Università di Siena

Teoria della Stima Set Membership: Modelli Parametrici

Andrea Garulli e Antonio Vicino

Dipartimento di Ingegneria dell'Informazione
Centro per lo Studio dei Sistemi Complessi
Università di Siena
www.dii.unisi.it/∼control

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Outline

- ◊ Problem formulation (IBC)
- ◊ Examples of estimation problems
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Set Membership estimation: the Information Based Complexity framework

Set Membership estimation: problem formulation

ESTIMATION PROBLEM

> *Given an unknown element x, find an estimate of $S(x)$ based on a priori information K and measurements of $F(x)$, corrupted by additive noise e.*

Main hypothesis:

Unknown-But-Bounded (UBB) noise

Noise e takes values in an assigned bounded set of the measurement space.

Information Based Complexity (IBC) framework

- $x \in X$: problem element
- $z = S(x) \in Z$: problem solution (to be estimated)
- Available information

  a priori: $x \in K \subseteq X$

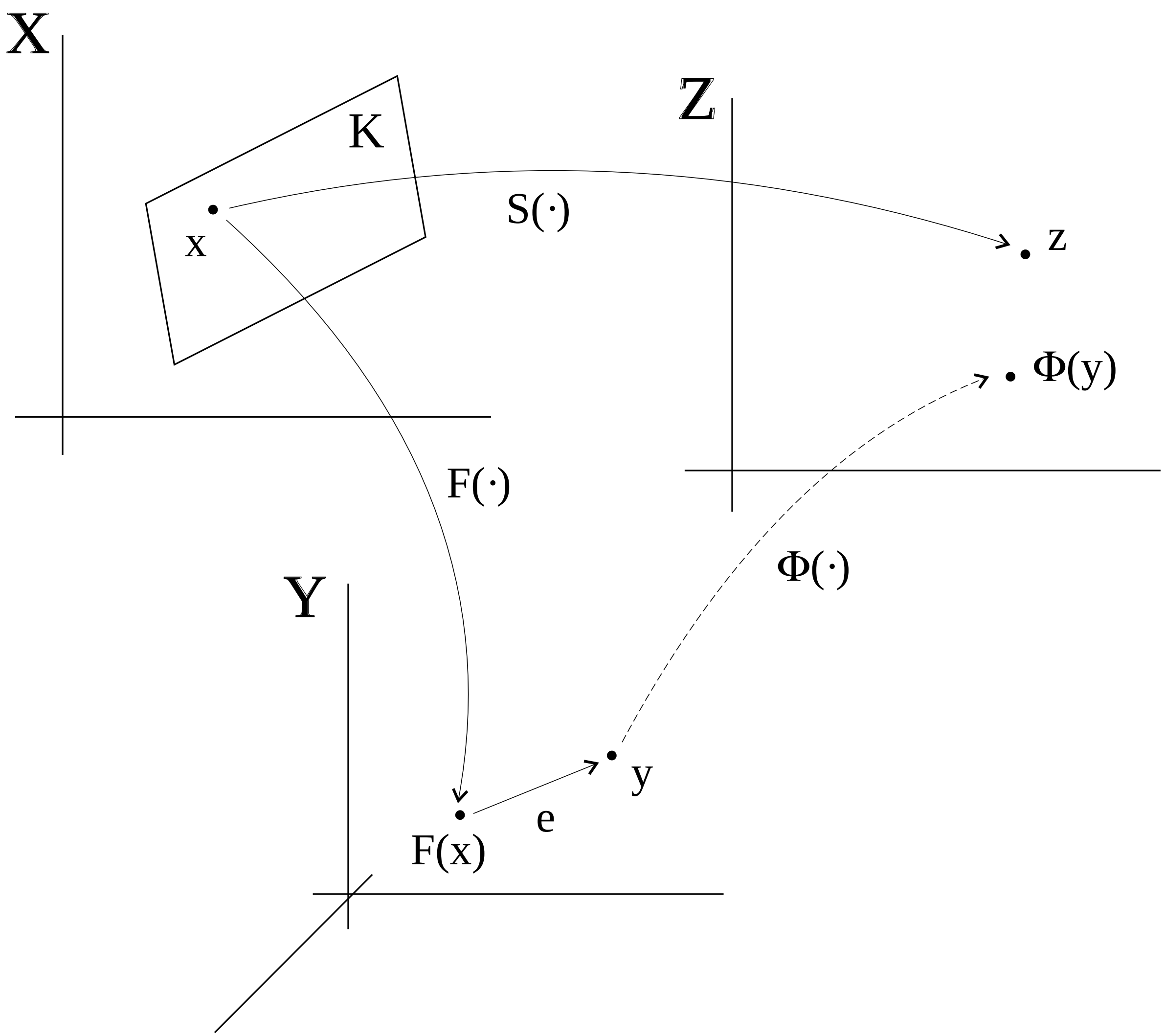
  a posteriori (measurements): $y = F(x) + e\ , \quad F(\cdot) : X \to Y$
- $e \in Y$: unknown-but-bounded (UBB) measurement noise

$$\|e\|_Y \leq \varepsilon$$

  with given $\varepsilon > 0$
- Estimation operator : $\Phi(\cdot) :\ Y \to Z$ such that

$$\Phi(y) \approx S(x)$$

IBC framework



Membership sets

Feasible (Problem element) Set:

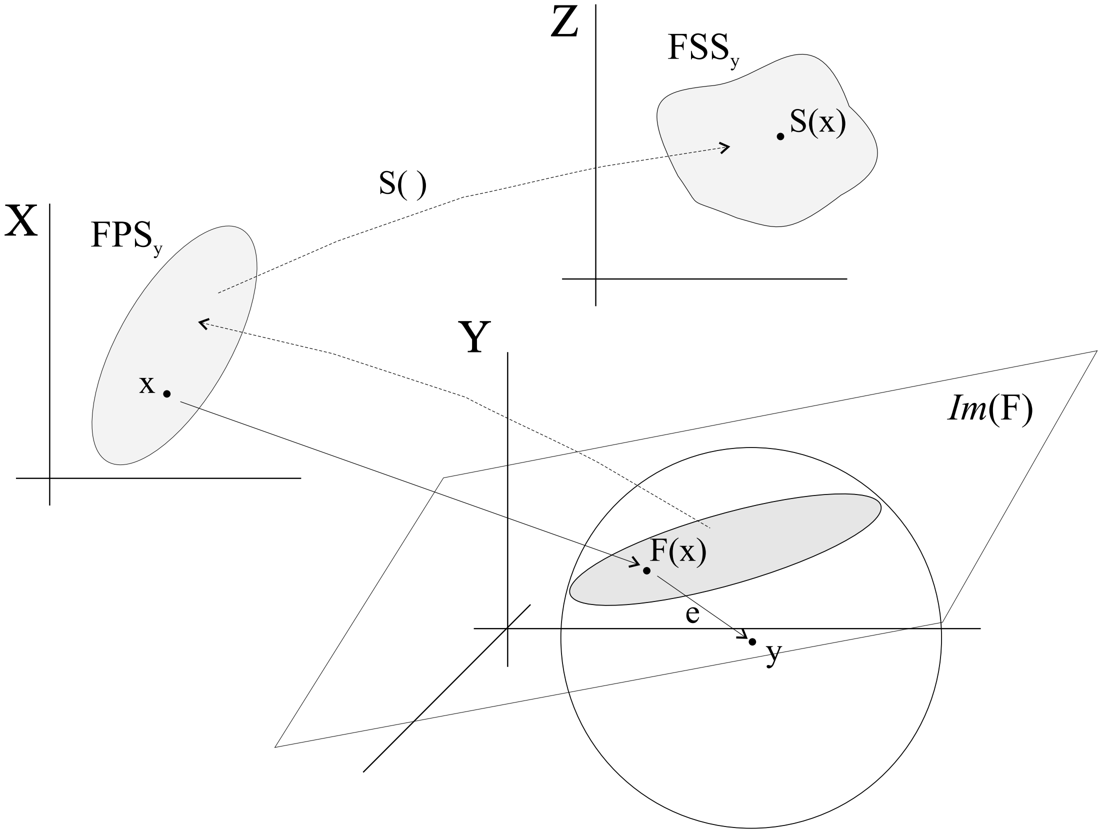
problem elements compatible with all the available information

$$FPS_y = \{x \in K : \; \|y - F(x)\|_Y \leq \varepsilon\}$$

Feasible Solution Set:

solutions consistent with the information

$$FSS_y = S(FPS_y)$$



Membership sets

General case $\longrightarrow$ complicated structure (nonconvex, disconnected,...)

Linear case: $F(x) = Fx, \;\; S(x) = x$

$\longrightarrow$ $\|\cdot\|_Y$-norm in the UBB noise assumption

$\|\cdot\|_Y$	FPS_y
ℓ_∞	polytope
ℓ_2	ellipsoid
ℓ_1	polytope

Example #1: parameter identification of ARX model (1/2)

ARX model

$$y_k = -\sum_{i=1}^{n_a} a_i y_{k-i} + \sum_{i=1}^{n_b} b_i u_{k-i} + e_k$$

with (weighted) ℓ_∞ bounded noise

$$|e_k| < \varepsilon_k \quad \forall\, k$$

Estimate the parameters a_i, b_i from measurements $\{y_k, u_k\}$, $k = 1, \ldots, N$.

In the IBC framework:

- problem element: $x = [a_1 \ \ldots \ a_{n_a} \ b_1 \ \ldots \ b_{n_b}]'$
- measurement vector: $y = [y_1 \ \ldots \ y_N]'$
- noise vector: $e = [e_1 \ldots e_N]'$, with $\|e\|_\infty^w \leq 1$

Example #1: parameter identification of ARX model (2/2)

- information operator

$$F(x) = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_N \end{pmatrix} x.$$

where $\phi_k = [-y_{k-1} \ \ldots \ -y_{k-n_a} \ \ u_{k-1} \ \ldots \ u_{k-n_b}]$.

Here $S(x) = x$, and $FSS_y = FPS_y$ is a convex polytope:

$$FPS_y = \{x : \ |y_k + \sum_{i=1}^{n_a} a_i y_{k-i} - \sum_{i=1}^{n_b} b_i u_{k-i}| \leq \varepsilon_k, \ k = 1, \ldots, N\}$$

Example #2: state estimation of linear systems (1/3)

Linear time-varying system

$$\begin{aligned} x_{k+1} &= A_k\, x_k + w_k \\ y_k &= C_k\, x_k + v_k \end{aligned}$$

with disturbances

a) ℓ_∞ bounded: $\|w_k\|_\infty \le \alpha_k, \quad \|v_k\|_\infty \le \beta_k, \quad \|x_0 - \hat{x}_0\|_\infty \le \gamma$

b) energy bounded: $\|x_0 - \hat{x}_0\|_2^2 + \sum_{k=0}^{N-1} \|w_k\|_2^2 + \sum_{k=0}^{N} \|v_k\|_2^2 \le \varepsilon^2$

In the IBC framework

- problem element: $x = [x_0' \ x_1' \ \ldots \ x_N']'$
- measurement vector: $y = [\hat{x}_0' \ 0 \ \ldots \ 0 \ y_0' \ \ldots \ y_N']'$
- noise vector: $e = [(\hat{x}_0 - x_0)' \ w_0' \ \ldots \ w_{N-1}' \ v_0' \ \ldots \ v_N']'$

Example #2: state estimation of linear systems (2/3)

- information operator

$$F(x) = \left(\begin{array}{ccccc} I_{n_x} & 0 & \cdots & & \\ A_0 & -I_{n_x} & 0 & \cdots & \\ 0 & A_1 & -I_{n_x} & \cdots & \\ & & \ddots & \ddots & \\ & & & A_{N-1} & -I_{n_x} \\ \hline C_0 & 0 & \cdots & & \\ 0 & C_1 & 0 & \cdots & \\ & 0 & C_2 & \cdots & \\ & & \ddots & \ddots & \\ & & & 0 & C_N \end{array}\right) x.$$

- solution: $x_N = S(x) = [0 \ \ldots \ 0 \ I_{n_x}]\, x$

Here FSS_y and FPS_y are: a) polytopes
b) ellipsoids

Example #2: state estimation of linear systems (3/3)

Recursive construction of FSS_y for ℓ_∞ bounded noise:

- A priori information:

$$x_0 \in \Xi(0|-1) = \hat{x}_0 + \gamma\,\mathcal{B}_\infty$$

  $\mathcal{B}_\infty$: unit ball in the ℓ_∞ norm

- For $k = 0, \ldots, N$

$$\begin{aligned} \Xi(k|k) &= \Xi(k|k-1) \bigcap \{x : \|y_k - C_k\,x\|_\infty \leq \beta_k\} \\ \Xi(k+1|k) &= A\,\Xi(k|k) + \alpha_k\,\mathcal{B}_\infty \end{aligned}$$

- $FSS_y = \Xi(N|N)$

$\bigcap$, $+$: set operators

Example #3: $\mathcal{H}_\infty$ identification (1/2)

Linear SISO system

$$H(z) = \sum_{j=0}^{\infty} h_j z^j$$

a) A priori knowledge

$$H(z) \in K = \{H(z) \in \mathcal{H}_\infty : \ |h_j| \leq M\rho^{-j}\}$$

b) Frequency measurements

$$y_k = H(e^{j2\pi k/N}) + v_k \qquad k = 1, \ldots, N$$

with bounded noise $|v_k| \leq \varepsilon_k$.

