Problem 3

[25 pts]

Consider a continuous-time signal $x_c(t)$, and assume that its continuous-time Fourier transform (CTFT) is as given in Figure 2.

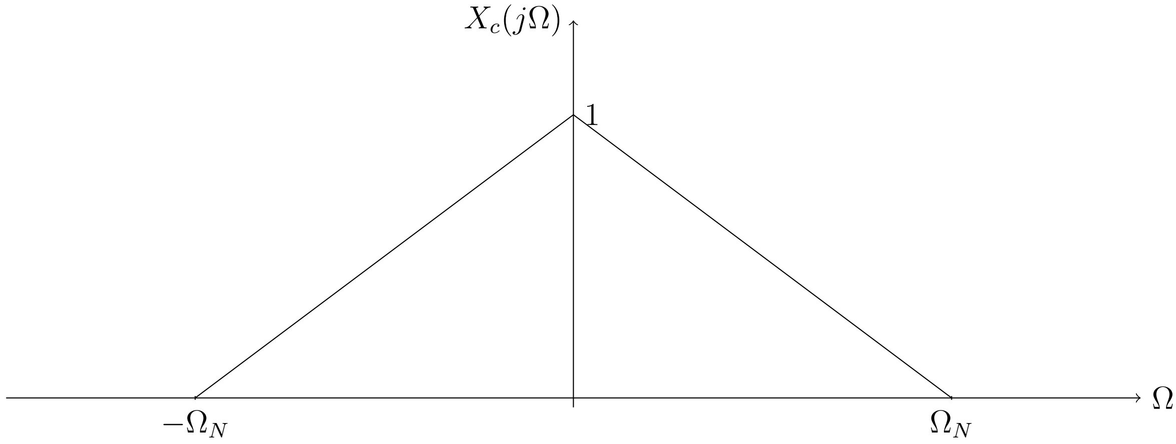

Figure 2: Spectrum of $x_c(t)$.

[*1pts*] (a) According to the sampling theorem, what is the smallest sampling frequency Ω_s at which reconstruction of $x_c(t)$ from the samples is possible?

[1pts] (b) Assume that we sample $x_c(t)$ at a sampling frequency $\Omega'_s = 3\Omega_N$. Let $u[n]$ be the sample sequence, *i.e.*, $u[n] = x_c(nT'_s)$, where $T'_s = \frac{2\pi}{\Omega'_s}$. Is perfect reconstruction of $x_c(t)$ possible from the sample sequence $u[n]$?

[4pts] (c) Sketch the discrete-time Fourier transform (DTFT) $U(e^{j\omega})$ of the sample sequence $u[n]$ for $\omega \in [-3\pi, 3\pi]$. Make sure to label all the important points on both axes.

[3pts] (d) Assume that we upsample $u[n]$ by a factor of 2 to obtain $v[n]$, *i.e*,

$$v[n] = \begin{cases} u[\frac{n}{2}] & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

Sketch the DTFT $V(e^{j\omega})$ of the new sequence $v[n]$ for $\omega \in [-3\pi, 3\pi]$. Again, label all the important points on both axes.

[6pts] (e) Now, we use sinc interpolation on $v[n]$ to construct a continuous-time signal $y_c(t)$ (using $\Omega'_s = 3\Omega_N$). In other words,

$$y_c(t) = \sum_{n=-\infty}^{\infty} v[n] \,\mathrm{sinc}(\frac{t - nT'_s}{T'_s}).$$

Sketch the CTFT $Y_c(j\Omega)$ of $y_c(t)$. Make sure to label all the important points on both axes.

[1pts] (f) According to the sampling theorem, what is the smallest sampling frequency $\tilde{\Omega}_s$ at which reconstruction of $y_c(t)$ from the samples is possible?

[3pts] (g) Assume that we sample $y_c(t)$ at a sampling frequency $\hat{\Omega}_s = 2\Omega_N$ to obtain the sample sequence $w[n]$. Sketch the DTFT $W(e^{j\omega})$ of the sample sequence $w[n]$ for $\omega \in [-3\pi, 3\pi]$. Make sure to label all the important points on both axes.

[6pts] (h) Is perfect reconstruction of $y_c(t)$ possible from the sample sequence $w[n]$? If yes, explain how. If not, explain why. (You can give your answer in terms of diagrams and some explanations.)

