So far, detection under:

- Neyman-Pearson criteria (max P_D s.t. P_{FA} = constant): likelihood ratio test, threshold set by P_{FA}
- minimize Bayesian risk (assign costs to decisions, have priors of the different hypotheses): likelihood ratio test, threshold set by priors+costs
    - minimum probability of error = maximum a posteriori detection
        - maximum likelihood detection = minimum probability of error with equal priors
- known deterministic signals in Gaussian noise: correlators

Now we look at detecting **random Gaussian signals**

Motivation

- Some processes are better represented as random (e.g. speech)
- rather than assume completely random, assume signal comes from a random process of known **covariance structure**

Consider a binary hypothesis testing model of the following form:

$$\begin{aligned} \mathcal{H}_0 &: x[n] = w[n] \\ \mathcal{H}_1 &: x[n] = s[n] + w[n], \end{aligned}$$

where $\mathbf{w} \sim \mathcal{N}(0, \mathbf{C_w})$ and $\mathbf{s} \sim \mathcal{N}(\mu_\mathbf{s}, \mathbf{C_s})$ and $\mathbf{s}, \mathbf{w}$ are independent. We have $n = 0, 1, \cdots N-1$ (N samples).

The problem

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We thus can discriminate between the two hypothesis based on both their means and covariances. Taking the likelihood ratio and simplifying, our test statistic $T(\mathbf{x})$ can be shown to be:

$$T(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T \left[\mathbf{C_w}^{-1}\mathbf{C_s}(\mathbf{C_s} + \mathbf{C_w})^{-1}\right] \mathbf{x} + \mathbf{x}^T(\mathbf{C_s} + \mathbf{C_w})^{-1}\mu_\mathbf{s} \tag{1}$$

The test statistic has a quadratic term in $\mathbf{x}$ (intuitively account for the different variances) as well as a linear term in $\mathbf{x}$ accounting for the different means.

Example 1: does this reduce to previous results for deterministic signals?

Deterministic signals? Take $\mathbf{C_s} = \mathbf{0}, \mu_s = s$ for $\mathbf{s}$ a known signal. Then our test statistic becomes $T(\mathbf{x}) = \mathbf{x}^T \mathbf{C_w}^{-1}\mathbf{s}$, the generalized matched filter!

Example 2: zero mean WSS signal in white noise

Energy detectors? Suppose we have WGN of variance σ^2 and a signal which is a zero-mean Wide Sense Stationary Gaussian process with variance σ_s^2?

Then $\mathbf{C_s} = \sigma_s^2\mathbf{I}, \mu_\mathbf{s} = \mathbf{0}$ and $\mathbf{C_w} = \sigma^2\mathbf{I}$. Then the test statistic becomes $T(\mathbf{x}) = \sum_{n-0}^{N-1} x^2[n]$ which is then compared to a threshold. This is just an energy detector, which makes sense as the only difference between the signal and the noise is its variance.

We can derive its performance as tails ($Q_{\mathcal{X}_N^2}(x)$) of chi-squared random variables with N degrees of freedom ($\mathcal{X}_N^2$).

Example 3: correlated signal covariance in white noise

Estimator-correlator? Suppose we have WGN of variance σ^2 and a signal of zero mean and covariance $\mathbf{C_s}$. Then the test statistic becomes $T(\mathbf{x}) = \sigma^2\mathbf{x}^T\left[\mathbf{C_s}(\mathbf{C_s}+\sigma^2\mathbf{I})^{-1})\mathbf{x}\right]$, which may be re-written as a new test statistic $T'(\mathbf{x} = \mathbf{x}^T\hat{\mathbf{s}}$ for $\hat{\mathbf{s}} = \mathbf{C_s}(\mathbf{C_s}+\sigma^2\mathbf{I})^{-1})\mathbf{x}$. In

Interestingly, $\hat{\mathbf{s}}$ is the Minimum Mean Squared Error Estimate of the signal $\mathbf{s}$ given the received data $\mathbf{x}$ (we will see this later). So what we are in essence doing is correlating the received signal with an *estimate* of the signal $\mathbf{s}$, hence the name *estimator-correlator.*

