1.I) $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -3 & -2 \end{bmatrix}$ $\quad \det(\lambda I - A) = \lambda^3 + 2\lambda^2 + 3\lambda =$

$= \lambda(\lambda^2 + 2\lambda + 3)$

Autovalori : $\lambda_1 = 0 \quad \lambda_{2,3} = -1 \pm j\sqrt{2}$

Modi : $\mathbb{1}(t)$, $e^{-t}\cos(\sqrt{2}t)$, $e^{-t}\sin(\sqrt{2}t)$

Sistema marginalmente stabile

1.II) $F : \det(\lambda I - A - BF) = \lambda(\lambda^2 + 4) = \lambda^3 + 4\lambda$

$$A + BF = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ f_1 & -3+f_2 & -2+f_3 \end{bmatrix}$$

$$\det(\lambda I - A - BF) = \lambda^3 + (2 - f_3)\lambda^2 + (3 - f_2)\lambda - f_1 = \lambda^3 + 4\lambda$$

$$\Rightarrow f_3 = 2,\ f_2 = -1,\ f_1 = 0 \quad \Rightarrow F = [0 \quad -1 \quad 2]$$

1.III) $\begin{cases} \bar{x}_2 = 0 \\ \bar{x}_3 = 0 \\ -3\bar{x}_2 - 2\bar{x}_3 + \dfrac{\alpha - e^{-\beta \bar{x}_1}}{\alpha + e^{-\beta \bar{x}_1}} = 0 \end{cases} \Rightarrow e^{-\beta \bar{x}_1} = \alpha$

$\bar{x}_1 = -\frac{1}{\beta} \ln \alpha$

Stato di equilibrio : $\bar{x} = \begin{bmatrix} -\frac{1}{\beta}\ln\alpha \\ 0 \\ 0 \end{bmatrix}$

1.IV) $J = \dfrac{\partial f}{\partial x}\Big|_{x = \bar{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \delta(\bar{x}_1) & -3 & -2 \end{bmatrix}$

dove $\delta(\bar{x}_1) = \dfrac{-\beta e^{-\beta x_1}(\alpha + e^{-\beta x_1}) + \beta e^{-\beta x_1}(e^{-\beta x_1} - \alpha)}{(\alpha + e^{-\beta x_1})^2} =$

$$= \frac{-2\alpha\beta e^{-\beta x_1}}{(\alpha + e^{-\beta x_1})^2}\Bigg|_{e^{-\beta \bar{x}_1} = \alpha} = -\frac{2\alpha^2 \beta}{4\alpha^2} = -\frac{\beta}{2}$$

$$\det(\lambda I - J) = \lambda^3 + 2\lambda^2 + 3\lambda + \frac{\beta}{2}$$

$$\begin{array}{c|cc}
3 & 1 & 3 \\
2 & 2 & \frac{\beta}{2} \\
1 & \frac{6-\frac{\beta}{2}}{2} & \\
\emptyset & \frac{\beta}{2} &
\end{array}
\quad \Rightarrow \quad
\begin{cases} 6 - \frac{\beta}{2} > 0 \\ \frac{\beta}{2} > 0 \end{cases}
\qquad \beta \in (0, 12)$$

2.I)

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & \gamma & 0 & 0 \\ 1 & 0 & \gamma & 0 \\ 1 & 0 & 0 & \gamma \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \gamma & \gamma^2 \\ 0 & 1 & \gamma & \gamma^2 \\ 0 & 1 & \gamma & \gamma^2 \end{bmatrix} \qquad \mathcal{X}^R = \mathcal{L}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right\}$$

2.II)

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 2 & 4 \end{bmatrix} \qquad \det R = 2 \neq 0$$

$\bar{x} \notin \operatorname{Im} R_3 \Rightarrow$ numero minimo di passi $= 4$

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} u(3) \\ u(2) \\ u(1) \\ u(0) \end{bmatrix} \Rightarrow \begin{array}{l} u(0) = \frac{1}{2} \\ u(1) = -\frac{1}{2} \\ u(2) = \emptyset \\ u(3) = \emptyset \end{array}$$

2.III)
$$\begin{pmatrix}0\\0\\0\\1\end{pmatrix}=\begin{bmatrix}1&0&0&0&0\\0&1&0&0&0\\0&1&1&1&1\\0&1&2&4&8\end{bmatrix}\begin{bmatrix}u(4)\\u(3)\\u(2)\\u(1)\\u(0)\end{bmatrix}$$

