}^{k} \left(\frac{s^2 + 2\zeta_i \omega_i s + \omega_i^2}{s^2 + \omega_i^2}\right)^{N_i} \cdot (1+\delta s)^{\partial \bar{\Pi}_2 - \partial \bar{\Pi}_1 - N_0 - 2\sum_{i=1}^{k} N_i} \tag{56}$$

for sufficiently small non-negative ε, δ and ζ_i, $i = 1, \ldots, k$.

Proof: see Appendix.

Note that in this case we can recover the stronger condition of Corollary 1 concerning the degree of D_F, since according to (56) and (50), $P_0(s)$ is a factor of the numerator of $\Phi(s)$ (recall from Lemma 8 that $P_0(s)$ is a factor of $\bar{\Pi}_1(s)$). Indeed, we have the following result.

Corollary 3: Let the assumptions in Theorem 4 be satisfied. Then,

$$\partial D_F \leq l - 1.$$

IV. Application examples

In this section we develop some numerical examples to illustrate the features of the results in Section 3. One specific goal is to show that the filter $F(s) = P_0(s)$, that is quite often used in several application contexts (see [6],[7]), is not an appropriate choice especially when ρ is close to ρ^*.

Example 1: Let

$$\mathcal{P}_\rho = \left\{P(s) = (s+1)^3 + q_1 s + q_2 \quad : \quad \|q\|_2 \leq \rho\right\}.$$

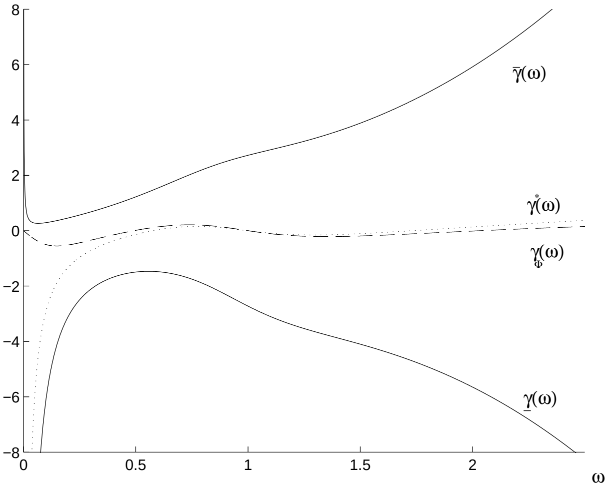


Fig. 2. Example 1: $\underline{\gamma}(\omega)$ and $\overline{\gamma}(\omega)$ (solid), $\gamma^*(\omega)$ (dotted), $\gamma_\Phi(\omega)$ (dashed).

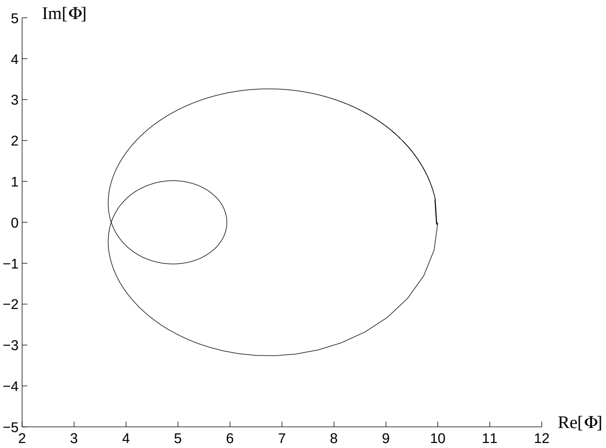


Fig. 3. Example 1: Nyquist plot of $\Phi(s)$.

The function $G(s)$ is given by

$$G(s) = \left[-\frac{s}{(s+1)^3} \quad -\frac{1}{(s+1)^3} \right]'$$

and, moreover,

$$\Omega_0 = \{0\}.$$

According to Lemma 2, the l_2 stability margin of $\mathcal{P}_\rho$ is given by $\rho^* = \rho_0 = 1$. The associated $\Pi(s)$ can be factorized as in Lemma 7 yielding

$$\Pi(s) = 3(-s)(s+1)^4(s^2 - 2/3s + 1)$$

and, therefore,

$$A = 3; \quad r = 1; \quad \overline{\Pi}_1(s) = (s+1)^4; \quad \overline{\Pi}_2(s) = (s^2 - 2/3s + 1).$$

From the application of Theorem 2, we get that, for sufficiently small positive ε and δ, the rational function

$$\Phi(s) = \frac{(s+1)^4}{(s+\varepsilon)(s^2+2/3s+1)(1+\delta s)}$$

solves the $RSPR$ problem for $\rho < 1$. The corresponding filter $F(s)$ is

$$F(s) = \frac{(s+\varepsilon)(s^2+2/3s+1)(1+\delta s)}{s+1}.$$

Note that the transfer function

$$\Phi_o^*(s) = \frac{(s+1)^4}{s(s^2+2/3s+1)}$$

is positive real, but not strictly positive real.

