Known signal in Gaussian noise - matched filter!

We consider detecting the presence of a *known* signal $s[n]$, $n = 0, 1, \cdots, N-1$ in *Gaussian* noise. This means, the received signal $x[n]$, for $n = 0, 1, \cdots N-1$, is

$$\mathcal{H}_0 : x[n] = w[n]$$
$$\mathcal{H}_1 : x[n] = s[n] + w[n],$$

where $w[n]$ is for now assumed to be white with variance σ^2.

Recall that this means its autocorrelation function $r_{ww}[k] = E(w[n]w[n+k]) = \sigma^2\delta[k]$, where $\delta[k] = 1$ for $k = 0$ and 0 otherwise.

Starting from the likelihood ratio test, you can simplify the test to deciding $\mathcal{H}_1$ if the *test statistic* $T(\mathbf{x})$ is above a threshold (threshold determined by P_{FA} in Neyman-Pearson detection and by the priors and costs in Bayesian detection),

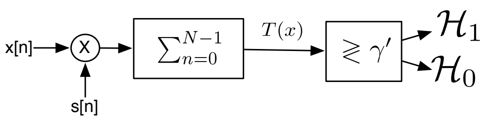
$$T(\mathbf{x}) = \sum_{n=0}^{N-1} x[n]s[n] > \gamma' \tag{1}$$

Correlator / replica-correlator / matched-filter

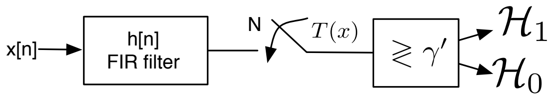
$$T(\mathbf{x}) = \sum_{n=0}^{N-1} x[n]s[n] > \gamma' \quad (1)$$

This test is called a *correlator* or *replica-correlator*. It is optimal in white Gaussian noise.

Another equivalent but more "signal processing" type approach to detecting $s[n]$ in the received $x[n]$ is to use a so-called *matched filter*. Here we view send the received signal $x[n]$ through a linear time invariant filter with a finite impulse response (FIR) $h[n] = s[N-1-n]$ for $n = 0, 1, \cdots N-1$. This impulse response is "matched" to the signal, it's a flipped version of it. We make our decision by sampling the output of the filter at time $N-1$ and comparing it with the threshold γ', as before.



"correlator"



"matched filter"

The matched filter maximizes SNR over linear filters

The correlator (or equivalently the matched filter) implementation of the NP detector weights the samples with more energy (larger values) more heavily than those of small energy. Equivalently, in the frequency domain it emphasizes the bands in which more signal power is located. The matched filter has the interesting property that it maximizes the SNR at the output of an FIR filter.

Furthermore, its performance (P_D) can be derived explicitly as a function of P_{FA} as

$$P_D = Q\left(Q^{-1}(P_{FA}) - \sqrt{\frac{\mathcal{E}}{\sigma^2}}\right), \quad (1)$$

where $\mathcal{E}$ is the energy in the signal, $\mathcal{E} = \sum_{n=0}^{N-1} s^2[n]$.

From (1) we see that the performance of the matched filter detector in *white* (uncorrelated) Gaussian noise is unaffected by the signal shape, this is not the case for colored (correlated) Gaussian noise.

Performance of matched filter

$$\mathcal{H}_0 : x[n] = w[n]$$
$$\mathcal{H}_1 : x[n] = s[n] + w[n],$$

where $w[n]$ is for now assumed to be white with variance σ^2.

The test statistic $T(\mathbf{x}) = \mathbf{x}^T\mathbf{s}$ has pdf $\mathcal{N}(0, \sigma^2\mathcal{E})$ under $\mathcal{H}_0$ and pdf $\mathcal{N}(\mathcal{E}, \sigma^2\mathcal{E})$ under $\mathcal{H}_1$. We can obtain P_{FA} and P_Q as follows:

$$P_{FA} = Pr\{T > \gamma'; \mathcal{H}_0\} \;\Rightarrow\; \gamma' = \sqrt{\sigma^2\mathcal{E}}Q^{-1}(P_{FA})$$
$$P_D = Pr\{T > \gamma'; \mathcal{H}_1\} = Q\left(\frac{\gamma' - \mathcal{E}}{\sqrt{\sigma^2\mathcal{E}}}\right) = Q\left(Q^{-1}(P_{FA}) - \sqrt{\frac{\mathcal{E}}{\sigma^2}}\right)$$