Problem 4

[17 pts]

Consider the system given in Figure 3, where $\mathcal{S}$ is a time-**variant** operator which acts on its input $u[n]$ as follows:

$$v[n] = \mathcal{S}\{u[n]\} = \begin{cases} u[n] & \text{if } n \text{ is even} \\ \frac{1}{2}u[n] + \frac{1}{2}u[n-1] & \text{if } n \text{ is odd.} \end{cases}$$

[7pts] (a) Compute $v[n]$ in terms of $x[n]$ and $y[n]$.
Hint: First write down $v[0]$, $v[1]$, $v[2]$ and $v[3]$, and then generalize.

[5pts] (b) Rewrite the system under the form given in Figure 4. All you are allowed to do is fill in the gaps in the rectangles and circles. All the filters that you use should be linear and time-invariant.
Hint: You are allowed to put $\delta[n]$ in some of the filters.

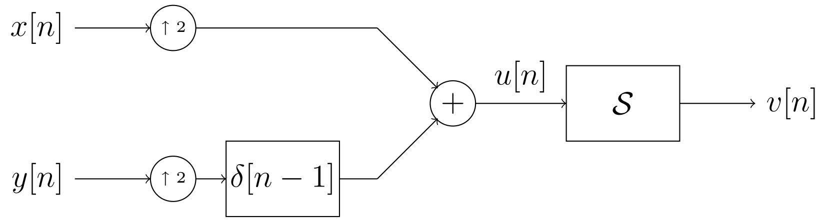

Figure 3: Multirate system

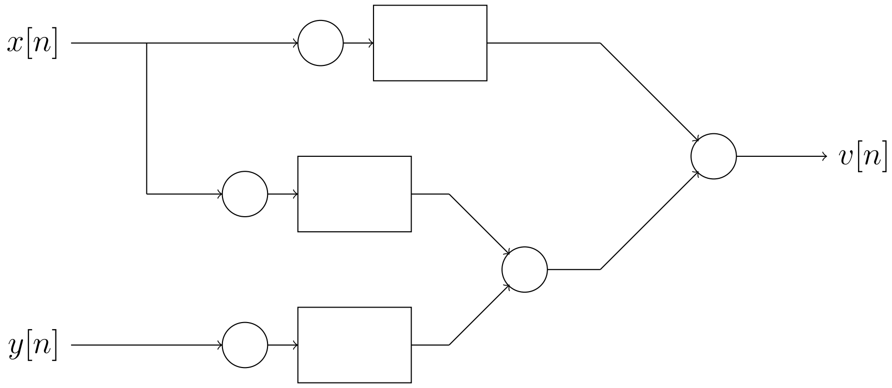

Figure 4: Equivalent multirate system

[*1pts*] (c) Compute the Z-transform $V(z)$ of the system output as a function of $X(z)$ and $Y(z)$.

[*2pts*] (d) Now we feed $v[n]$ into the system given in Figure 5. Compute $a[n]$ and $b[n]$. Express the result in terms of $x[n]$ and $y[n]$.

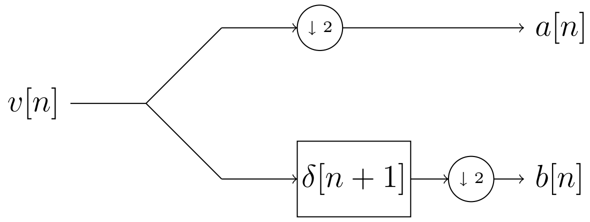

Figure 5: Continuation of the multirate system

[*2pts*] (e) Find a system that inverts the system in Figure 3, *i.e.*, your system should have input $v[n]$ and outputs $x[n]$ and $y[n]$. Draw the system diagram and determine all the operations and filters used. All the filters that you use should be linear and time-invariant.

Problem 3

(a) From the sampling theorem, we know that $\Omega_s = 2\Omega_N$ is such that any sampling frequency larger than Ω_s allows perfect reconstruction of $x_c(t)$.

(b) Yes, perfect reconstruction is possible, because $\Omega_s' \geq \Omega_s$.

(c) The DTFT of the sample sequence $u[n]$ is given in Figure 2.

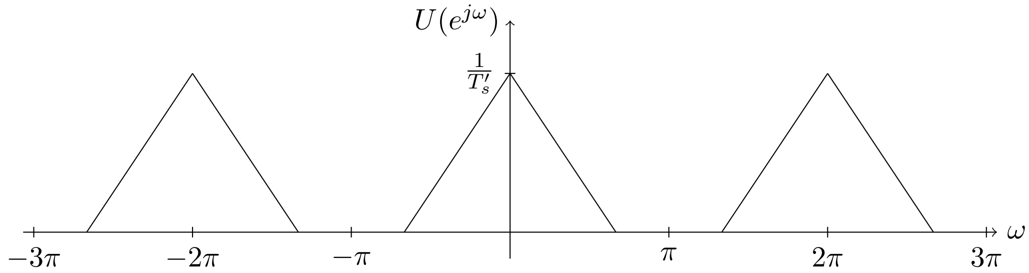

Figure 2: Problem 3 (c)

(d) Upsampling by a factor of 2 has the effect of "contracting" the DTFT horizontally by a factor of 2. Hence, we obtain the DTFT in Figure 3.