In the IBC framework

- problem element: $x = h \triangleq \{h_j\}_{j=0}^{\infty}$
- measurement vector: $y = [y_1 \ldots y_N]' \in \mathbb{C}^N$

Example #3: $\mathcal{H}_\infty$ identification (2/2)

- noise vector: $e = [v_1 \dots v_N]'$, with $\|e\|_\infty^w \leq 1$
- information operator

$$F(h) = [H(e^{j2\pi k/N}),\ k = 1, \dots, N] = \begin{pmatrix} 1 & z_1 & z_1^2 & \dots \\ 1 & z_2 & z_2^2 & \dots \\ \vdots & \vdots & \vdots & \vdots \\ 1 & z_N & z_N^2 & \dots \end{pmatrix} h$$

with $z_k = e^{j2\pi k/N}$.

Here $S(h) = h$, and the set of feasible systems is given by

$$FPS_y = \{h \in K : \ \|F(h) - y\|_\infty^w \leq 1\}$$

and it is a bounded convex set in an infinite dimensional space.

Class of estimators

✓ **Set-valued estimators**

Return a set in the solution space

$$\Phi(y) \;=\; \Omega \;\subset\; Z$$

✓ **Pointwise estimators**

Return a single element in the solution space

$$\Phi(y) \;=\; \hat{z} \;\in\; Z$$

Set-valued estimators

Set-valued estimators

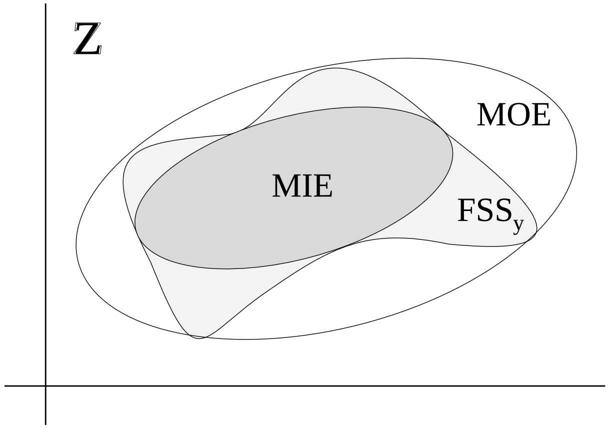
A set-valued estimator provides a set in the solution space

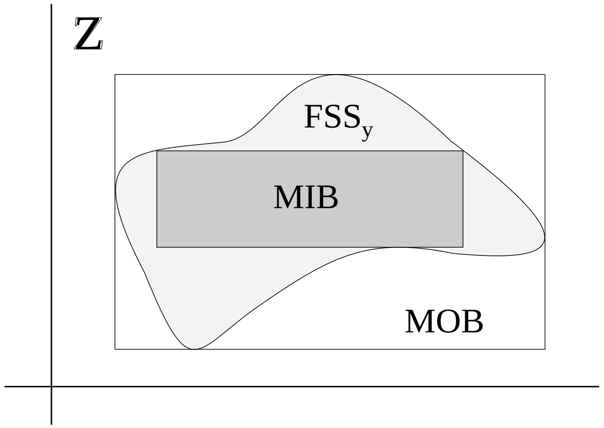
$$\Phi(y) = \Omega \subset Z$$

✓ Exact algorithms

$$\Phi(y) = FSS_y$$

✓ Approximation algorithms
- inner/outer
- approximating regions





Exact algorithms: energy bounded noise

Linear setting, ℓ_2 bounded noise $\Rightarrow$ FPS_y is an ellipsoid
(rank$(F) = n$)

$$FPS_y = \{x : (x - x^c)'(F'F)(x - x^c) \leq \varepsilon^2 - d^2\}$$

where

$$x^c = (F'F)^{-1}F'y$$

$$\begin{aligned} d &= (y'y - y'F(F'F)^{-1}F'y)^{1/2} \\ &= \text{distance of } y \text{ from the subspace } \mathcal{R}(F) \end{aligned}$$

$$FPS_y \neq \emptyset \;\Leftrightarrow\; \varepsilon \geq d$$

Exact algorithms: instantaneous bounds

Linear setting, ℓ_∞ bounded noise $\Rightarrow$ FPS_y is a polytope

$$FPS_y = \{x : \|y - Fx\|_\infty \leq \varepsilon\} = \bigcap_{i=1}^{N} \mathcal{S}_i$$

where $\mathcal{S}_i = \{x : \ |y_i - f_i' x| \leq \varepsilon\}$.

Let $FPS_y^k = \bigcap_{i=1}^{k} \mathcal{S}_i$, then $\quad FPS_y^k = FPS_y^{k-1} \bigcap \mathcal{S}_k$

Recursive characterization of a polytope: $(FPS_y^{k-1}, \mathcal{S}_k) \rightarrow FPS_y^k$

Several solutions: - edge algorithm `(Walter & Pronzato)`

- facet algorithm `(Broman & Shensa)`
- vertex algorithm `(Mo & Norton)`

Approximate algorithms

Recursive outer approximation problem

Given:

- *a class* $\mathcal{R}$ *of approximating regions*
- *a set* $\mathcal{R}_{k-1} \in \mathcal{R}$ *such that*

$$\mathcal{R}_{k-1} \supseteq FPS_y^{k-1} \ ,$$

find a set $\mathcal{R}_k \in \mathcal{R}$ *satisfying*

$$\mathcal{R}_k \supseteq \mathcal{R}_{k-1} \bigcap \mathcal{S}_k$$

and hence $\mathcal{R}_k \supseteq FPS_y^k$.

Crucial issue: choice of $\mathcal{R}$

Recursive outer approximation

✓ Choices of $\mathcal{R}$:

– ellipsoids `(Fogel & Huang, Dasgupta & Huang, Deller)`

– parallelotopes `(Vicino & Zappa, Chisci et al.)`

– limited-complexity polytopes `(Broman & Shensa, Chisci et al.)`

✓ Criterion used for computing $\mathcal{R}_k$

minimize the "size" of $\mathcal{R}_k$

Typical measure for the "size" of $\mathcal{R}_k$ is the *volume*.

✓ Computation of $\mathcal{R}_k \;\Rightarrow\;$ optimization problems

✓ *Suboptimality*: minimizing the volume at each time step does not provide the minimum volume set in the class $\mathcal{R}$ containing the exact FPS_y (but dramatically reduces the computational complexity!)

Ellipsoidal approximations

An ellipsoid is defined as

$$\mathcal{E}(x^c, \Sigma) = \{x : \ (x - x^c)'\Sigma^{-1}(x - x^c) \leq 1\}$$

- x^c: center
- $\Sigma = \Sigma' > 0$
- The volume of $\mathcal{E}(x^c, \Sigma)$ is given by: $\quad \mathrm{vol}(\mathcal{E}) = \dfrac{\pi^{n/2}}{\gamma(\frac{n}{2} + 1)}\sqrt{\det \Sigma}$

Approximation Problem:

Let

$$\begin{aligned} \mathcal{E}_{k-1} \supseteq FPS_y^{k-1} \quad &: \quad \text{approximation at time } k - 1 \\ y_k \quad &: \quad \text{measurement at time } k \end{aligned}$$

Solve

$$\begin{aligned} &\min \ \mathrm{vol}(\mathcal{E}_k) \\ &\ \text{s.t.} \\ &\mathcal{E}_k \supseteq (\mathcal{E}_{k-1} \bigcap \mathcal{S}_k) \end{aligned}$$

A generalized algorithm (Fogel & Huang, Deller et al.)

Let

$$\begin{aligned}\mathcal{E}_{k-1} &= \{x:\ (x-x^c_{k-1})'\Sigma^{-1}_{k-1}(x-x^c_{k-1})\le 1\}\\ \mathcal{S}_k &= \{x:\ |y_k - f'_k x|\le\varepsilon\}\end{aligned}$$

where $\Sigma_{k-1}=\eta_{k-1}P_{k-1}$ (for $k=1$: $\eta_0=1,\ P_0=\Sigma_0$).

Procedure:

$$\begin{aligned}x^c_k &= x^c_{k-1}+\beta_k P_k f_k \nu_k && \longleftarrow \text{ RLS-like updating eqn.}\\ \Sigma_k &= \eta_k P_k\end{aligned}$$

where

$$\begin{aligned}P^{-1}_k &= \alpha_k P^{-1}_{k-1}+\beta_k f_k f'_k\\ \eta_k &= \alpha_k\eta_{k-1}+\beta_k\varepsilon^2-\frac{\alpha_k\beta_k\nu_k}{\alpha_k+\beta_k\mu_k}\\ \nu_k &= y_k - f'_k x^c_{k-1} && \longleftarrow \text{ one step prediction error}\\ \mu_k &= f'_k P_{k-1} f_k\\ \alpha_k &> 0 && \longleftarrow \text{ forgetting factor}\\ \beta_k &> 0 && \longleftarrow \text{ data weighting factor}\end{aligned}$$

Then

$$\mathcal{E}_k=\{x:\ (x-x^c_k)'\Sigma^{-1}_k(x-x^c_k)\le 1\}\supseteq\mathcal{E}_{k-1}\bigcap\mathcal{S}_k.$$

Ellipsoidal approximations: comments

◊ Existing procedures differ in two aspects

- optimization criterion for the choice of $\mathcal{E}_k$
- choice of sequences $\{\alpha_k\}$, $\{\beta_k\}$

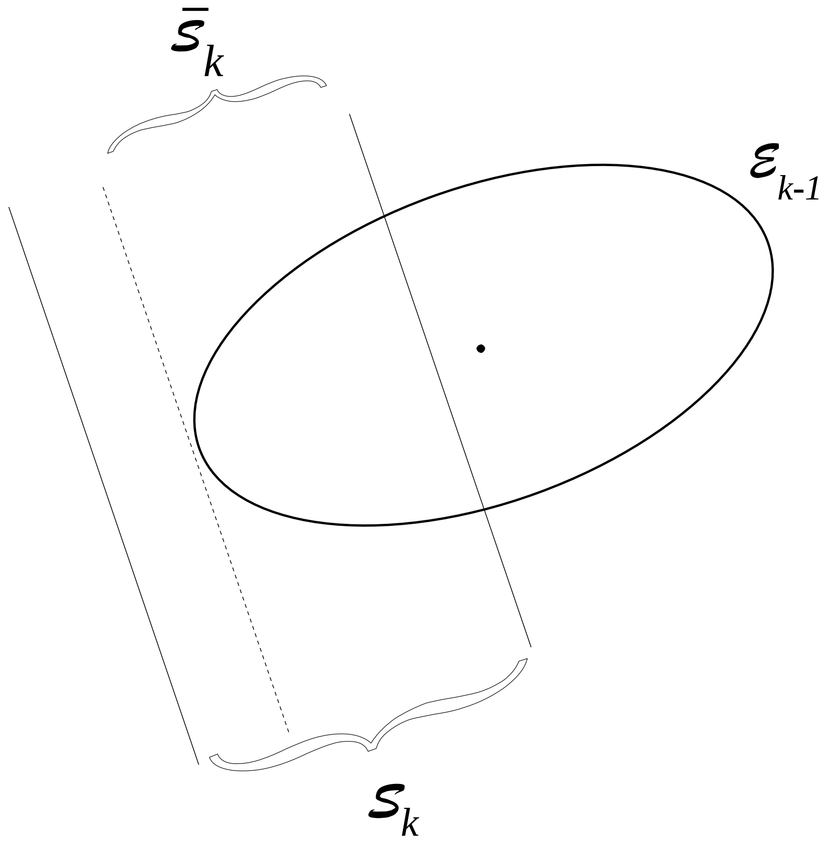
Algorithm	$\alpha_k(\lambda_k)$	$\beta_k(\lambda_k)$
`Fogel & Huang`	$1/\eta_{k-1}$	λ_k/ε^2
`Dasgupta & Huang`	$1-\lambda_k$	λ_k
`Deller`	1	λ_k

The free parameter λ_k is computed according to the selected criterion

◊ In `(Fogel & Huang)`, λ_k is chosen to minimize $\det \Sigma_k$ or $\mathrm{tr}\, \Sigma_k$.