Concerning the selection of ε and δ, we can proceed as discussed in Remark 3. Since all the singularities of $\Phi_o^*(s)$ are at $\omega = 1$, we can select $\varepsilon = 0.1$ and $\delta = 0.1$. This choice allows for the solution of the $RSPR$ problem for ρ very close to one. As an example, Fig. 2 shows the functions $\underline{\gamma}(\omega)$ and $\overline{\gamma}(\omega)$ (solid line) calculated for $\rho = 0.97$ along with $\gamma^*(\omega)$ (dotted line) and $\gamma_\Phi(\omega)$ (dashed line). In Fig. 3 the Nyquist plot of such $\Phi(s)$ is depicted, thus showing its SPR.

Note that in this case the filter $F(s) = P_0(s) = (s+1)^3$ is a solution of the $RSPR$ problem, since $\gamma(\omega) = 0$ lies within the band defined by $\overline{\gamma}(\omega)$ and $\underline{\gamma}(\omega)$ for all $\omega \geq 0$.

Example 2: Let

$$\mathcal{P}_\rho = \left\{ P(s) = (s+1)^3 + q_1 s^2 + q_2 s \quad : \quad \|q\|_2 \leq \rho \right\}.$$

We get

$$\rho^* = \bar{\rho} = \sqrt{7}$$
$$\Omega_0 = \{0\}$$

and, according to Lemma 7,

$$\Pi(s) = -s^2(s+1)^4(s^2 - 0.78s + 3.54)(s^2 - 0.22s + 0.28).$$

The application of Theorem 2 leads to the rational function

$$\Phi(s) = \frac{(s+1)^4}{(s^2+0.78s+3.54)(s^2+0.22s+0.28)},$$

which solves the $RSPR$ problem for $\rho < \sqrt{7}$. The corresponding filter $F(s)$ is given by

$$F(s) = \frac{(s^2+0.78s+3.54)(s^2+0.22s+0.28)}{s+1}.$$

Note that in this case $\Phi(s) = \Phi_e^*(s)$, i.e. $\Phi_e^*(s)$ turns out to be strictly positive real.

The function $\gamma_\Phi(\omega) = \gamma^*(\omega)$ for $\rho = 2.63$ is shown in Fig. 4. Note that the filter $F(s) = P_0(s)$ does not solve the $RSPR$ problem.

Example 3: Consider

$$\mathcal{P}_\rho = \left\{ P(s) = (s+1)^3 + q_1 s^2 + q_2 \quad : \quad \|q\|_2 \leq \rho \right\}.$$

We have

$$\Omega_0 = \{0, \sqrt{3}\}$$
$$\rho^* = \rho_0 = 1$$

and, according to Lemma 8,

$$\Pi(s) = (-s)(s^2+3)(s+1)^3(s^2+\sqrt{2}s+1)(s^2-\sqrt{2}s+1)$$

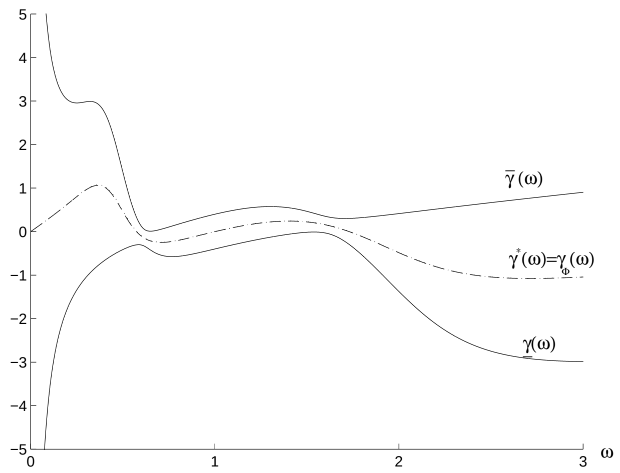


Fig. 4. Example 2: $\underline{\gamma}(\omega)$ and $\overline{\gamma}(\omega)$ (solid), $\gamma^*(\omega)$ (dotted), $\gamma_\Phi(\omega)$ (dashed).

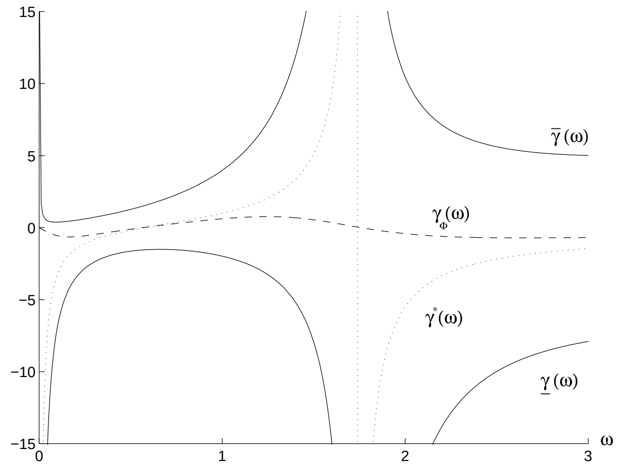


Fig. 5. Example 3: $\underline{\gamma}(\omega)$ and $\overline{\gamma}(\omega)$ (solid), $\gamma^*(\omega)$ (dotted), $\gamma_\Phi(\omega)$ (dashed).