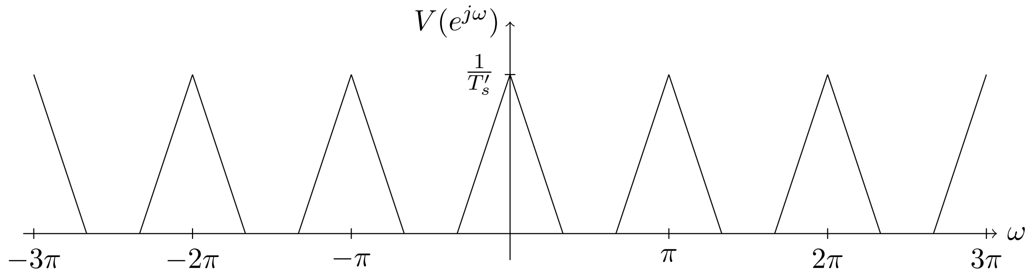

Figure 3: Problem 3 (d)

(e) Consider first the continuous-time function

$$y_s(t) = \sum_{n=-\infty}^{\infty} v[n]\delta(t - nT_s').$$

This is just a sequence of delta-pulses, whose amplitudes are $v[0], v[1], v[2], \ldots$ (we call this a pulse-train). The CTFT of $y_s(t)$ is given in Figure 4. The sinc interpolation of $v[n]$ is

$$y_c(t) = \sum_{n=-\infty}^{\infty} v[n] \operatorname{sinc}\left(\frac{t - nT_s'}{T_s'}\right).$$

Hence, it can be shown that $y_c(t) = y_s(t) * \operatorname{sinc}(\frac{t}{T_s'})$. In the frequency domain, this convolution corresponds to the multiplication

$$Y_c(j\Omega) = Y_s(j\Omega)T_s'\mathbf{1}_{[-\frac{\Omega_s'}{2}, \frac{\Omega_s'}{2}]},$$

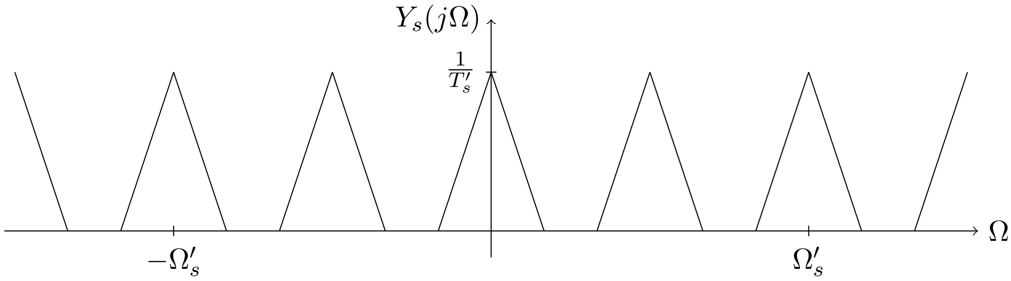

Figure 4: Problem 3 (e)

where the box function $T'_s \mathbf{1}_{[-\frac{\Omega'_s}{2}, \frac{\Omega'_s}{2}]}$ is simply the CTFT of $\text{sinc}(\frac{t}{T'_s})$. The resulting CTFT $Y_c(j\Omega)$ is given in Figure 5.

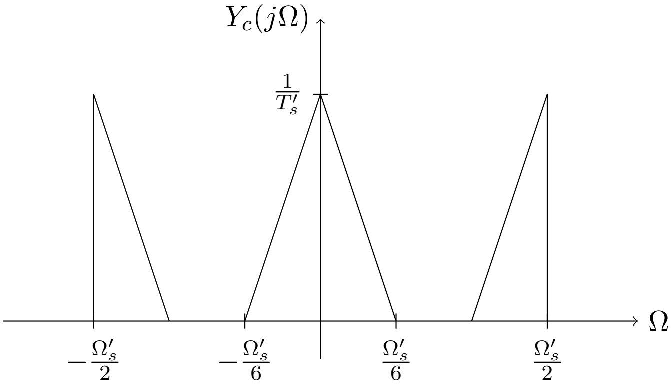

Figure 5: Problem 3 (e)

(f) Here, the spectrum $Y_c(j\Omega)$ has support $[-\frac{\Omega'_s}{2}, \frac{\Omega'_s}{2}]$. According to the sampling theorem, the limit sampling frequency is equal to the width of this support, namely $\tilde{\Omega}_s = 2\frac{\Omega'_s}{2} = \Omega'_s$.