◊ In `(Dasgupta & Huang)` and `(Deller)` different criteria are adopted which are finalized to "improve" the asymptotic behaviour of the sequence $\{x_k^c\}$ or $\mathrm{vol}(\mathcal{E}_k)$
(e.g.: $\lim_{k\to\infty} \nu_k^2 \leq \varepsilon^2$, $\lim_{k\to\infty} \lambda_k = 0$, $\lim_{k\to\infty} \mathrm{vol}(\mathcal{E}_k) = 0$).

◊ The generalized algorithm does not guarantee that $\mathcal{E}_k$ is the minimum volume ellipsoid containing $\mathcal{E}_{k-1} \bigcap \mathcal{S}_k$.
This problem can be overcome by an appropriate replacement of certain strips $\mathcal{S}_k$ with equivalent $\bar{\mathcal{S}}_k$.



Parallelotopic approximations

Key idea: the information provided by the i-th measurement is a "strip" in the problem element space $\mathcal{S}_i = \{x : \ |y_i - f_i' x| \leq \varepsilon\}$.

A *parallelotope* is defined as

$$\mathcal{P}(x^c, P) = \{x : \ \|P(x - x^c)\|_\infty \leq 1\}, \qquad \det P \neq 0$$

where

$$\begin{array}{l} x^c \ : \ \text{center} \\ \text{columns of } T = P^{-1} \ : \ \text{edges} \end{array}$$

$\mathcal{P}(x^c, P)$ is the intersection of n strips, where $n = \dim(X)$.

Recursive Optimization: among all the parallelotopes satisfying $\mathcal{P}_k \supseteq \mathcal{P}_{k-1} \bigcap \mathcal{S}_k$, select the minimum volume one, i.e. solve

$$\begin{array}{l} \max \ |\det P_k| \\ \text{ s.t.} \\ \mathcal{P}(x_k^c, P_k) \supseteq \mathcal{P}(x_{k-1}^c, P_{k-1}) \bigcap \mathcal{S}_k \end{array}$$

Parallelotopic approximations

The procedure is based on two steps

- strip tightening
- selection of n out of $n+1$ strips with minimal volume intersection

Comments:

◊ The parallelotopic approximation algorithm has the same computational complexity of standard recursive identification algorithms (RLS, generalized ellipsoidal algorithm, etc.), i.e. $O(n^2)$ operations at each iteration

◊ Parallelotopes have more degrees of freedom than ellipsoids:

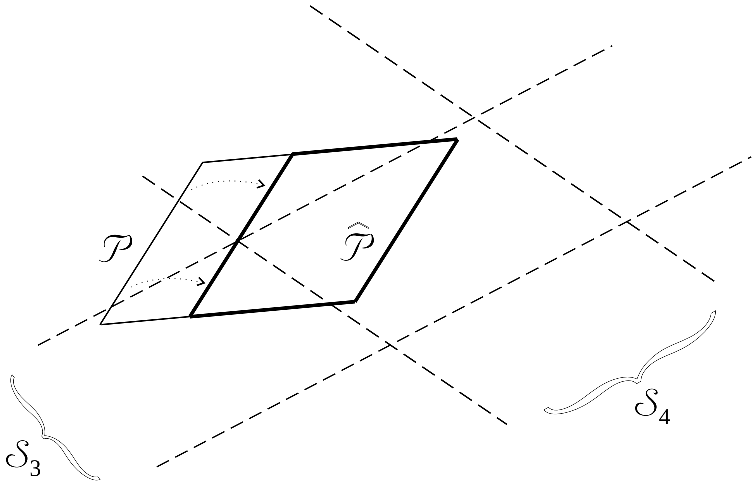
$$\# \text{ of free parameters: } \begin{cases} n^2 + n & \text{(parallelotope)} \\ \dfrac{n(n+1)}{2} + n & \text{(ellipsoid)} \end{cases}$$

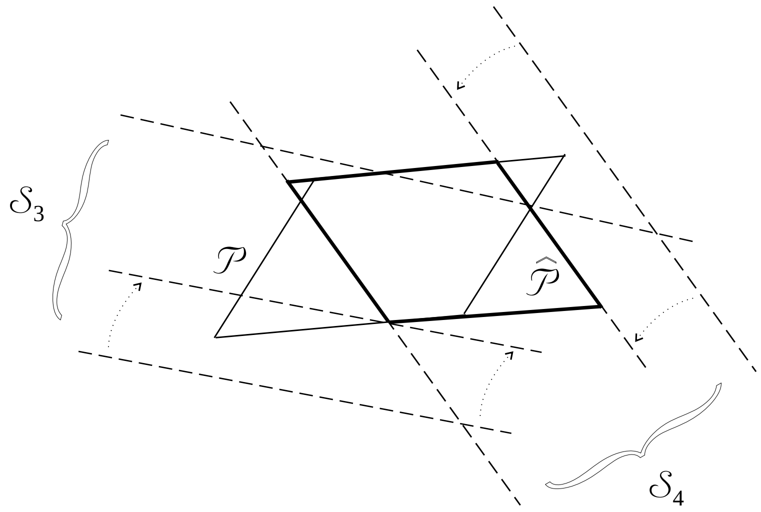
Parallelotopic approximations: properties and results

- Recursive optimization: closed form solution (no linear programming)
- Low computational complexity (equivalent to least-squares, Kalman filter, etc..)
- More accurate approximations with respect to algorithms of the same complexity (ellipsoids, boxes)
- *Block processing* of two or more measurements gives remarkable volume reductions (trade-off approximation and complexity)
- Applications:
  - Parameter identification of ARX models
  - Errors-in-variables identification
  - Recursive state estimation

Parallelotopic approximations: block processing

Key idea: processing more than one strip at a time provides a remarkable improvement of the approximation ($\rightarrow$ uncertainty reduction)





ROBPq algorithm

At time k, let the parallelotope $\mathcal{P}_{k-1} \supset FPS_y^{k-1}$ and the strips $\mathcal{S}_k^1, \ldots, \mathcal{S}_k^q$ be given.

i) *Tightening*. Tighten the $n+q$ strips of the polytope $\mathcal{P}_{k-1} \bigcap \mathcal{S}_k^1 \bigcap \cdots \bigcap \mathcal{S}_k^q$.

ii) *Selection*. Among the $n+q$ tightened strips, select the n giving the minimal volume parallelotope $\mathcal{P}_k$.

Batch approximation algorithms

◊ Optimal outer approximation of a polytope in the ℓ_∞ norm requires the solution of $2n$ LP problems `(Milanese & Belforte)`

◊ Batch approximations can be very difficult to compute
Example: computing the minimum volume ellipsoid containing a polytope described by linear inequalities is NP-hard `(Boyd et al.)`

◊ Inner approximations:

- useful to evaluate the level of goodness of outer approximations
- (weighted) ℓ_2 or ℓ_∞ norm approximations require the solution of an LP problem, provided weights are fixed
- if only the orientation is fixed, convex optimization allows to solve the problem
- nonconvex (polynomial) optimization is needed if the weights are free `(Vicino & Milanese)`.

Pointwise estimators

Pointwise estimators

A pointwise estimator provides a single element in the solution space

$$\Phi(y) = \hat{z} \in Z$$

- Central estimators (*Chebishev center*)

$$\Phi_c(y) = \mathrm{cen}(FSS_y) \triangleq \arg \inf_{z \in Z} \sup_{\tilde{z} \in FSS_y} \|z - \tilde{z}\|_Z$$

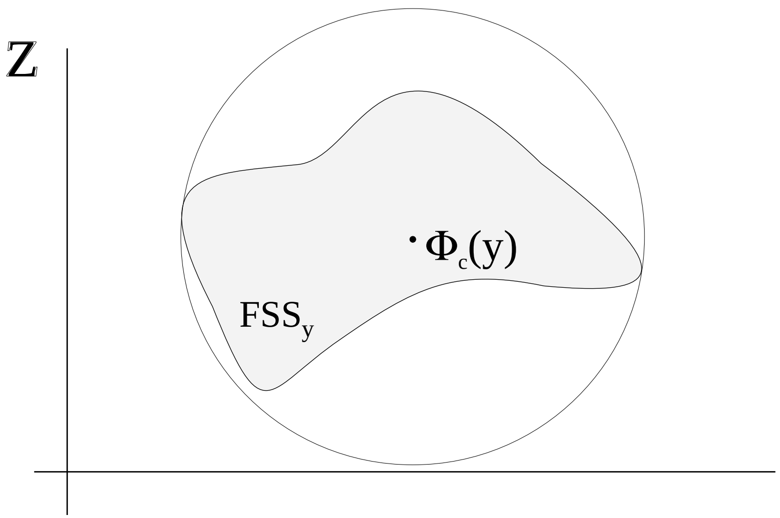
- Projection estimators

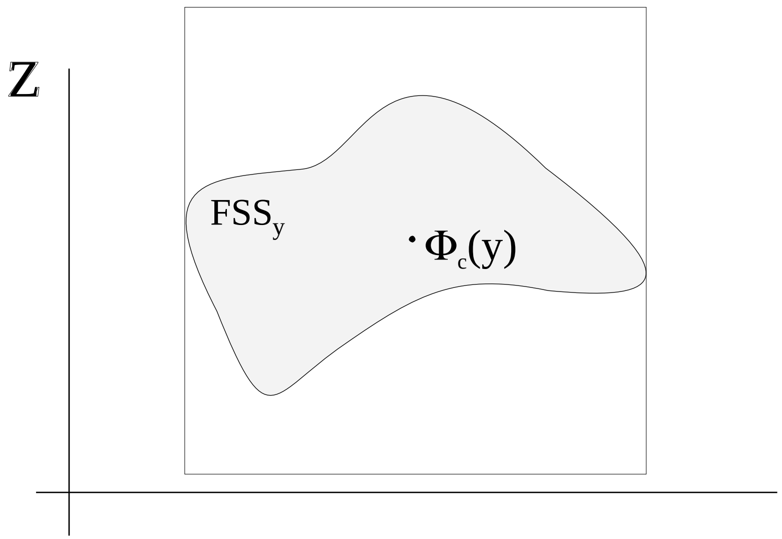
$$\Phi_p(y) = S(x_p), \qquad x_p = \arg \min_{x \in K} \|y - F(x)\|_Y$$