Note that the two imaginary roots of $\Pi(s)$ are simple. Hence, the solution of the $RSPR$ problem can be obtained by applying Theorem 4. For sufficiently small positive ε and ζ, the rational function

$$\Phi(s) = \frac{(s+1)^3}{(s+\varepsilon)(s^2 + 2\sqrt{3}\zeta s + 3)}$$

solves the $RSPR$ problem for $\rho < 1$ and the corresponding filter $F(s)$ turns out to be a polynomial

$$F(s) = (s+\varepsilon)(s^2 + 2\sqrt{3}\zeta s + 3).$$

The diagram of Figure 5 is obtained for $\rho = 0.97$, $\varepsilon = 0.1$, and $\zeta = 0.2$. Note that $\underline{\gamma}(\omega)$, $\overline{\gamma}(\omega)$, and $\gamma^*(\omega)$ are unbounded for $\omega = \sqrt{3}$, while the nominal filter $F(s) = P_0(s)$ is a valid solution.

Example 4: Let

$$\mathcal{P}_\rho = \big\{ P(s) = s^4 + 3s^3 + 5.5s^2 + 4.5s + 5.5+ \\ +q_1(s^2+s+3) + q_2(s^3+s-1) \ : \ \|q\|_2 \le \rho \big\}\,.$$

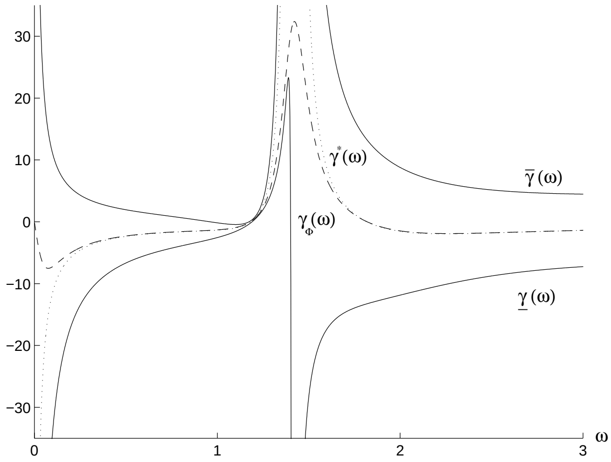


Fig. 6. Example 4: $\underline{\gamma}(\omega)$ and $\overline{\gamma}(\omega)$ (solid), $\gamma^*(\omega)$ (dotted), $\gamma_\Phi(\omega)$ (dashed).

We have

$$\rho^* = \rho_0 = 1.0607$$
$$\Omega_0 = \{0, \sqrt{2}\}$$

and, according to Lemma 8,

$$\begin{aligned}\Pi(s) &= (-s)(s^2+2)^2(s^4+3s^3+5.5s^2+4.5s+5.5)\cdot \\ &\cdot(s^5+3.5s^3+3s^2+0.5s+8.5) = \\ &= (-s)(s^2+2)^2(s^4+3s^3+5.5s^2+4.5s+5.5)\cdot \\ &\cdot(s+1.3569)(s^2-0.1306s+3.2591)(s^2-1.2263s+1.9220).\end{aligned}$$

In this case assumption 3 of Theorem 4 does not hold, since the roots at $s = \pm j\sqrt{2}$ are double and $\mathrm{Re}[\Pi(s)/(s+2)^2]\big|_{s=j\sqrt{2}} = 0$.
Thus, we have to apply Theorem 3. First, according to (50), we compute the positive real rational function

$$\Phi^*(s) = \frac{(s^4+3s^3+5.5s^2+4.5s+5.5)(s+1.3569)}{s(s^2+0.1306s+3.2591)(s^2+1.2263s+1.9220)}$$

according to (50). Then, for sufficiently small ε, the rational function

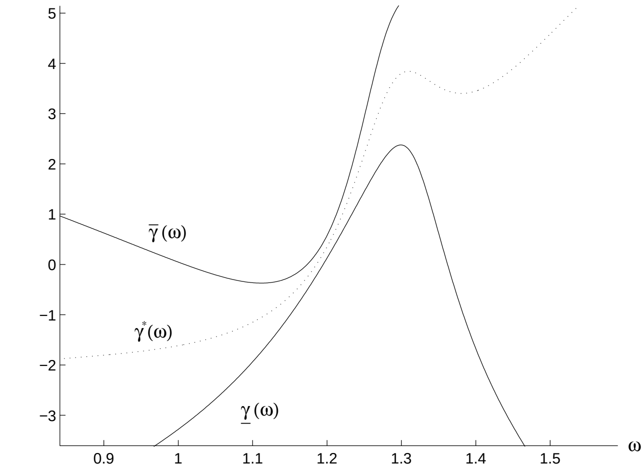
$$\Phi(s) = \Phi^*(s+\varepsilon)$$

solves the $RSPR$ problem for $\rho < 1.0607$.
The plot in Figure 6 is calculated for $\rho = 1$ and $\varepsilon = 0.005$. Note that in this case, $F(s) = P_0(s)$ is not a solution of the $RSPR$ problem.