Generalized matched filter

In white Gaussian noise, the noise samples were $\mathbf{w} := [w[0], w[1], \cdots, w[N-1]]^T$ were distributed according to $\mathcal{N}(\mathbf{0}, \sigma^2\mathbf{I})$. Here we assume that, more generally, $\mathbf{w} \sim \mathcal{N}(\mathbf{0}, \mathbf{C})$ for an arbitrary covariance matrix $\mathbf{C}$. Recall that $\mathbf{C}_{mn} = cov(w[m], w[n]) = E(w[m]w[n]) = r_{ww}[m-n]$ when the noise is zero mean.

Starting from the likelihood ratio test one arrives at the test statistic $T(\mathbf{x})$ which is compared to a threshold γ'. We thus decide $\mathcal{H}_1$ if

$$T(\mathbf{x}) := \mathbf{x}^T\mathbf{C}^{-1}\mathbf{s} > \gamma' \tag{1}$$

When $\mathbf{C}$ is positive semi-definite, $\mathbf{C}^{-1}$ exists and is also positive semi-definite, and so may be factored as $\mathbf{C}^{-1} = \mathbf{D}^T\mathbf{D}$ for $\mathbf{D}$ a non-singular $N \times N$ matrix called the *prewhitening matrix.* If you process your received signal $\mathbf{x}$ by multiplying it by $\mathbf{D}$ (and similarly for your known signal $\mathbf{s}$, or form $\mathbf{x}' = \mathbf{D}\mathbf{x}$ and $\mathbf{s}' = \mathbf{D}\mathbf{s}$, then the generalized correlator in (1) looks like

$$T(\mathbf{x}) = \mathbf{x}^T\mathbf{C}^{-1}\mathbf{s} = \mathbf{x}^T\mathbf{D}^T\mathbf{D}\mathbf{s} = \mathbf{x}'^T\mathbf{s}'$$

Correlated noise and whitening

When $\mathbf{C}$ is positive semi-definite, $\mathbf{C}^{-1}$ exists and is also positive semi-definite, and so may be factored as $\mathbf{C}^{-1} = \mathbf{D}^T\mathbf{D}$ for $\mathbf{D}$ a non-singular $N \times N$ matrix called the *prewhitening matrix.* If you process your received signal $\mathbf{x}$ by multiplying it by $\mathbf{D}$ (and similarly for your known signal $\mathbf{s}$, or form $\mathbf{x}' = \mathbf{D}\mathbf{x}$ and $\mathbf{s}' = \mathbf{D}\mathbf{s}$, then the generalized correlator in (1) looks like

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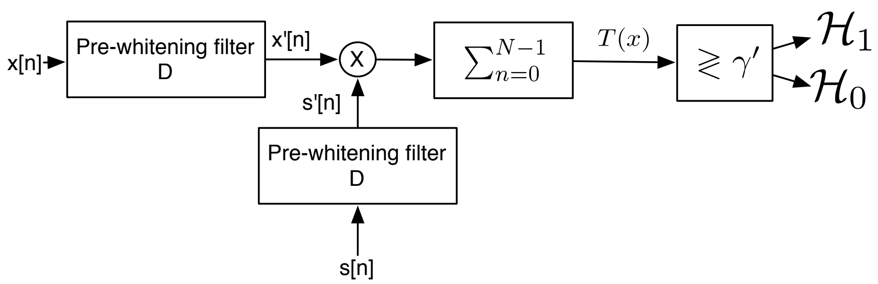


$\mathrm{C}^{-1} = \mathrm{D}^T\mathrm{D}$ for D a non-singular $N \times N$ matrix called the *prewhitening matrix.*

Performance of generalized matched filters

$$\mathcal{H}_0 : x[n] = w[n]$$
$$\mathcal{H}_1 : x[n] = s[n] + w[n],$$

where $w[n]$ is for now assumed to be correlated Gaussian noise with covariance matrix C.