Example 4: canonical form

The canonical form of the estimator-correlator? When dealing with matched filters with colored noise, we could "whiten" it, when we have a signal with a general covariance matrix $\mathbf{C_s}$ we can try to de-correlate the received data $\mathbf{x}$ before using a variant of an energy detector on it. That is, suppose the signal covariance matrix $\mathbf{C_s}$ has eigendecomposition $\mathbf{C_s V} = \wedge_\mathbf{s}\mathbf{V}$, where $\wedge_\mathbf{s}$ is a diagonal matrix with eigenvalues $\lambda_{s_0}, \cdots \lambda_{s_{N-1}}$ of $\mathbf{C_s}$ on the diagonal and $\mathbf{V}$ is a matrix with the corresponding eigenvectors as its columns. Then if we take the received data $\mathbf{x}$ and multiply it to obtain $\mathbf{y} = \mathbf{V}^T\mathbf{x}$ we can show that the test statistic becomes

$$T(\mathbf{x}) = \mathbf{y}^T \wedge_s \left[\wedge_\mathbf{s} + \sigma^2\mathbf{I}\right]^{-1}\mathbf{y} = \sum_{n=0}^{N-1}\frac{\lambda_{s_n}}{\lambda_{s_n}+\sigma^2}\,y^2[n]$$

We have a weighted energy detector!

Example 5: correlated signal in colored noise

Estimator-correlator with colored noise? We now have $\mathbf{w} \sim \mathcal{N}(\mathbf{0}, \mathbf{C_w})$ and $\mathbf{s} \sim \mathcal{N}(\mathbf{0}, \mathbf{C_s})$. The test statistic becomes

$$T(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T \left[\mathbf{C_s}(\mathbf{C_s} + \mathbf{C_w})^{-1})\mathbf{x}\right] = \frac{1}{2}\mathbf{x}^T \mathbf{C_w}^{-1}\hat{\mathbf{s}}$$

This looks like the generalized matched filter (matched filter in colored noise), where $\hat{\mathbf{s}}$ is now an estimate of the signal given by $\hat{\mathbf{s}} = \mathbf{C_s}(\mathbf{C_s} + \mathbf{C_w})\mathbf{x}$ rather than the known signal we had before.

Example 6: Linear Model

Linear model? We now have

$$\begin{aligned}\mathcal{H}_0 &: \mathbf{x} = \mathbf{w} \\ \mathcal{H}_1 &: \mathbf{x} = \mathbf{H}\theta + \mathbf{w},\end{aligned}$$

where $\mathbf{w} \sim \mathcal{N}(0, \mathbf{C_w})$, $\mathbf{H}$ is a known $N \times p$ observation matrix, and $\theta \sim \mathcal{N}(\mathbf{0}, \mathbf{C}_\theta)$ and $\theta, \mathbf{w}$ are independent. We have $n = 0, 1, \cdots N-1$ (N samples).

The test statistic becomes

$$\begin{aligned}T(\mathbf{x}) &= \mathbf{x}^T \left[\mathbf{C_s}(\mathbf{C_s} + \mathbf{C_w})^{-1}\mathbf{x}\right] \\ &= \mathbf{x}^T \mathbf{H}\mathbf{C}_\theta\mathbf{H}^T(\mathbf{H}\mathbf{C}_\theta\mathbf{H}^T + \mathbf{C_w})^{-1}\hat{\mathbf{x}}\end{aligned}$$

Example of linear model: Rayleigh Fading Sinusoid

When the signal is present, we observe

$$\begin{aligned} x[n] &= A\cos(2\pi f_0 n + \phi) + w[n], \quad n = 0, 1, \cdots N-1 \\ &= a\cos(2\phi f_0 n) + b\sin(2\pi f_0 n), \quad \text{where } a = A\cos(\phi), b = -A\sin(\phi) \end{aligned}$$

where $0 < f_0 < 1/2$ and $w[n]$ is WGN of variance σ^2. Due to fading characteristics of the wireless channel, we assume the following statistics on the fading coefficients $[a\, b]^T$:

$$\theta = \begin{bmatrix} a \\ b \end{bmatrix} \sim \mathcal{N}(\mathbf{0}, \sigma_s^2 \mathbf{I})$$

Find the detector.