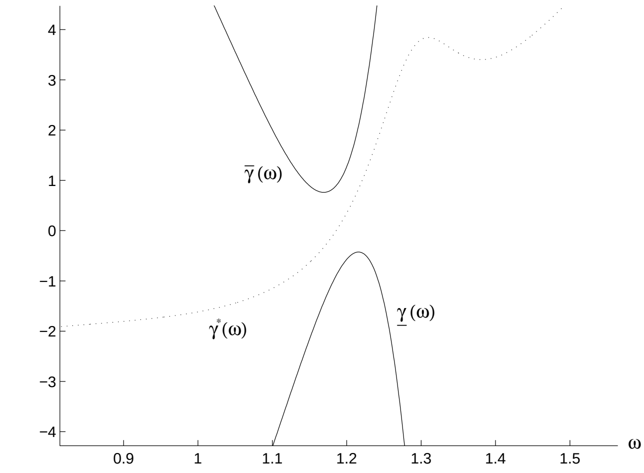
Example 5: In this example we show that the filter $F(s) = P_0(s)$ is not in general a solution of the $RSPR$ problem, especially for values of ρ close to ρ^*. This has been already pointed out in Example 4, where however the considered problem led to a peculiar form of $\Pi(s)$ that forced to use Theorem 3. Indeed, consider the set

$$\mathcal{P}_\rho = \big\{ P(s) = s^4 + 3s^3 + 5.5s^2 + 4.5s + 5.5+ \\ +q_1(s^2+s+3) + q_2(s^3+s-0.5) \ : \ \|q\|_2 \le \rho \big\}\,.$$

which is a slight modification of the one of the previous example (only $P_2(0)$ has been changed).



(a)



(b)

Fig. 7. Example 5: $\underline{\gamma}(\omega)$ and $\overline{\gamma}(\omega)$ (solid), $\gamma^*(\omega)$ (dotted), $\gamma_\Phi(\omega)$ (dashed); (a) $\rho = 0.89$; (b) $\rho = 0.3$.

In this case, we have $\Omega_0 = \{0\}$ and $\rho^* = \bar{\rho} \approx 0.99$ and we can apply Theorem 2. We get

$$\Phi^*(s) =$$
$$= \frac{(s^4 + 3s^3 + 5.5s^2 + 4.5s + 5.5)(s + 1.32)(s^2 + 0.26s + 1.81)}{s(s^2 + 0.11s + 4.02)(s^2 + 1.27s + 1.64)(s^2 + 0.20s + 1.76)}$$

and

$$F(s) = (s + \varepsilon)\cdot$$
$$\cdot \frac{(s^2 + 0.11s + 4.02)(s^2 + 1.27s + 1.64)(s^2 + 0.20s + 1.76)}{(s + 1.32)(s^2 + 0.26s + 1.81)}.$$

The plots of $\overline{\gamma}(\omega)$, $\underline{\gamma}(\omega)$ and $\gamma^*(\omega)$ are depicted in Fig. 7 (a) for $\rho = 0.89$, making it clear that the filter $F(s) = P_0(s)$ is not working. Indeed, it turns out the solution $\Phi(s) = 1$ can be used as long as $\rho < 0.32$. As an example, the case $\rho = 0.3$ is reported in Fig. 7 (b).

V. Conclusion

This paper has considered the continuous-time robust SPR problem when the uncertain family of polynomials is assumed to be an ellipsoid in the coefficient space.

It has been first shown that the stability of all the polynomials of the uncertain family is a necessary and sufficient condition for the solution of the robust SPR problem. This result exactly parallels the one available when the uncertain family is a polyhedron in coefficient space. More importantly, contrary to the available results on the polyhedral case, it has been shown that the solution of the $RSPR$ problem is given by a rational function having a known degree. In particular, the degree is less than twice the degree of the polynomials of the uncertain family. Furthermore, the rational filter is obtained in closed form via the factorization of a suitable polynomial. Finally, several application examples have been given to illustrate the features of the approach.