Performance of generalized matched filter: the test statistic $T(\mathbf{x}) = \mathbf{X}^T\mathbf{C}^{-1}\mathbf{s}$ has pdf $\mathcal{N}(0\mathbf{1}, \mathbf{s}^T\mathbf{C}^{-1}\mathbf{s})$ under $\mathcal{H}_0$ and pdf $\mathcal{N}(\mathbf{s}^T\mathbf{C}^{-1}\mathbf{s}, \mathbf{s}^T\mathbf{C}^{-1}\mathbf{s})$ under $\mathcal{H}_1$. We can obtain P_{FA} and P_Q as follows:

$$P_{FA} = Pr\{T > \gamma'; \mathcal{H}_0\} \Rightarrow \gamma' = (\mathbf{s}^T\mathbf{C}^{-1}\mathbf{s})^{1/2}Q^{-1}(P_{FA})$$
$$P_D = Pr\{T > \gamma'; \mathcal{H}_1\} = Q\left(\frac{\gamma' - \mathbf{s}^T\mathbf{C}^{-1}\mathbf{s}}{(\mathbf{s}^T\mathbf{C}^{-1}\mathbf{s})^{1/2}}\right) = Q\left(Q^{-1}(P_{FA}) - \sqrt{\mathbf{s}^T\mathbf{C}^{-1}\mathbf{s}}\right)$$

From the performance metric P_D we see that in colored noise, unlike in white Gaussian noise, the shape of the signal $\mathbf{s}$ *can* affect the performance.

Designing signals

From the performance metric P_D we see that in colored noise, unlike in white Gaussian noise, the shape of the signal $\mathbf{s}$ *can* affect the performance. The question is now how to design $\mathbf{s}$ to maximize P_D, subject to a given desired power of the signal $\mathcal{E} = \mathbf{s}^T\mathbf{s}$ (else best performance will result from taking a signal of infinite energy).

Attack?

Using Lagrangians, we find that if λ_{min} is the smallest eigenvalue of $\mathbf{C}$ with corresponding normalized eigenvector $\mathbf{v}_{\min}$ then the signal $\mathbf{s}$ that maximizes P_D under an energy constraint of $\mathcal{E}$ is $\mathbf{s} = \sqrt{\mathcal{E}}\mathbf{v}_{\min}$.

Be able to show this!

Example: Signal design for uncorrelated noise with unequal variance

If $w[n] \sim \mathcal{N}(0, \sigma_n^2)$ and the $w[n]$'s are uncorrelated, the $\mathbf{C} = \text{diag}(\sigma_0^2, \sigma_1^2, \cdots, \sigma_{N-1}^2)$ and $\mathbf{C}^{-1} = \text{diag}(1/\sigma_0^2, 1/\sigma_1^2, \cdots 1/\sigma_{N-1}^2)$.

Hence, we decide $\mathcal{H}_1$ if

$$T(\mathbf{x}) = \sum_{n=0}^{N-1} \frac{x[n]s[n]}{\sigma_n^2} > \gamma'.$$

Example: Signal design for correlated noise, very simple

If $w[n] \sim \mathcal{N}(\mathbf{0}, \mathbf{C})$ with

$$\mathbf{C} = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix},$$

where ρ is the correlation coefficient which satisfies $\rho \leq 1$.

What does the test statistic become?

What does the deflection coefficient become?

Application: linear model

Linear model: $\mathbf{x}$ is an $N \times 1$ received signal, $\mathbf{H}$ is an $N \times p$ *known* full-rank matrix, θ is a $p \times 1$ set of parameters (known or unknown) and $\mathbf{w}$ is an $N \times 1$ noise vector $\sim \mathcal{N}(0, \mathbf{C})$, $\mathbf{C}$ a covariance matrix. Assume $\theta = \theta_1$ for some known set of parameters θ_1.