- Interpolatory estimators

$$\Phi_p(y) = S(x_i), \qquad x_i \in FPS_y$$

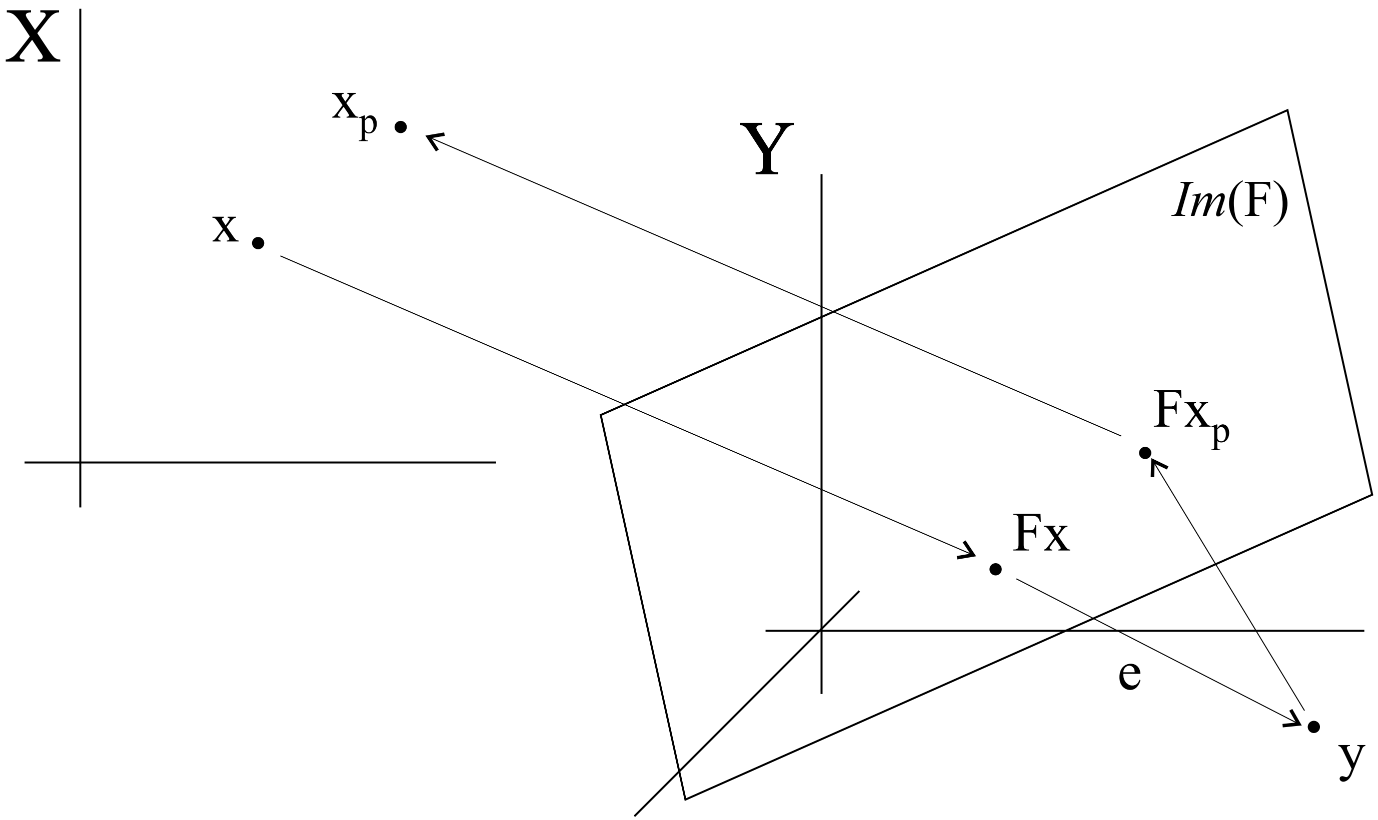
Central estimators for different norms





$\|\cdot\|_Z = \ell_2$ $\qquad$ $\|\cdot\|_Z = \ell_\infty$

Projection estimator



$$\|\cdot\|_Y = \ell_2 \ , \quad S(x) = x$$

Worst-case estimation errors

Local error: $$E_y(\Phi,\varepsilon) = \sup_{x\in FPS_y} \|S(x)-\Phi(y)\|_Z$$

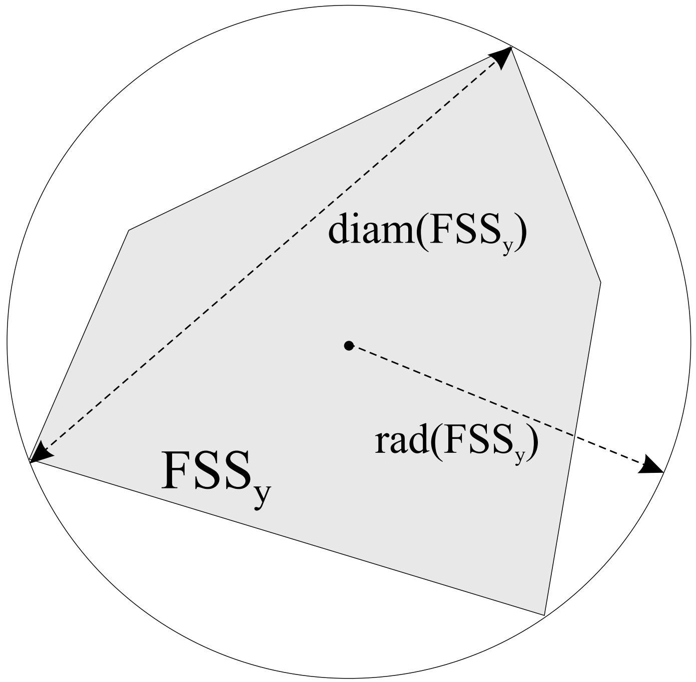
Global error: $$E(\Phi,\varepsilon) = \sup_{y\in Y_0} E_y(\Phi,\varepsilon) \qquad Y_0 = \{y :\ FPS_y \neq \emptyset\}$$

Radius of information:

$$R(\varepsilon) \triangleq \inf_{\Phi} E(\Phi,\varepsilon) = \sup_{y\in Y_0} \mathrm{rad}(FSS_y)$$

Diameter of information:

$$D(\varepsilon) = \sup_{y\in Y_0} \sup_{x_1,x_2\in FPS_y} \|S(x_1)-S(x_2)\|_Z = \sup_{y\in Y_0} \mathrm{diam}(FSS_y)$$



Properties of set membership estimators

Optimality:

$$\text{local} \longrightarrow E_y(\Phi^*, \varepsilon) \leq E_y(\Phi, \varepsilon) \quad \forall y \in Y_0, \ \forall \Phi$$

$$\text{global} \longrightarrow E(\Phi^*, \varepsilon) \leq E(\Phi, \varepsilon) \quad \forall \Phi$$

local optimality $\Rightarrow$ global optimality

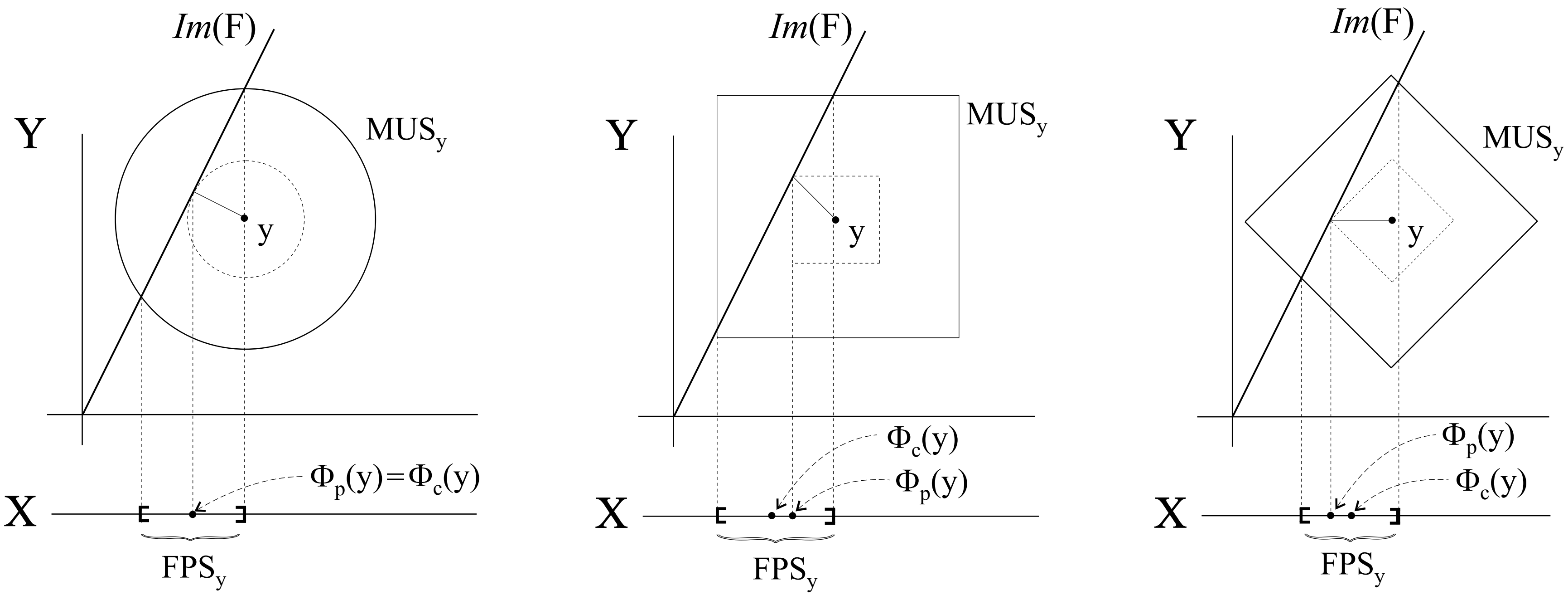
Almost-optimality (k-optimality): $\quad E(\Phi, \varepsilon) \leq k \cdot R(\varepsilon) \qquad (k > 1)$

Correctness: $\quad \Phi(F(x)) = S(x) \quad \forall x \in X$

Robust Convergence: $\quad \lim_{\varepsilon \to 0} \lim_{m \to \infty} E(\Phi, \varepsilon, m) = 0 \qquad m = \dim\{y\}$

Example: projection estimators

$$X = \mathbb{R}, \quad Y = \mathbb{R}^2, \quad \begin{cases} y_1 = x + e_1 \\ y_2 = 2x + e_2 \end{cases} \quad S(x) = x$$



$\|e\|_2 \le \varepsilon$ | $\|e\|_\infty \le \varepsilon$ | $\|e\|_1 \le \varepsilon$

$$E(\Phi_c, \varepsilon) = E(\Phi_p, \varepsilon) = \frac{1}{\sqrt{5}}\varepsilon$$

$$E(\Phi_c, \varepsilon) = \frac{1}{2}\varepsilon \qquad E(\Phi_p, \varepsilon) = \frac{2}{3}\varepsilon$$

$$E(\Phi_c, \varepsilon) = \frac{1}{3}\varepsilon \qquad E(\Phi_p, \varepsilon) = \frac{1}{2}\varepsilon$$

Pointwise estimators: general results

✓ For any norm in Y and Z, a *central* estimator Φ_c is locally optimal

$$E_y(\Phi_c, \varepsilon) \leq E_y(\Phi, \varepsilon) \quad \forall y, \ \forall \Phi$$

and hence also globally optimal.

✓ For a *projection* estimator Φ_p

$$E_y(\Phi_p, \varepsilon) \leq 2\,\mathrm{rad}(FSS_y) \leq 2\,E_y(\Phi, \varepsilon) \quad \forall y, \ \forall \Phi, \ \forall \varepsilon.$$

hence Φ_p is 2-optimal.

✓ All *interpolatory* estimators are 2-optimal.
In particular: Φ_p is an interpolatory estimator.

Pointwise estimators: linear setting

Let $F(x) = Fx$, $F \in \mathbb{R}^{m\times n}$, $m > n$ and $S(x) = Sx$, $S \in \mathbb{R}^{p\times n}$

✓ For any norm $\|\cdot\|_Y$ such that FPS_y has a symmetry center, the projection estimator Φ_p is linear, correct, central, optimal (in any sense).