VI. Appendix

Proof of Lemma 1: It is easily checked that $\Phi(s)$ satisfies condition 1. of Definition 2 for any $\varepsilon, \delta > 0$. To prove that also condition 2. holds, we proceed as follows.
It can be shown (see [16, pp. 63–65]) that $\Phi^*(s)$ being positive real implies $\mathrm{Re}[\Phi^*(j\omega + \varepsilon)] > 0 \quad \forall \omega \geq 0$ for some small $\varepsilon > 0$. Therefore, if $\partial P_1 = \partial P_2$ the proof is already concluded.
On the contrary, suppose that $\partial P_2 - \partial P_1 = 1$ and let

$$\begin{aligned} \Phi(s) &= \Phi^*(s + \varepsilon)(1 + \delta s) \\ R_\varepsilon(\omega) &= \mathrm{Re}[\Phi^*(j\omega + \varepsilon)] \\ I_\varepsilon(\omega) &= \mathrm{Im}[\Phi^*(j\omega + \varepsilon)]. \end{aligned}$$

We have

$$\mathrm{Re}[\Phi(j\omega)] = R_\varepsilon(\omega) - \delta\omega I_\varepsilon(\omega).$$

Then, $\mathrm{Re}[\Phi(j\omega)] > 0 \quad \forall \omega \geq 0$ if and only if

$$\frac{1}{\delta} > \sup_{\omega \geq 0} \frac{\omega I_\varepsilon(\omega)}{R_\varepsilon(\omega)}.$$

Hence, there exists $\delta > 0$ such that $\Phi(s)$ is SPR if and only if

$$\sup_{\omega \geq 0} \frac{\omega I_\varepsilon(\omega)}{R_\varepsilon(\omega)} < +\infty. \tag{57}$$

Now, since $\Phi^*(s + \varepsilon)$ is by construction a minimum phase (i.e., all its poles and zeros have negative real part) relative degree one rational function, it turns out that $I_\varepsilon(\omega)$ is bounded for any finite $\omega \geq 0$ and negative for $\omega \to +\infty$. Taking into account that $R_\varepsilon(\omega) > 0$ for all $\omega \geq 0$, we get that (57) holds.
A similar argument applies for $\partial P_2 - \partial P_1 = -1$. ■

Proof of Lemma 2: The proof can be found in [2, pp. 125–127]. For completeness of the paper, we give a sketch of the proof. Since $\mathcal{P}_\rho$ is a degree-invariant set of polynomials and $P_0(s) \in \mathcal{H}$, the zero-exclusion principle yields that the l_2 stability margin amounts to satisfying the following optimization problem

$$\begin{array}{c}\rho^* = \inf_{\omega\geq 0}\min_q q'q \\ P_0(j\omega)+\sum_{i=1}^n q_i P_i(j\omega)=0\end{array}$$

which is equivalent to

$$\begin{array}{c}\rho^* = \inf_{\omega\geq 0}\min_q q'q \\ 1-q'R(\omega)=0 \\ q'I(\omega)=0\end{array}\quad .$$

By applying Lagrangian multipliers to the above problem, the proof can be readily obtained. ■

Proof of Lemma 3: Taking $q=0$ in (19) yields (20a). Therefore, (19) can be rewritten equivalently as

$$\begin{array}{ll}(a) & \mathrm{Re}[\Phi(j\omega)]>0 \\ (b) & q'\left[R(\omega)-\gamma_\Phi(\omega)I(\omega)\right]<1\end{array}\quad \forall\omega\geq 0\quad \forall q:\|q\|_2\leq\rho\ . \tag{58}$$

By a standard property of the 2-norm, (58b) holds for all q such that $\|q\|_2\leq\rho$ if and only if (20b) holds. ■

Proof of Lemma 4:

1. Let us rewrite (22) as

$$\|I(\omega)\|_2^2\,\gamma^2(\omega)-2R'(\omega)I(\omega)\gamma(\omega)+\|R(\omega)\|_2^2-\frac{1}{\rho^2}<0. \tag{59}$$

Note that, for each fixed ω, the left hand side term of (59) is a second order polynomial with respect to $\gamma(\omega)$. Moreover, since $\rho<\rho^*$, Lemma 2 ensures that $\rho<\overline{\rho}$ (see (16)) and it is therefore straightforward to verify (see (13)-(15)) that, for all $\omega\in\overline{\Omega}_0$, inequality (59) holds for any $\gamma(\omega)$ satisfying

$$\underline{\gamma}(\omega)<\gamma(\omega)<\overline{\gamma}(\omega) \tag{60}$$

where

$$\underline{\gamma}(\omega)=\min\left\{\gamma^*(\omega)\pm\frac{\sqrt{\Delta(\omega)}}{\|I(\omega)\|_2^2}\right\}$$
$$\overline{\gamma}(\omega)=\max\left\{\gamma^*(\omega)\pm\frac{\sqrt{\Delta(\omega)}}{\|I(\omega)\|_2^2}\right\}$$

and

$$\Delta(\omega)=\left[R'(\omega)I(\omega)\right]^2-\|I(\omega)\|_2^2\left[\|R(\omega)\|_2^2-\frac{1}{\rho^2}\right].$$

Finally, again from Lemma 2 (see (12) and (16)), it turns out that, for all $\omega\in\Omega_0$, inequality (59) holds for $\gamma(\omega)$ being any real value.