Our hypotheses are:

$$\begin{aligned} \mathcal{H}_0 &: \ \mathbf{x} = \mathbf{w} \\ \mathcal{H}_1 &: \ \mathbf{x} = \mathbf{H}\theta + \mathbf{w} \end{aligned}$$

Our old NP or ML detector for a known signal $\mathbf{s}$ in colored Gaussian noise with covariance matrix $\mathbf{C}$ reduces to the test statistic $T(\mathbf{x}) = \mathbf{x}^T\mathbf{C}^{-1}\mathbf{s}$, which is compared to a threshold $\gamma' = \ln(\gamma) + \frac{1}{2}\mathbf{s}^T\mathbf{C}^{-1}\mathbf{s}$. We can directly apply this by taking $\mathbf{s} = \mathbf{H}\theta_1$.

Multiple Deterministic Signals in Gaussian Noise

Before:

$$\mathcal{H}_0 : \mathbf{x} = \mathbf{w}$$
$$\mathcal{H}_1 : \mathbf{x} = \mathbf{s} + \mathbf{w}$$

Now:

$$\mathcal{H}_0 : \mathbf{x} = \mathbf{s_0} + \mathbf{w}$$
$$\mathcal{H}_1 : \mathbf{x} = \mathbf{s_1} + \mathbf{w}$$

Multiple Deterministic Signals in Gaussian Noise

We are interested in simple ML detection in white Gaussian noise, i.e. selecting the hypothesis $\mathcal{H}_i$ for which $p(\mathbf{x}|\mathcal{H}_i)$ is maximal. When you want to determine which of many signals was sent the general approach is to correlate the received signal with each of the possible hypothesis signals $\mathbf{s}_i = [s_i[0], s_i[1], \cdots s_i[N-1]]^T$, adjust for the signal energy $\mathcal{E}_i = \sum_{i=0}^{N-1} s_i^2[n]$, and select the hypothesis which resulted in the maximal correlator output. That is, pick the $\mathcal{H}_i$ for which the following is maximum:

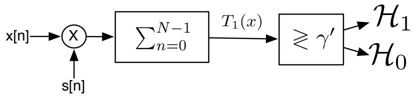
$$T_i(\mathbf{x}) := \sum_{n=0}^{N-1} x[n]s_i[n] - \frac{1}{2}\mathcal{E}_i \quad (1)$$

Let's derive this decision rule!

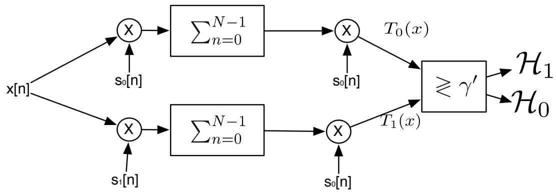
Geometric meaning?

Multiple Deterministic Signals in Gaussian Noise

$\mathcal{H}_0 : \ \mathbf{x} = \mathbf{w}$
$\mathcal{H}_1 : \ \mathbf{x} = \mathbf{s} + \mathbf{w}$



$\mathcal{H}_0 : \ \mathbf{x} = \mathbf{s_0} + \mathbf{w}$
$\mathcal{H}_1 : \ \mathbf{x} = \mathbf{s_1} + \mathbf{w}$



Geometric interpretation

$$T_i(\mathbf{x}) := \sum_{n=0}^{N-1} x[n]s_i[n] - \frac{1}{2}\mathcal{E}_i \tag{1}$$

This test statistic is identical to selecting the hypothesis $\mathcal{H}_i$ for which the Euclidean distance D_i^2 of the received vector to the known signal $s_i[n]$ is smallest, where D_i^2 is

$$\text{Minimum distance receiver:} \quad D_i^2 := \sum_{n=0}^{N-1} (x[n] - s_i[n])^2 = ||\mathbf{x} - \mathbf{s_i}||^2$$

Example: M-ary case

How would you choose which of M deterministic signals $\{s_0[n], s_1[n], \cdots, s_{M-1}[n]\}$, with equal prior probabilities, occurred upon receiving $x[n] = s_i[n] + w[n]$ under hypothesis $\mathcal{H}_i$?

Find the test statistic $T(x)$.

Find the probability of error P_e.