$u(4)=0 \quad u(3)=0 \quad u(2)+u(1)+u(0)=\varnothing \quad 2u(2)+4u(1)+8u(0)=1$

$\rightarrow$ sequenze del tipo $u(0)=\alpha \quad u(1)=\dfrac{1-6\alpha}{2} \quad u(2)=\dfrac{4\alpha-1}{2}$

$u(3)=u(4)=\varnothing$

Affinché $|u(k)|\leq 1$ occorre:

$$\begin{cases}|\alpha|\leq 1\\ |1-6\alpha|\leq 2\\ |4\alpha-1|\leq 2\end{cases} \Rightarrow \begin{cases}-1\leq\alpha\leq 1\\ -2\leq 1-6\alpha\leq 2\\ -2\leq 4\alpha-1\leq 2\end{cases} \Rightarrow \begin{matrix}-1\leq\alpha\leq 1\\ -\frac{1}{6}\leq\alpha\leq\frac{1}{2}\\ -\frac{1}{4}\leq\alpha\leq\frac{3}{4}\end{matrix}$$

$\Rightarrow \forall\alpha\in\left[-\frac{1}{6},\frac{1}{2}\right]$ (ad es., $\alpha=0$)

3.I)
$$\begin{cases}\dot{q}(t)=k\,u(t)=k\,(r(t)-w(t))\\ \dot{y}(t)+y(t)=q(t)\\ \dot{w}(t)+3w(t)=y(t)\end{cases}$$

$$\begin{bmatrix}\dot{q}(t)\\\dot{y}(t)\\\dot{w}(t)\end{bmatrix}=\begin{bmatrix}\varnothing&\varnothing&-k\\1&-1&0\\\varnothing&1&-3\end{bmatrix}\begin{bmatrix}q(t)\\y(t)\\w(t)\end{bmatrix}+\begin{bmatrix}k\\\varnothing\\\varnothing\end{bmatrix}r(t)$$

$$y(t)=\begin{bmatrix}\varnothing&1&\varnothing\end{bmatrix}\begin{bmatrix}q(t)\\y(t)\\w(t)\end{bmatrix}+[\varnothing]\,r(t)$$

3.II) $\dot{\hat{x}}(t)=A\hat{x}(t)+Br(t)+L(y(t)-C\hat{x}(t))$

con L: $\det(\lambda I-A+LC)=(\lambda+1)(\lambda+2)(\lambda+3)$

Ponendo $k=1$, in MATLAB si ha

```
A=[0 0 -1; 1 -1 0; 0 1 -3];
C=[0 1 0];
L = place(A', C', [-1 -2 -3])'
```

restituisce $L = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$.

3.III) $W(s) = \dfrac{\frac{k}{s(s+1)}}{1 + \frac{k}{s(s+1)(s+3)}} = \dfrac{k(s+3)}{s^3 + 4s^2 + 3s + k}$

$$\begin{array}{c|cc} 3 & 1 & 3 \\ 2 & 4 & k \\ 1 & 12-k & \\ 0 & k & \end{array} \quad \Rightarrow \quad k \in (0, 12)$$

$$|W(j)| = \frac{k \cdot |j+3|}{|k-4+j(3-1)|} = \frac{k\sqrt{10}}{\sqrt{(k-4)^2+4}}$$

$$|W(j)| > 3 \quad \Rightarrow \quad \frac{k^2 \cdot 10}{(k-4)^2 + 4} > 9$$

$$10k^2 > 9k^2 - 72k + 180$$

$$k^2 + 72k - 180 > 0 \quad \Rightarrow \quad k > -36 + \sqrt{1476} \cong 2.42$$

Conclusione : $\boxed{k \in (2.42,\ 12)}$

3.IV) $W(s) = \dfrac{12(s+3)}{s^3+4s^2+3s+12}$

Utilizziamo MATLAB per tracciare il diagramma di Bode

```
s = tf('s');
W= 12*(s+3)/(s^3+4*s^2+3*s+12);
bode(W)
```

Dal grafico si osserva che $|W(j\omega)| \geq 0\,dB$ $\forall\, \omega \in (0,\ 3{,}67)$.