2. Under the assumption $\rho<\rho_0$, it can be easily verified that for each $\omega_0\in\Omega_0$ there exists a neighborhood $\mathcal{N}(\omega_0)$ of ω_0 such that, for all $\omega\in\mathcal{N}(\omega_0)\setminus\{\omega_0\}$, $\underline{\gamma}(\omega)$ and $\overline{\gamma}(\omega)$ are continuous functions of opposite sign. Moreover, as stated above, $\gamma(\omega_0)$ can be any real value. Thus, any bounded $\gamma(\omega)$ satisfying (60) for $\omega\in\bar{\Omega}_0$ can be extended to a continuous solution of (22) for all $\omega\geq 0$. Hence, Γ is nonempty. ■

Proof of Lemma 5: Exploiting (7), we rewrite (27) as

$$\Pi(s)=P_0(s)[P_0(-s)G'(-s)][P_0(-s)P_0(s)G(s)]_o.$$

Thus, $\Pi(j\omega)$ can be calculated as

$$\begin{array}{r}\Pi(j\omega)=P_0(j\omega)P_0(-j\omega)G'(-j\omega)\cdot \\ \cdot\, j\mathrm{Im}\{P_0(-j\omega)P_0(j\omega)G(j\omega)\}= \\ =[P_0(j\omega)P_0(-j\omega)]^2[R'(\omega)-jI'(\omega)]\cdot jI(\omega)= \\ =[P_0(j\omega)P_0(-j\omega)]^2[I'(\omega)I(\omega)+jR'(\omega)I(\omega)].\end{array} \tag{61}$$

This proves property 1.. Property 2. directly follows from (61) and the fact that $I(\omega)\neq 0$ for $\omega\in\bar{\Omega}_0$, while property 3. derives from (61) and (23). ■

Proof of Lemma 6: From (30) and(31) we get

$$\gamma_{\Phi^*}(\omega)=\frac{\mathrm{Im}\left[\Pi_1(j\omega)\Pi_2(-j\omega)\right]}{\mathrm{Re}\left[\Pi_1(j\omega)\Pi_2(-j\omega)\right]}=\frac{\mathrm{Im}\left[\Pi(j\omega)\right]}{\mathrm{Re}\left[\Pi(j\omega)\right]}=$$
$$=\gamma^*(\omega)\quad\forall\omega\in\bar{\Omega}_0.$$

Moreover, we have

$$\mathrm{Re}[\Phi^*(j\omega)]=\frac{\mathrm{Re}[\Pi(j\omega)]}{|\Pi_2(j\omega)|^2}.$$

Hence, $\mathrm{Re}[\Phi^*(j\omega)]>0\quad\forall\omega\in\bar{\Omega}_0$ follows from (29) in Lemma 5. ■

Proof of Theorem 3: First, it can be easily verified that $\Phi(s)$ in (53) satisfies (17) by construction. Lemma 3 states that condition (18) is equivalent to condition (20)-(21). Thus, we have to prove that $\Phi(s)$ satisfies this condition.

We start with the following consideration: if $\Phi^*(s)$ in (50) satisfies (20) for all $\omega\in\bar{\Omega}_0$, then, for sufficiently small non-negative ε, (20) also holds for $\Phi^*(s+\varepsilon)$ and $\omega\in\bar{\Omega}_0$. Moreover, assumption 1 implies that $\rho<\rho_0$, and therefore from (12) it follows that $\Phi^*(s+\varepsilon)$ satisfies (20b) for $\omega\in\Omega_0$, too. Hence, all we have to show amounts to:

i) $\Phi^*(s)$ satisfies (20) for all $\omega\in\bar{\Omega}_0$;

ii) for sufficiently small $\varepsilon,\delta>0$, $\Phi(s)$ is a SPR function.

Let us rewrite the non-negative integers r_i in Lemma 8 as $r_i=2p_i+q_i$, where p_i is a non-negative integer and $q_i\in\{0,1\}$. Accordingly, (48) has the form

$$\Pi(s)=As^{r_0}\prod_{i=1}^k\left\{\left(s^2+\omega_i^2\right)^{2p_i}\left(s^2+\omega_i^2\right)^{q_i}\right\}\bar{\Pi}_1(s)\bar{\Pi}_2(s).$$

As in Theorem 2, the next step is to factorize $\Pi(s)$ in a suitable way. Taking into account (29), $\Pi(s)$ can be expressed as

$$\begin{array}{c}\Pi(s)=C_0\ s^{q_0}\prod_{i=1}^k\left(s^2+\omega_i^2\right)^{q_i}\cdot \\ \cdot\left[|A|^{1/2}\ \bar{\Pi}_1(s)\ s^{p_0}\ \prod_{i=1}^k\left(s^2+\omega_i^2\right)^{p_i}\right]\cdot \\ \cdot\left[|A|^{1/2}\ \bar{\Pi}_2(-s)\ (-s)^{p_0}\ \prod_{i=1}^k\left(s^2+\omega_i^2\right)^{p_i}\right]\end{array}$$

where $C_0=1$ if $q_0=0$ and $C_0=\mathrm{sgn}A\ (-1)^{p_0}$ if $q_0=1$.