Corollary [Hilbert norms]
If $\|\cdot\|_Y$ is a Hilbert norm, the least squares estimator $\Phi_p = \Phi_{LS}$ is correct, central, optimal (in any sense).
When F is full rank

$$\Phi_p(y) = \Phi_{LS}(y) \stackrel{\triangle}{=} S(F'F)^{-1}F'y$$

Optimality of pointwise estimators

Estimator	Y-norm	Z-norm	Y-local optimality	global optimality
central estimator	general	general	yes (by def.)	yes
ℓ_2 projection (LS)	ℓ_2	general	yes	yes
ℓ_1 projection	ℓ_1	whatever	no	no
ℓ_∞ projection	ℓ_∞	whatever	no	no
S projection	convex norm S	whatever	no	no

Remark: “no” means “not generally true”, as counterexamples can be found.

Pointwise estimators: convergence results

Let $F(x) = F_m x, \;\; F_m \in \mathbb{R}^{m \times n}, \;\; m > n.$

Sufficient information: $\exists m_0$ s.t. $\mathrm{rank}(F_{m_0}) = n$. Then

$$\lim_{\varepsilon \to 0} \lim_{m \to \infty} R(\varepsilon, m) = 0$$

Consequences:

✓ A globally optimal or almost-optimal algorithm is robustly convergent.

✓ Central estimators, projection estimators, interpolatory estimators, are all robustly convergent.

Example: Robust identification (1/2)

Linear SISO system

$$H(z) = \sum_{k=0}^{\infty} h_k z^k$$

a) A priori knowledge

Ex.: exponential decay

$$H(z) \in K = \{H(z): \ |h_j| \leq M\rho^{-j}\}$$

b) Measurements

Ex.: frequency domain $\quad \eta_k = H(e^{j2\pi k/N}) + w_k \qquad k = 1, \ldots, N$

time domain $\quad y_t = \sum_{i=0}^{t} h_i u_{t-i} + v_t \qquad t = 0, \ldots, N-1$

c) bounded noise

Ex.: weighted ℓ_∞ bounds

$$|w_k| \leq \varepsilon_k \qquad |v_t| \leq \delta_t$$

Example: Robust identification (2/2)

In the IBC framework

- problem element: $x = h \triangleq \{h_k\}_0^\infty$
- measurement vector: $y = [\eta_1 \ldots \eta_N]' \in \mathbb{C}^N$ or $y = [y_0 \ldots y_{N-1}]' \in \mathbb{R}^N$
- information operator: $F(h) = [H(e^{j2\pi k/N}),\ k = 1, \ldots, N]$ or
  $F(h) = UT^N h$ $\quad U$: input Toeplitz matrix

The set of feasible systems is given by

$$FPS_y = \{h \in K :\ \|F(h) - y\|_\infty^\omega \leq 1\}$$

For suitable K, it is a bounded convex set in an infinite dimensional space.

Pointwise estimator: $\hat{h} = \Phi(y)$

Error norm:

- $\|\cdot\|_X = \mathcal{H}_\infty \quad \rightarrow \quad \mathcal{H}_\infty$ identification
- $\|\cdot\|_X = \ell_1 \quad \rightarrow \quad \ell_1$ identification

⇛ Several different settings

- A priori assumptions on the true system
- Time-domain and/or frequency-domain measurements
- Prior information on noise (noise bounds, deterministic correlation, ...)

⇛ Convergence results

✓ There exist no robustly convergent linear algorithms for $\mathcal{H}_\infty$ and ℓ_1 identification `(Partington, Mäkilä)`

✓ $\mathcal{H}_\infty$ robustly convergent nonlinear algorithms → two-stage procedures: inverse Fourier transform + Nehari approximation
`(Helmicki et al., Gu & Khargonekar, ...)`

✓ Robust convergence in ℓ_1 requires K relatively compact (i/o stability is not sufficient) → several nonlinear algorithms `(Mäkilä, Tse, ...)`

⇒ $\mathcal{H}_\infty$ and ℓ_1 identification algorithms

✓ Interpolatory and nearly interpolatory identification algorithms (Chen, Gu, ...)

✓ Computable upper and lower bounds of $R(\varepsilon) \rightarrow$ reliable evaluation of the actual suboptimality level of interpolatory identification algorithms (Milanese & Taragna)

✓ If K is a polytope, interpolatory and projection estimators in ℓ_1 can be obtained via linear programming

Drawback: all these algorithms generally produce high-order models

Restricted-complexity models ⇒ *Conditional* set membership identification

Conditional set membership estimation

Conditional estimation: problem formulation

Let $x \in X$ *and*

$$y = F(x) + e \qquad \textit{with} \qquad \|e\|_Y \leq \varepsilon.$$

Find an estimate $z = \Phi(y) \in X$ *of the unknown problem element* x*, such that*

$$z \in \mathcal{M}$$

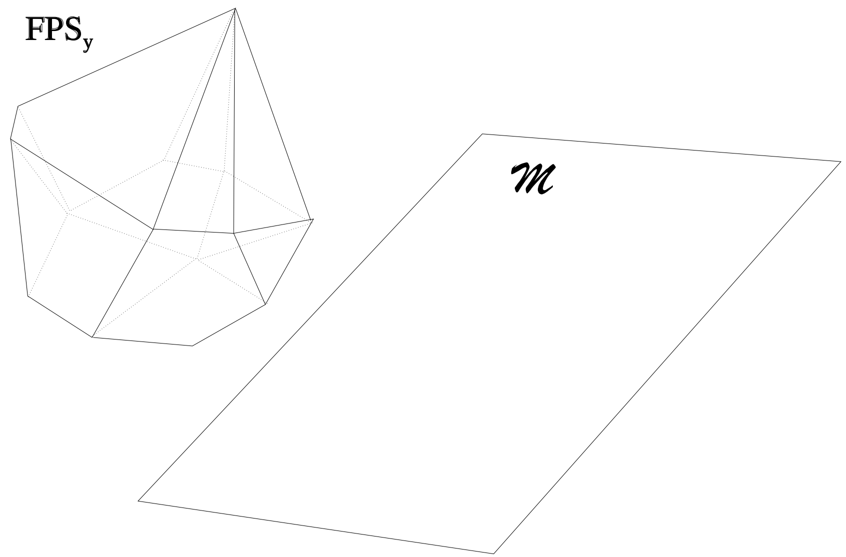
where $\mathcal{M}$ *is an assigned set in* X*.*

Typical setting:

- complicated feasible sets (including different types of a priori and a posteriori information)
- highly structured estimate sets (linear, finitely parameterized, etc.)
- quality of the estimate $\rightarrow$ *worst-case* estimation error

Conditional estimation

Problem: selection of a point estimate in a pre-specified set of fixed structure



$\Rightarrow$ Example: Restricted-complexity identification

Example: Restricted complexity identification (1/3)

Linear SISO system

$$y_k = \sum_{i=0}^{\infty} h_i^o u_{k-i} + e_k \ , \qquad \|e\|_Y \leq \varepsilon$$

According to the identification experiment the *feasible system set* is

$$FPS_y = \{h \in K : \ \|y^N - U_N h^N\|_Y \leq \varepsilon\}$$

where

$$\begin{aligned} y^N &= [y_0 \ y_1 \ \ldots \ y_{N-1}]' \\ h^N &= [h_0 \ h_1 \ \ldots \ h_{N-1}]' \\ U_N &= \begin{bmatrix} u_0 & 0 & \ldots & 0 \\ u_1 & u_0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ u_{N-1} & u_{N-2} & \ldots & u_0 \end{bmatrix} \end{aligned}$$

Example: Restricted complexity identification (2/3)

Define a p-dimensional linear manifold

$$\mathcal{M} = \{h \in \mathcal{H} : \; h = \overline{h} + M\alpha, \quad \alpha \in \mathbb{R}^q, \; M : \mathbb{R}^q \to \mathcal{H}\}$$

The basis of $\mathcal{M}$ is typically a collection of impulse responses of linear filters (FIR, Laguerre, Kautz, etc.)

Restricted complexity estimator $\quad \Phi : \; Y \to \mathcal{M}, \;\; \Phi(y) = \hat{\alpha}$

Quality of estimate (*worst-case* error)

$$E_y(\mathcal{M}, \Phi) = \sup_{h \in FPS_y} \|h - M\Phi(y^N)\|_{\mathcal{H}}$$

Objective: find an estimate of h,

$$\hat{h} \; = \; \overline{h} + M\,\Phi(y) \;\; \in \mathcal{M}$$

that minimizes $E_y(\mathcal{M}, \Phi)$

Example: Restricted complexity identification (3/3)

Finite dimension conditional estimation problem

- Residual a priori information: K is a constraint on the "tail" of h
- For several norms $\|\cdot\|_{\mathcal{H}}$ (ℓ_r, $1 \le r < \infty$), it can be shown that

$$E_y(T_N\mathcal{M}, \phi) \le E_y(\mathcal{M}, \phi) \qquad \forall \phi,\ \forall y \tag{1}$$

T_N: truncation operator

$\Rightarrow$ It is appropriate to select $\mathcal{M} \subset T_N\mathcal{H}$:

$$\mathcal{M} = \{h^N \in \mathbb{R}^N : \ h^N = M\alpha, \ \ \alpha \in \mathbb{R}^q, \ q < N\}$$

For the norms for which (1) holds, there exists an integer r such that

$$E_y(\mathcal{M}, \phi) = \left(\sup_{h^N \in FPS_y^N} \|h^N - M\phi(y^N)\|_{\mathcal{H}}^r + \sup_{h \in K} \|(I - T_N)h\|_{\mathcal{H}}^r \right)^{1/r}.$$

$FPS_y^N = T_N(FPS_y)$

$\Uparrow$ **to be minimized** $\qquad$ $\Uparrow$ **computed a priori**

Conditional estimators

The class of conditional estimators: $\Phi_{\mathcal{M}} = \{\Phi : \ \Phi : Y \rightarrow \mathcal{M}\}$

Minimum achievable local error (*Conditional radius of information*)

$$R_c(\varepsilon) \triangleq \inf_{\Phi \in \Phi_{\mathcal{M}}} E_y(\Phi, \varepsilon)$$

Conditional Central estimator (*Conditional Chebishev center*)

$$z_{cc} = \Phi_{cc}(y) \triangleq \arg \inf_{z \in \mathcal{M}} \sup_{x \in FPS_y} \|z - x\|_X$$

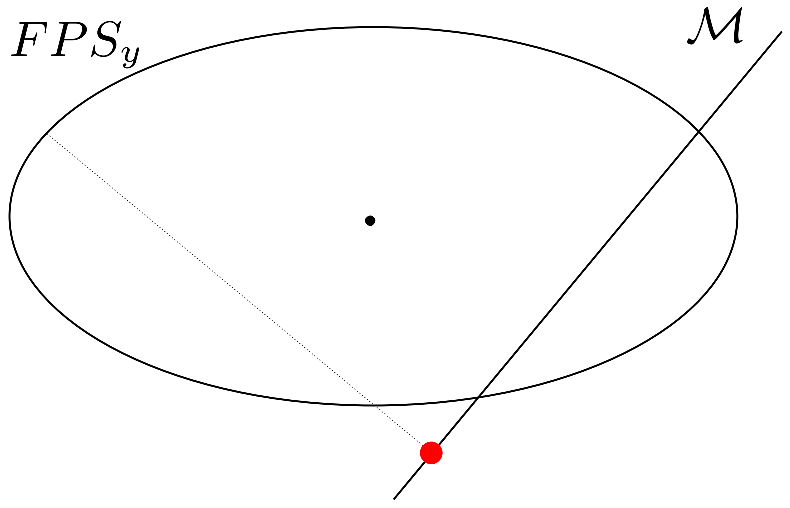
Φ_{cc} is locally optimal in $\Phi_{\mathcal{M}}$, i.e. $E_y(\Phi_{cc}, \varepsilon) = R_c(\varepsilon)$