(g) To find out how to sketch the DTFT of $w[n]$, we observe that the support of $Y_c(j\Omega)$ is $[-\frac{\Omega'_s}{2}, \frac{\Omega'_s}{2}] = [-\frac{3}{2}\Omega_N, \frac{3}{2}\Omega_N]$. Now, we sample $y_c(t)$ at $\hat{\Omega}_s = 2\Omega_N$, which will correspond to 2π in the DTFT sketch. Hence, the support of one "copy" of the spectrum in the DTFT is $[-\frac{3}{2}\pi, \frac{3}{2}\pi]$. The result is shown in Figure 6.

(h) Yes, perfect reconstruction is possible. We can see this because there is no aliasing in the DTFT in Figure 6. First, we sketch the CTFT of $\hat{y}_s(t)$, which is the pulse-train that corresponds to $w[n]$. This sketch is shown in Figure 7.

To recover $y_c(t)$, we can use a filter that combines a low-pass and a band-pass filter. Such a filter $H_r(j\Omega)$ is shown in Figure 8.

Mathematically, the frequency response of the filter is

$$H_r(j\Omega) = \begin{cases} \hat{T}_s & \text{if } |\Omega| \in [\frac{1}{2}\hat{\Omega}_s, \frac{3}{4}\hat{\Omega}_s], \\ \hat{T}_s & \text{if } |\Omega| \in [0, \frac{1}{4}\hat{\Omega}_s], \\ 0 & \text{otherwise.} \end{cases}$$

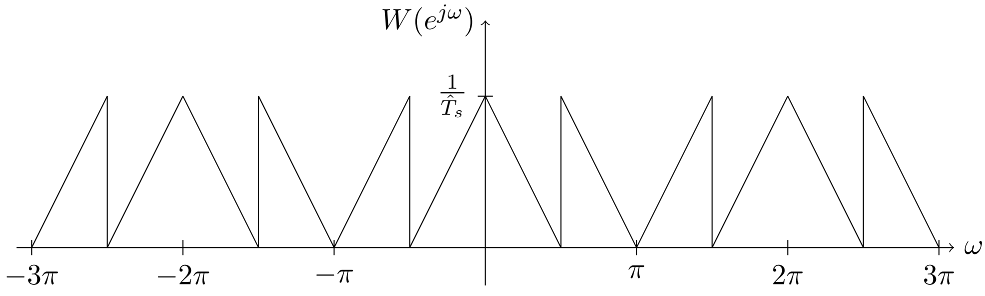

Figure 6: Problem 3 (g)

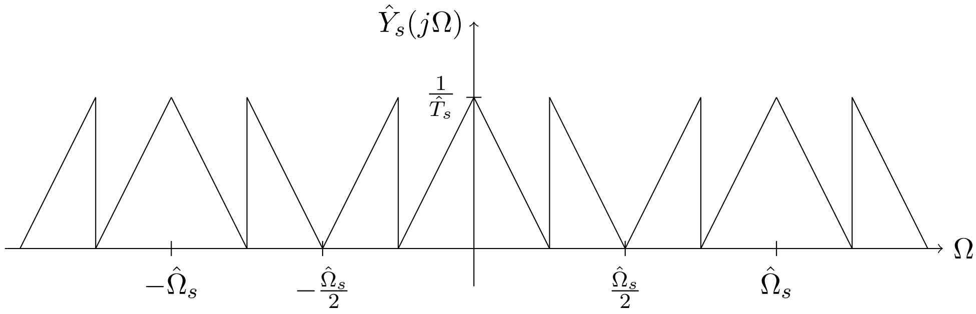

Figure 7: Problem 3 (h)

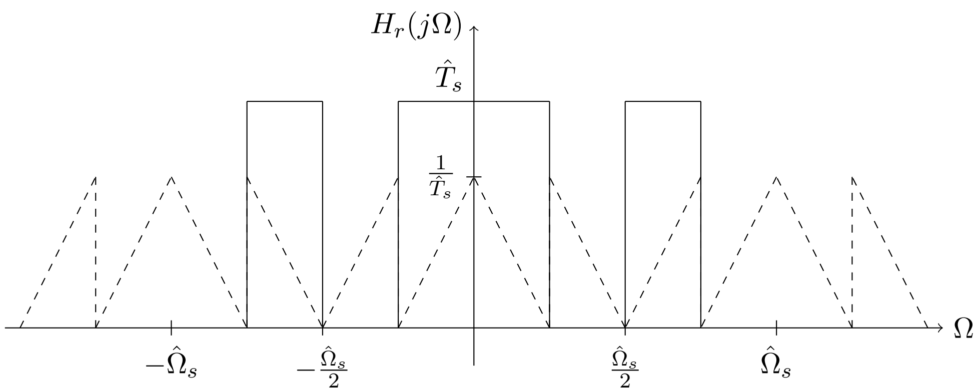

Figure 8: Problem 3 (h)