By applying Lemmas 4, 5, 6, it can be verified that the rational function

$$\Phi^*(s)=\frac{\bar{\Pi}_1(s)}{\bar{\Pi}_2(s)}s^{N_0}\prod_{i=1}^k(s^2+\omega_i^2)^{N_i}$$

satisfies (20) for all $\omega \in \bar{\Omega}_0$. This completes the proof of point i).

In order to prove that $\Phi(s)$ is SPR for sufficiently small ε and $\delta > 0$, we employ Lemma 1. Hence, it suffices to check the positive real character of $\Phi^*(s)$.

We note that $\Phi^*(s)$ is analytic for $\mathrm{Re}[s] > 0$ and that $\mathrm{Re}\,[\Phi^*(j\omega)] \geq 0$ for all ω such that $\Phi^*(s)$ is analytic in $s = j\omega$. Since $\Phi^*(s)$ is positive real if and only if $\Phi^{*-1}(s)$ is, all we have to prove is that both $\Phi^*(s)$ and $\Phi^{*-1}(s)$ possess real positive residues in their respective finite imaginary poles, which are all simple by construction.

To this purpose, we first introduce a useful result concerning the rational functions $\tilde{\Pi}_i(s)$, $i = 1, \ldots, k$, defined in (49). Note that for any $i \in \{1, \ldots, k\}$, the following equality holds

$$\Pi(j\omega) = (\omega_i^2 - \omega^2)^{r_i} \left\{ \mathrm{Re}\left[\tilde{\Pi}_i(j\omega)\right] + j\mathrm{Im}\left[\tilde{\Pi}_i(j\omega)\right] \right\} . \tag{62}$$

Since Lemma 5 ensures that $\mathrm{Re}\,[\Pi(j\omega)] \geq 0$ for all $\omega \geq 0$, it can be shown that $\tilde{\Pi}_i(s)$ must satisfy the following condition

$$r_i \text{ odd} \quad \Longrightarrow \quad \mathrm{Re}\left[\tilde{\Pi}_i(j\omega_i)\right] = 0 \text{ and } \mathrm{Im}\left[\tilde{\Pi}_i(j\omega_i)\right] \neq 0. \tag{63}$$

Let us consider the singularities of $\Phi^*(s)$ on the imaginary axis and their corresponding residues.

From (50) and (51), $s = 0$ is a singularity of $\Phi^*(s)$ if r_0 is odd and $\mathrm{sgn}A\ (-1)^{(r_0-1)/2} = -1$. The corresponding residue is given by

$$\mathrm{Res}[\Phi^*(s), 0] = \frac{\bar{\Pi}_1(0)}{\bar{\Pi}_2(0)} \prod_{i=1}^{k} \omega_i^{2N_i},$$

and it is positive since $\bar{\Pi}_1(s)$ and $\bar{\Pi}_2(s)$ are Hurwitz polynomials.

From (50) and (52), $s = \pm j\omega_h$ is a singularity if r_h is odd and $\mathrm{Im}\left[\tilde{\Pi}_h(j\omega_h)\right] > 0$. The corresponding residue is given by

$$2\mathrm{Res}[\Phi^*(s), j\omega_h] = \frac{1}{j\omega_h} \frac{\bar{\Pi}_1(j\omega_h)}{\bar{\Pi}_2(j\omega_h)} (j\omega_h)^{N_0} \prod_{h \neq i=1}^{k} \left(\omega_i^2 - \omega_h^2\right)^{N_i},$$

and it can be rewritten as

$$\begin{aligned} 2\mathrm{Res}[\Phi^*(s), j\omega_h] &= \frac{1}{j\omega_h} \frac{\tilde{\Pi}_h(j\omega_h)}{R_h(j\omega_h)R_h(-j\omega_h)} = \\ &= \frac{1}{\omega_h} \frac{\mathrm{Im}\left[\tilde{\Pi}_h(j\omega_h)\right]}{R_h(j\omega_h)R_h(-j\omega_h)}. \end{aligned}$$

by introducing the non-zero quantity $R_h(j\omega_h)$, whose complete expression is omitted for brevity, and using (49) in the first equality, and exploiting condition (63) in the last equality. Thus, the residue is positive since $\mathrm{Im}\left[\tilde{\Pi}_h(j\omega_h)\right] > 0$.