✓ Characterization of conditional radius of information:
metric complexity and n-widths `(Zames, Wang, ...)`

Conditional estimators

Conditional Central estimates are generally very difficult to compute
$\Rightarrow$ Min-max nonconvex optimization problems

Computationally efficient solutions in special cases:
$F(x) = Fx$, FPS_y is an ellipsoid ($\|\cdot\|_Y = \ell_2$), linear $\mathcal{M}$ `(Garulli et al.)`



General case $\Rightarrow$ Turn to suboptimal estimators

Suboptimal conditional estimators

- Central Projection estimator

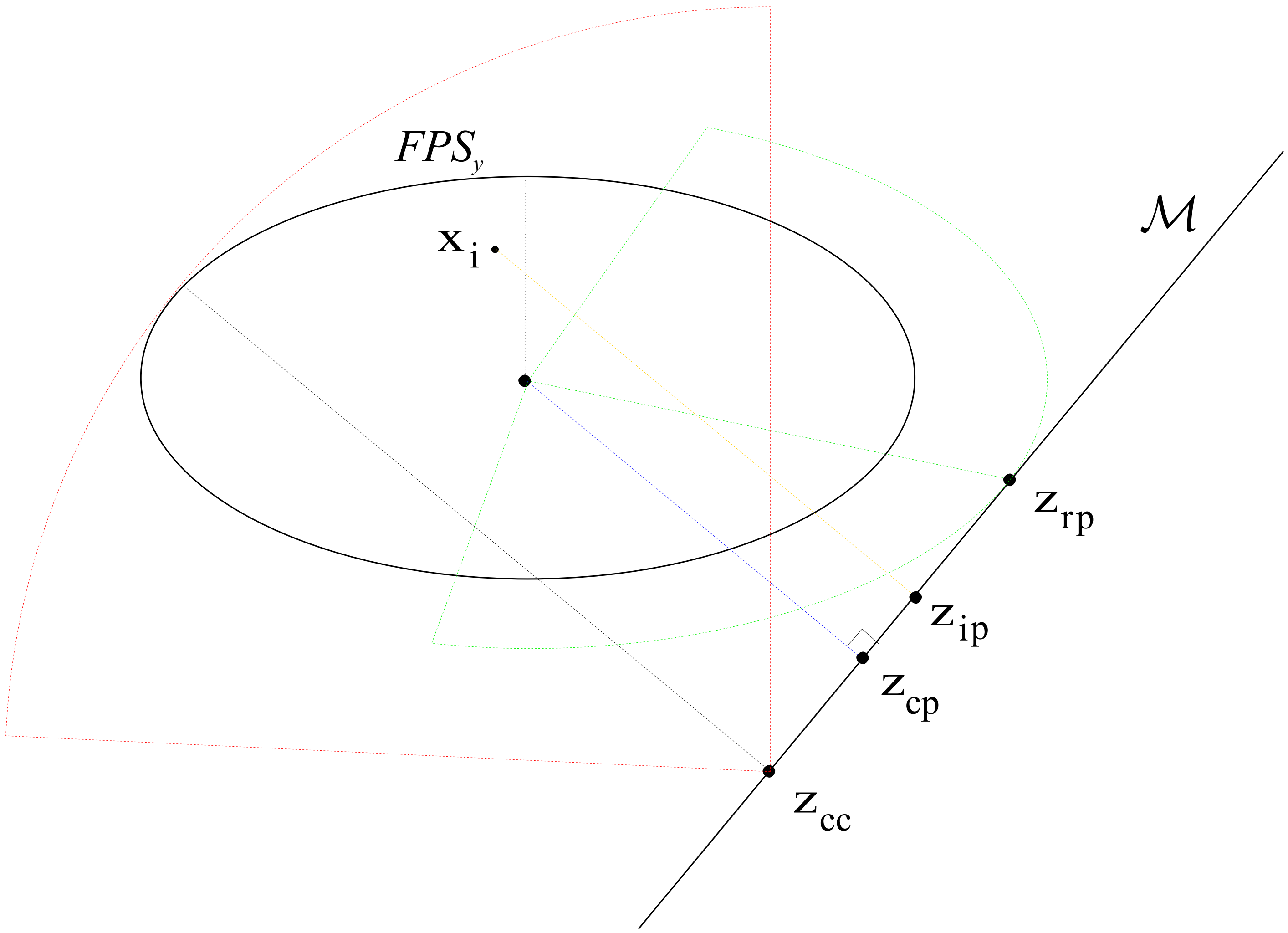
$$z_{cp} = \Phi_{cp}(y) \triangleq \arg \inf_{z \in \mathcal{M}} \|z - z_y\|_X \qquad z_y = \mathrm{cen}(FPS_y)$$

- Interpolatory projection estimator

$$z_{ip} = \Phi_{ip}(y) \triangleq \arg \inf_{z \in \mathcal{M}} \|z - x_i\|_X \qquad x_i \in FPS_y$$

- Restricted projection estimator

$$z_{rp} = \Phi_{rp}(y) \triangleq \arg \inf_{z \in \mathcal{M}} \|F(z) - y\|_Y$$



Example of optimal and suboptimal estimators

Suboptimality level of conditional estimators

Φ: conditional estimator

Problem: for given classes of $\mathcal{E}$ and $\mathcal{M}$ and given norms $\|\cdot\|_Y = \|\cdot\|_X = \ell_r$, compute the minimum γ_r such that

$$\frac{E_y(\Phi, \varepsilon)}{R_c(\varepsilon)} \leq \gamma_q$$

for all admissible $\mathcal{E}$ and $\mathcal{M}$.

Setting for the computation of tight error bounds:

- linearly parameterized estimates

$$\mathcal{M} = \{z \in \mathbb{R}^n : \; z = z^o + M\alpha, \quad \alpha \in \mathbb{R}^q\}$$

- different worst-case error norms: ℓ_2, ℓ_∞, ℓ_1

Suboptimality level of conditional estimators

Estimator	ℓ_2	ℓ_1	ℓ_∞
Φ_{cp}	$\sqrt{\frac{4}{3}}$	3	2
Φ_{ip}	2	3	3
Φ_{rp}	∞	∞	∞

Tight bounds on the worst-case estimation error
for different error norms

Conditional estimation: choice of the parameterization

Problem: choice of the set $\mathcal{M}$ to which the estimate z must belong

Assume a family Υ of sets $\mathcal{M}_p$ is given, where p is a free parameter whose choice determines the parameterization of the estimate

$$\Upsilon = \{\mathcal{M}_p : \ p \in \mathbb{R}^m\}$$

Find the optimal parameterization p^* such that

$$p^* \ = \ \arg \ \inf_{p} \ \inf_{z \in \mathcal{M}_p} \ \sup_{x \in FPS_y} \ \|z - x\|_X$$

→ Optimal parameterization is hard to compute

→ Suboptimal choices are pursued

Example: Choice of basis functions in restricted complexity identification (1/2)

- Restricted complexity identification: how to choose the basis functions
- Typical choice: linearly parameterized model class

$$\mathcal{M} = \{h : \; h = M_p \theta, \quad \theta \in \mathbb{R}^q\}$$

  where M_p is a linear operator, $M_p : \mathbb{R}^q \rightarrow \mathcal{H}$, and $\theta \in \mathbb{R}^q$ is the parameter vector to be identified, $q < N$.
- The model class $\mathcal{M}$ depends on some variables $p \in \mathbb{R}^m$.

Examples:

✓ For Laguerre filters, p is the real Laguerre pole ($m = 1$).

✓ For Kautz functions, p denotes the pair of complex conjugate poles ($m = 2$).

Example: Choice of basis functions in restricted complexity identification (2/2)

Optimal choice of the model class

$$p^*(FPS_y) = \arg\inf_p \inf_{\theta\in\mathbb{R}^q} \sup_{h\in FPS_y} \|h - M_p\,\theta\|_{\mathcal{H}}$$

Conditional Central Algorithm: $\quad \phi^*_{cc}(y) = h^*_{cc} = M_{p^*}\theta_{cc}$

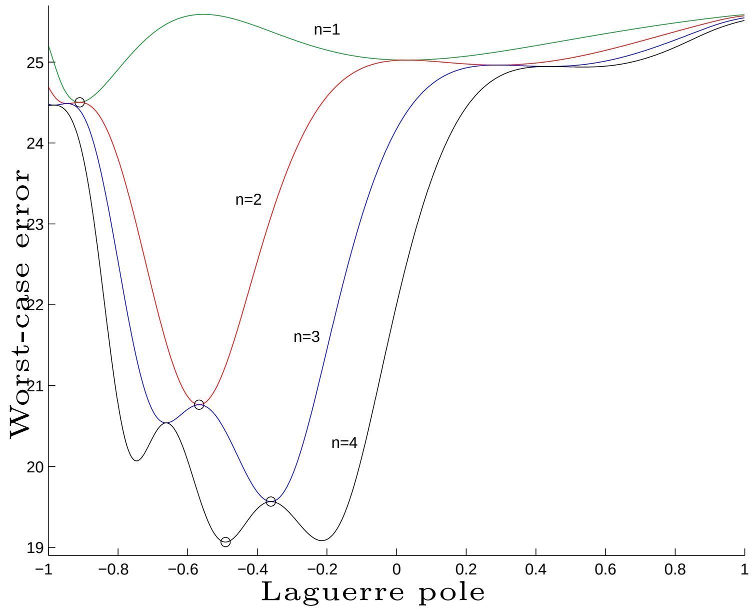
where $\theta_{cc} = \arg\inf_{\theta\in\mathbb{R}^q} \sup_{h\in FPS_y} \|h - M_{p^*}\theta\|_{\mathcal{H}}$.

Example:

$H(z) = \frac{5+10.7\,z^{-1}+5.002\,z^{-2}}{1+2.3\,z^{-1}+2.06\,z^{-2}+0.72\,z^{-3}}$, N=50.

Noise $\{v_k\}_{k=0}^{N-1}$ gaussian random sequence, such that $(1/\sqrt{N})\|v\|_2 \leq 1$.

$\mathcal{M}$: Laguerre filters model class.



Suboptimal choice of the parameterization

- Assume parameterization in p is linear

$$\mathcal{M} = \{z : \; z = M_p\theta, \quad \theta \in \mathbb{R}^q\}$$

- Given $x \in X$, define the optimal pole p^* (w.r.t x) as

$$p^*(x) = \arg \inf_{p} \inf_{\theta \in \mathbb{R}^q} \|x - M_p\,\theta\|_X$$

- Let Π_q be the optimal (with respect to p) projection operator

$$\Pi_q x = M_{p^*(x)}\theta^*(x)$$

where $\theta^*(x) = \arg\inf_{\theta\in\mathbb{R}^q} \|x - M_{p^*(x)}\theta\|_X.$

Idea: Replace optimal pole w.r.t. FPS_y (difficult to compute!), with optimal pole w.r.t. suitably selected elements in FPS_y.