Problem 4

(a) Since the system from $x[n]$ and $y[n]$ to $u[n]$ is LTI, we can simply write

$$u[n] = U_2(x[n]) + U_2(y[n]) * \delta[n-1].$$

So, is it clear that

$$u[n] = \begin{cases} x[\frac{n}{2}] & \text{if } n \text{ is even} \\ y[\frac{n-1}{2}] & \text{if } n \text{ is odd.} \end{cases}$$

Therefore, after the time-variant operator, $\mathcal{S}$, we get

$$v[n] = \mathcal{S}\{u[n]\} = \begin{cases} x[k] & \text{if } n = 2k \\ \frac{1}{2}x[k] + \frac{1}{2}y[k] & \text{if } n = 2k+1. \end{cases} \quad (1)$$

(b) The system shown in Fig. 9 is equivalent to the system considered in the problem. Note that all the filters used in the equivalent system are linear and time-invariant.

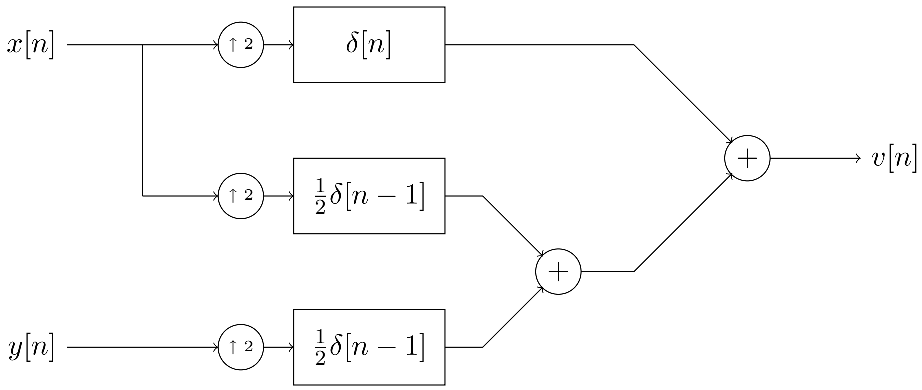

Figure 9: Equivalent multirate system

(c) It is clear from the equivalent system that

$$v[n] = U_2(x[n]) * \delta[n] + U_2(x[n]) * \frac{1}{2}\delta[n-1] + U_2(y[n]) * \frac{1}{2}\delta[n-1].$$

Therefore,

$$\begin{aligned} V(z) &= X(z^2) + \frac{1}{2}z^{-1}X(z^2) + \frac{1}{2}z^{-1}Y(z^2) \\ &= (1 + \frac{1}{2}z^{-1})X(z) + \frac{1}{2}z^{-1}Y(z). \end{aligned}$$

(d) $a[n]$ is just the down-sampled version of $v[n]$ by a factor 2, *i.e.*, it only contains the even-index samples of $v[n]$. So, from (1), it is clear that

$$a[n] = v[2n] = x[n].$$

We can can also easily see that $b[n]$ takes the odd-indices of the input sequence. Therefore,

$$b[n] = D_2(v[n] * \delta[n+1]) = D_2(v[n+1]) = v[2n+1] = \frac{1}{2}x[n] + \frac{1}{2}y[n].$$

(e) Having the system in part (d), it is easy to do a little modification to come up with the inverse system. Note that $x[n]$ is already produced at one of the output branches, and we only have to scale the other output, and subtract $x[n]$. Hence, we obtain the following system.

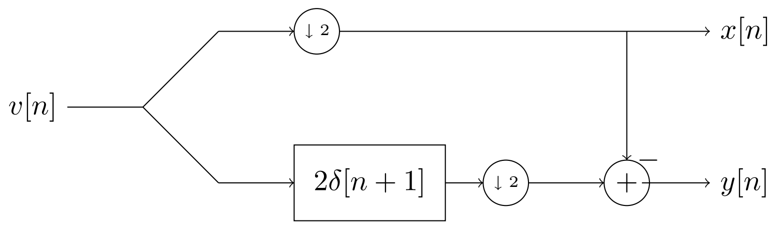

Figure 10: Inverse system