A similar analysis can be performed for $\Phi^{*-1}(s)$. It turns out that the residues are all positive as summarized below:

- $s = 0$ is a singularity if r_0 is odd and $\mathrm{sgn}A\ (-1)^{(r_0-1)/2} = 1$. Its residue satisfies

$$\mathrm{Res}[\Phi^{*-1}(s), 0] = \frac{\bar{\Pi}_2(0)}{\bar{\Pi}_1(0)} \prod_{i=1}^{k} \omega_i^{-2N_i} = \frac{\bar{\Pi}_2(0)}{\bar{\Pi}_1(0)} \prod_{i=1}^{k} \omega_i^{-2N_i} > 0$$

- $s = \pm j\omega_h$ is a singularity if r_h is odd and $\mathrm{Im}\left[\tilde{\Pi}_h(j\omega_h)\right] < 0$. Its residue satisfies

$$\begin{aligned} &2\mathrm{Res}[\Phi^{*-1}(s), j\omega_h] = \\ &= \frac{1}{j\omega_h} \frac{\bar{\Pi}_2(j\omega_h)}{\bar{\Pi}_1(j\omega_h)} (j\omega_h)^{-N_0} \prod_{h \neq i=1}^{k} \left(\omega_i^2 - \omega_h^2\right)^{-N_i} = \\ &= \frac{1}{j\omega_h} \frac{R_h(j\omega_h)R_h(-j\omega_h)}{\tilde{\Pi}_h(j\omega_h)} = -\frac{1}{\omega_h} \frac{R_h(j\omega_h)R_h(-j\omega_h)}{\mathrm{Im}\left[\tilde{\Pi}_h(j\omega_h)\right]} > 0. \end{aligned}$$

■

Proof of Corollary 2: Proceeding the same way as in Corollary 1 and observing that in general $P_0(s)$ and $\bar{\Pi}_1(s + \varepsilon)$ have no common factors, the following limitation on ∂D_F can be obtained

$$\partial D_F \leq 2l - 1 + \frac{e}{2} + \frac{N_0}{2} + \sum_{i=1}^{k} N_i - \frac{r_0}{2} - \sum_{i=1}^{k} r_i,$$

where e denotes the pole-zero excess in $\Phi^*(s)$. Since $\Phi^*(s)$ is positive real and assuming suitable worst case bounds on other parameters one obtains

$$\partial D_F \leq 2l - 1 + \frac{1}{2} + \frac{1}{2} + k - \frac{1}{2} - k,$$

which in turn proves (55). ■

Proof of Theorem 4: By looking at equations (53) and (56), it is clear that the two rational functions $\Phi(s)$ in Theorems 3 and 4 are generated by perturbing the same $\Phi^*(s)$ of (50) in two slightly different ways.

Therefore, from the proof of Theorem 3, it is clear that we have only to show that $\Phi(s)$ in (56) is strictly positive real. Observe that $\Phi(s)$ satisfies (17) by construction and, again from the proof of Theorem 3, it turns out that $\mathrm{Re}[\Phi^*(j\omega)] > 0$ for all $\omega \in \bar{\Omega}_0$. Hence, it remains to prove that

$$\mathrm{Re}[\Phi(j\omega_i)] > 0 \tag{64}$$

for all $\omega_i \in \Omega_0$ and some sufficiently small ϵ, δ, and ζ_j, $j = 1, \ldots, k$.

It can be verified that $\Phi(s)$ can be rewritten as

$$\Phi(s) = \Psi_i(s)\left(1 + \Delta\Psi_i(s; \epsilon, \delta, \zeta_1, \ldots, \zeta_{i-1}, \zeta_{i+1}, \ldots, \zeta_k)\right)$$

where

$$\Psi_i(s) = (s^2 + 2\zeta_i\omega_i s + \omega_i^2)^{N_i} \tilde{\Pi}_i(s), \tag{65}$$

being $\tilde{\Pi}_i(s)$ given in (49), and

$$\Delta\Psi_i(s; \epsilon, \delta, \zeta_1, \ldots, \zeta_{i-1}, \zeta_{i+1}, \ldots, \zeta_k)$$

is a rational function, whose value at $s = j\omega_i$ is continuous with respect to the parameters ϵ, δ, ζ_j, $j = 1, \ldots, k$, $j \neq i$, and such that

$$\Delta\Psi_i(j\omega_i; 0, 0, 0, \ldots, 0, 0, \ldots, 0) = 0.$$

Hence, it suffices to prove that the function $\Psi_i(s)$ satisfies

$$\mathrm{Re}\left[\Psi_i(j\omega_i)\right] > 0 \qquad \text{for some} \quad \zeta_i > 0. \tag{66}$$

Suppose r_i is odd. We get

$$\mathrm{Re}\left[\Psi_i(j\omega_i)\right] = 2\zeta_i\omega_i^2 j^{N_i+1}\mathrm{Im}\left[\tilde{\Pi}_i(j\omega_i)\right],$$

and therefore (66) follows from (52).
If r_i is even, from (65), (52) and (29) it turns out that

$$\mathrm{Re}\left[\Psi_i(j\omega_i)\right] = \mathrm{Re}\left[\tilde{\Pi}_i(j\omega_i)\right] \geq 0.$$

Hence, (66) follows from Assumption 3. ■

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