Suboptimal estimators

⇒ Central Projection Estimator:

$$\phi^*_{cp}(y) = x_{cp} \triangleq \Pi_q x_c \quad , \qquad p = p^*(x_c)$$

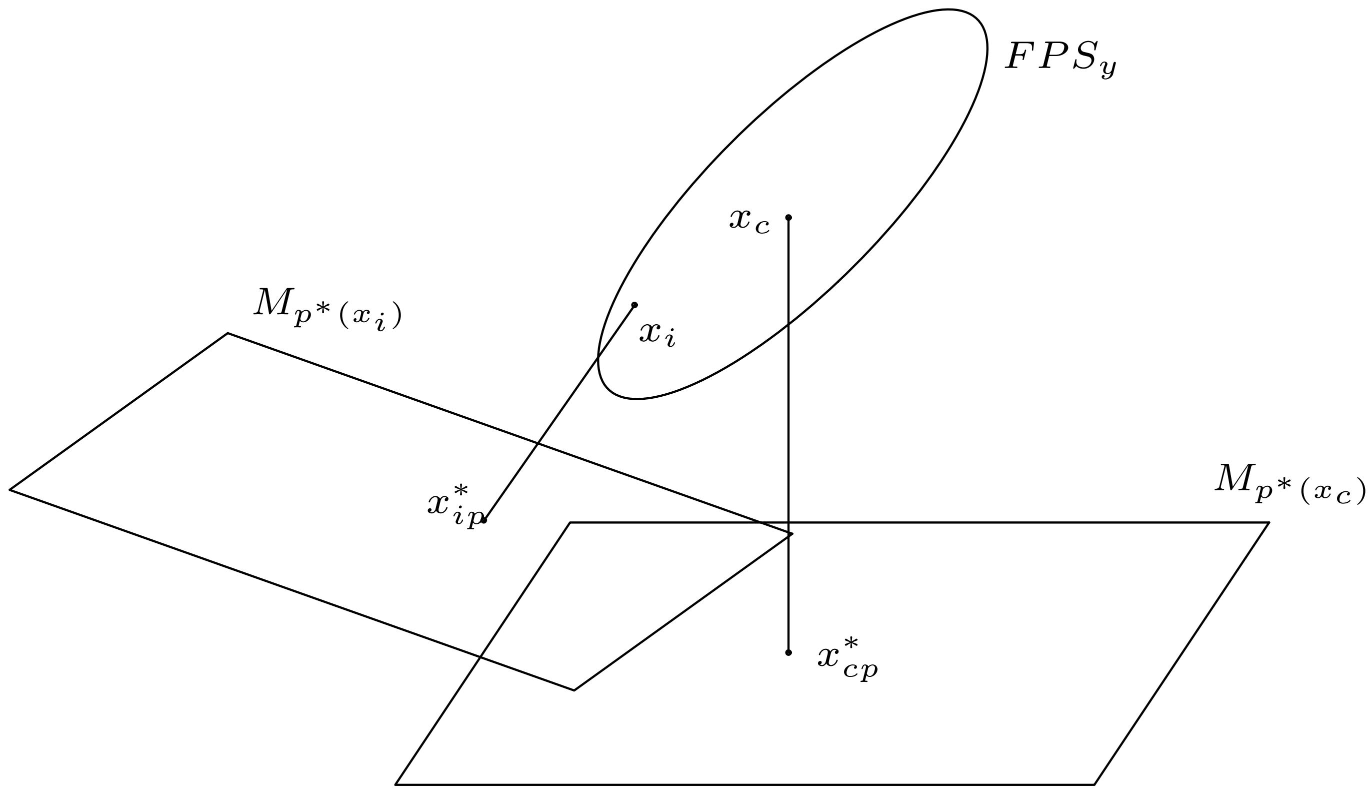
where x_c is the Chebishev center of FPS_y.

⇒ Interpolatory Projection Estimator:

$$\phi^*_{ip}(y) = x_{ip} \triangleq \Pi_q x_i \quad , \qquad p = p^*(x_i)$$

where $x_i \in FPS_y$.

Example: suboptimal choice of parameterization



x_{cp} : central projection estimate

x_{ip} : interpolatory projection estimate

Error bounds for suboptimal parameterization

- Central Projection Estimator

  Let $\|\cdot\|_X = \ell_2$. Let $r = \mathrm{rad}(FPS_y)$ and $d = \|x_c - x_{cp}\|_2$. Then,

$$\frac{E\left[\phi^*_{cp}\right]}{E\left[\phi^*_{cc}\right]} \leq \sqrt{2 - \frac{(r-d)^2}{r^2+d^2}}$$

- Interpolatory Projection Estimator

  Let $\|\cdot\|_X = \ell_2$. Let $r = \mathrm{rad}(FPS_y)$ and $d_i = \|x_i - x_{ip}\|_2$. Then,

$$\frac{E\left[\phi^*_{ip}\right]}{E\left[\phi^*_{cc}\right]} \leq \min\left\{2 + \frac{d_i}{r},\, 1 + \frac{2\,r}{d_i}\right\}$$

Current and future research

- Smart set approximation techniques for specific structures of the feasible sets (driven by applications)
- Optimal input selection in set membership system identification
- Set membership identification of *nonlinear systems*

Identification of Piecewise Affine systems

A PieceWise Affine (PWA) system is defined as

$$y_k = f(x_k) + e_k$$

where $x_k \in \mathbb{R}^n$ is the regressor vector, $y_k \in \mathbb{R}$ is the output, $e_k \in \mathbb{R}$ is the output error and $f(\cdot)$ is a piecewise affine function

$$f(x) = \begin{cases} \theta_1' \begin{bmatrix} x \\ 1 \end{bmatrix} & \text{se} \quad x \in \mathcal{X}_1 \\ \vdots & \qquad \vdots \\ \theta_s' \begin{bmatrix} x \\ 1 \end{bmatrix} & \text{se} \quad x \in \mathcal{X}_s \end{cases}$$

where $\theta_i \in \mathbb{R}^{n+1}$, i=$1, \ldots, s$, are parameter vectors and $\{\mathcal{X}_i\}_{i=1}^{s}$ is a polyhedral partition of the regressor space $\mathcal{X} \subseteq \mathbb{R}^n$.
For ease of notation, let $\varphi_k = \left[\; x_k' \quad 1 \;\right]'$.

Motivations

⇒ Identification of hybrid systems

⇒ Approximation of nonlinear maps

⇒ Identification of nonlinear dynamic systems using models of piecewise affine structure (Example: PWARX models)

Set membership approach

Basic assumption: UBB measurement error

$$|e_k| \leq \varepsilon$$

Problem: Given N datapoints (y_k, φ_k), $k = 1, \dots, N$, find an estimate of the partition $\{\mathcal{X}_i\}_{i=1}^{s}$ and s sets of feasible parameter vectors $\{\Theta_i\}_{i=1}^{s}$, compatible with the data and the assumption on the noise.

Example: identification of PWARX models (1/3)

When the regression vector x_k is a collection of past inputs and outputs

$$x_k = [y_{k-1} \dots y_{k-n_a} \; u_{k-1} \dots u_{k-n_b}]$$

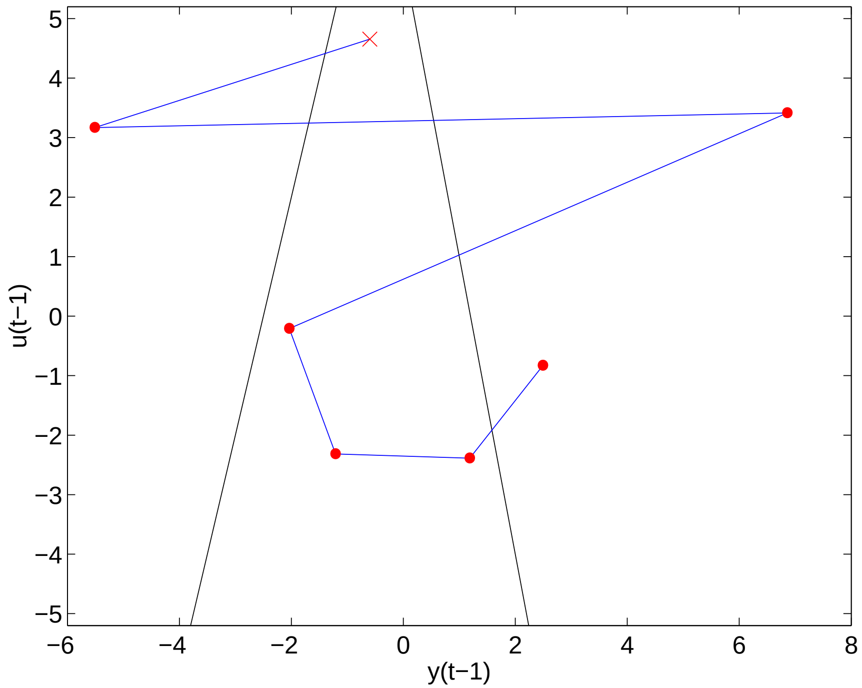
one has a PWARX (PieceWise affine AutoRegressive eXogenous) model.

Example: a PWARX system

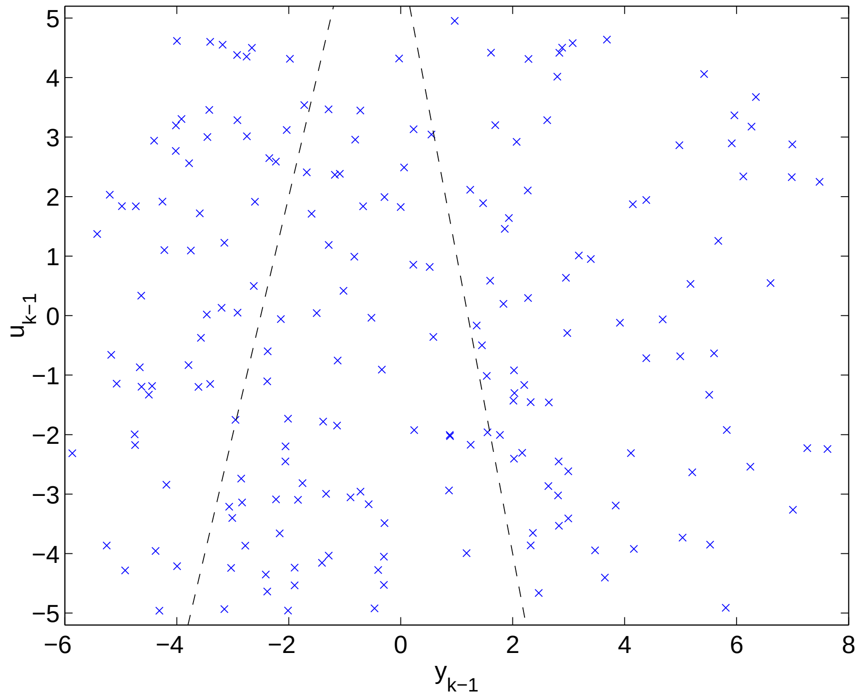
$$y_k = \begin{cases} \begin{bmatrix} -0.4 & 1 & 1.5 \end{bmatrix} \varphi_k + \varepsilon_k & \text{se } \begin{bmatrix} 4 & -1 & 10 \end{bmatrix} \varphi_k < 0 \\ \begin{bmatrix} 0.5 & -1 & -0.5 \end{bmatrix} \varphi_k + \varepsilon_k & \text{se } \begin{bmatrix} -4 & 1 & -10 \\ 5 & 1 & -6 \end{bmatrix} \varphi_k \leq 0 \\ \begin{bmatrix} -0.3 & 0.5 & -1.7 \end{bmatrix} \varphi_k + \varepsilon_k & \text{se } \begin{bmatrix} -5 & -1 & 6 \end{bmatrix} \varphi_k < 0 \end{cases}$$

with $\varphi_k = [y_{k-1} \; u_{k-1} \; 1]'$ and $s = 3$.

Example: identification of PWARX models (2/3)

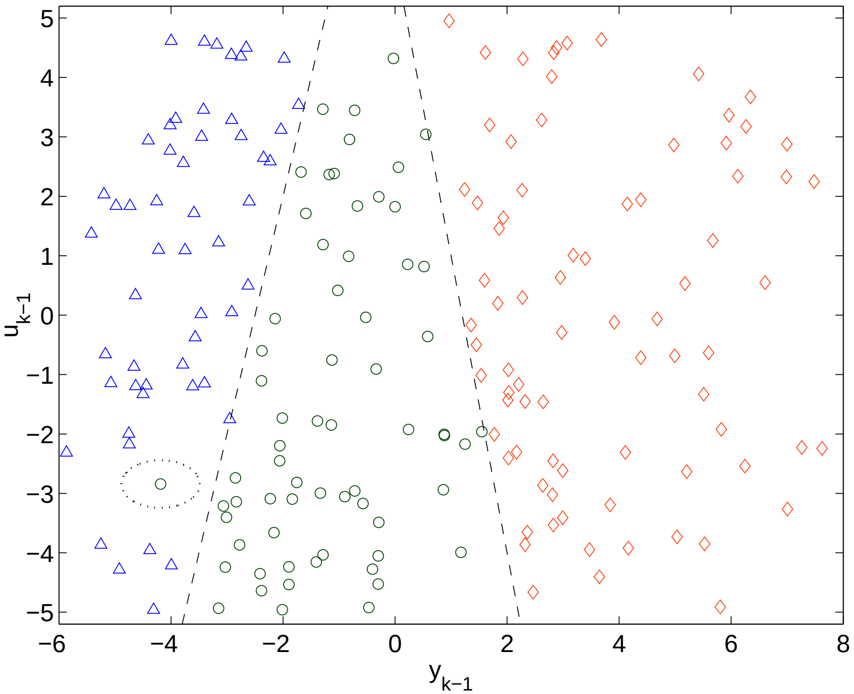


Partition of regressor space

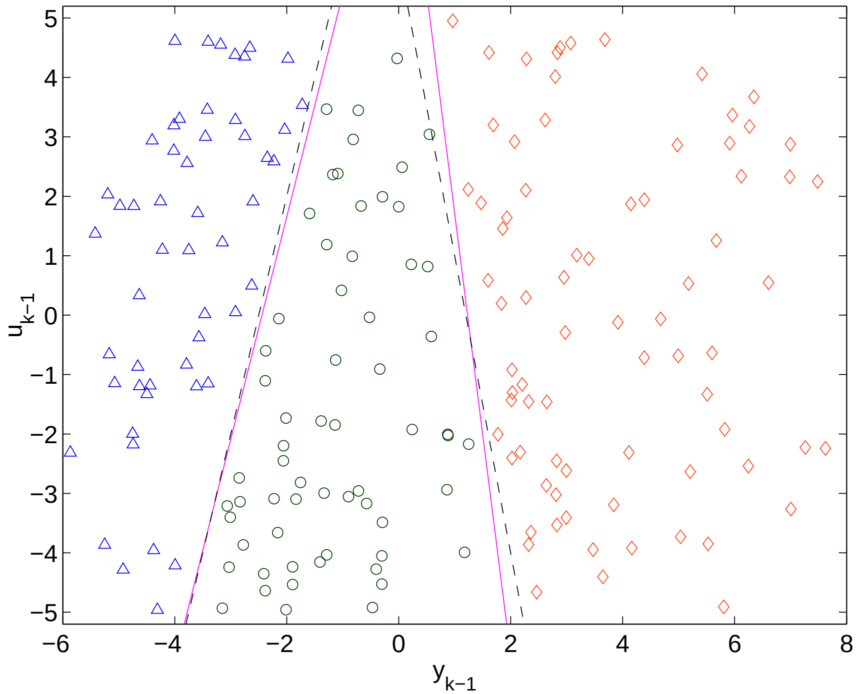


Data points

Example: identification of PWARX models (3/3)



Final data classification



Partition estimation

Set Membership Estimation: Applications

Applications of SM estimation

- ✓ Signal processing
  - Short signal classification for speech labelling `(Deller)`
  - Adaptive blind equalization for multichannel systems `(Huang)`
  - Estimation of time-varying magnitude harmonics `(Yaz)`
  - Audio frequency filter identification
- ✓ Information theory
  - Symbol-by-symbol decoding for binary data transmission `(Wells)`
  - Model reduction for channel models
- ✓ Fault detection
  - Fault detection and isolation in chemical MIMO plants
  - Non destructive testing of material structures

✓ Image processing and computer vision

- Tomographic feature detection and classification `(Hero III)`
- Image reconstruction with mixed stochastic/bounded error `(Chaumette)`
- Recursive time-to-collision estimation `(Vicino)`

✓ Semiconductor manufacturing

- Nonlinear filtering `(Khargonekar, Tsakalis)`

✓ Ecology

- Prediction of lake eutrophication `(Keesman)`
- Greenhouse energy requirement prediction `(Maksarov)`
- Long-term stability of shallow lakes

✓ Mobile robotics and autonomous navigation

- Real-time robot localization `(Sabater, Hanebeck, Vicino,...)`
- Obstacle modeling `(Ruiz, Meizel)`
- Robust observers for robot tracking `(Walter, Meizel)`
- Simultaneous localization and mapping `(Vicino)`

A case study:
Localization and mapping for autonomous navigation of mobile robots

Dynamic localization of autonomous mobile robots

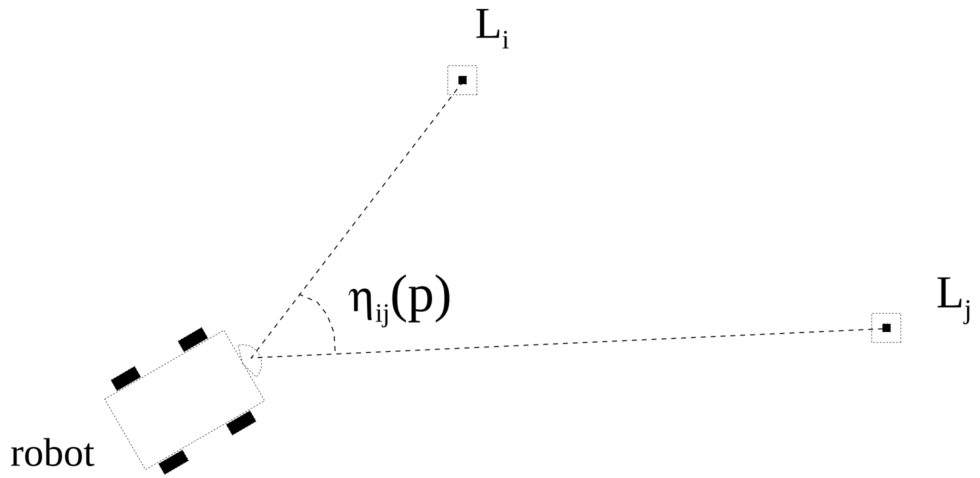
Autonomous navigation in an unstructured outdoor environment: the navigating robot periodically computes its coordinates in the reference system by exploiting geometric properties and measurements information based on landmarks

Available Information	Uncertainty Sources
- Environment map	- Map errors
- Landmarks	- Landmarks misidentification
- **Sensor measurements**	- **Measurements errors**

Robot dynamics implies error propagation

Visual angle measurements

For each pair of landmarks L_i, L_j the navigator is able to measure the *visual angle*

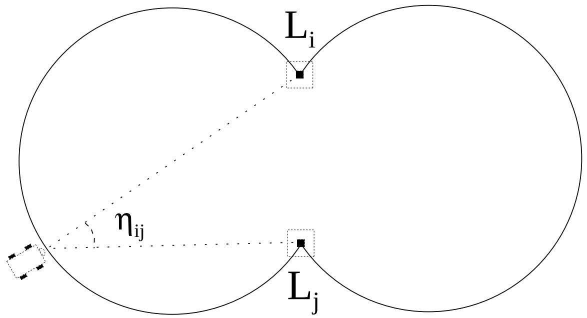


- Visual angles are nonlinear functions of the robot position p
- Visual angle measurements are affected by errors

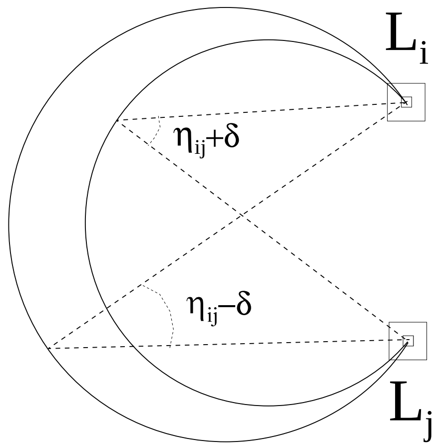
$$\theta_{ij} = \eta_{ij}(p) + v_{ij}$$

Information from visual angles

Locus of points from which a given visual angle η_{ij} is seen



UBB measurement errors $\rightarrow$ The robot is constrained to lie on a
$(|v_{ij}| \leq \delta)$ *thickened ring*



Dynamic model

LTV model of the vehicle dynamics

$$\xi(k+1) = A(k)\xi(k) + B(k)u(k) + G(k)w(k) \qquad k = 0, 1, \ldots$$

Measurement equation

$$\theta_{ij}(k) = \eta_{ij}(\Pi\xi(k)) + v_{ij}(k) \qquad i, j = 1, \ldots, n, \quad i < j$$

where:

- $\xi(k) = [x(k) \;\; y(k) \;\; \dot{x}(k) \;\; \dot{y}(k)]'$: state vector
- $u(k)$: driving command
- $w(k)$: unknown forces acting on the vehicle
- $v(k)$: measurement noise
- $\Pi = [I_2 \; 0_2]$, $\quad \Pi\xi$: vehicle position

Localization Problem

Nominal localization: Let $\hat{\xi}(0)$ be an estimate of the initial position and velocity of the vehicle. At each time instant k, compute an estimate $\hat{p}(k)$ of the vehicle position $p(k) = \Pi\xi(k)$.

Classical approach $\rightarrow$ Extended Kalman filter

Alternative approach $\rightarrow$ Set membership filter (UBB noise)

$$
\begin{array}{rcll}
|w_i(k)| & \leq & \varepsilon_i^w(k) & i = 1,2 \\
|v_{ij}(k)| & \leq & \varepsilon_{ij}^v(k) & i,j = 1,\ldots,n,\ i < j
\end{array}
$$

Set membership localization: Let $\Xi(0) \subset \mathbb{R}^4$ be a set containing the initial position and velocity vector $\xi(0)$. At each time instant k, find a set $\mathcal{L}(k)$ containing the vehicle position

$$p(k) \in \mathcal{L}(k), \qquad k = 1,2,\ldots$$

Set Membership Localization

Thickened ring

$$\mathcal{C}_{ij} = \mathcal{C}(\theta_{ij}, \varepsilon_{ij}^v) = \{p \in \mathbb{R}^2 : \ \theta_{ij} - \varepsilon_{ij}^v \leq \eta_{ij}(p) \leq \theta_{ij} + \varepsilon_{ij}^v\}$$

Measurement set

$$\mathcal{M}(k) = \bigcap_{\substack{i,j=1 \\ i<j}}^{n} \mathcal{C}_{ij}(k)$$

Exact recursion

$$\begin{aligned}
\Xi(0|0) &= \Xi(0), \\
\Xi(k|k-1) &= A(k)\Xi(k-1|k-1) + B(k)u(k) + G(k)\mathrm{Diag}\{\varepsilon^w(k)\}\mathcal{B}_\infty, \\
\Xi(k|k) &= \Xi(k|k-1) \bigcap \ \{\xi : \ \Pi\xi \in \mathcal{M}(k)\}, \\
\mathcal{L}(k) &= \Pi\,\Xi(k|k).
\end{aligned}$$

$\Rightarrow \quad p(k) \in \mathcal{L}(k)$: *feasible position set*

Recursive approximation of the feasible position set

$\mathcal{R}$: class of approximating regions
$\overline{\mathcal{R}}(\mathcal{S})$: minimum volume set in the class $\mathcal{R}$ containing the set $\mathcal{S}$

Let $\mathcal{R}(0|0) \supseteq \Xi(0)$.
Define the approximating sets

$$\mathcal{R}(k|k-1),\ \mathcal{R}(k|k) \subset \mathbb{R}^4 \quad \text{and} \quad \mathcal{R}_\Pi(k|k) \subset \mathbb{R}^2$$

through the recursion

$$\begin{aligned}
\mathcal{R}(k|k-1) &= \overline{\mathcal{R}}\left(A(k)\mathcal{R}(k-1|k-1) + B(k)u(k) + G(k)\mathrm{Diag}\{\varepsilon^w(k)\}\mathcal{B}_\infty\right) \\
\mathcal{R}_\Pi(k|k) &= \overline{\mathcal{R}}\left(\Pi\mathcal{R}(k|k-1) \bigcap \mathcal{M}(k)\right) \\
\mathcal{R}(k|k) &= \overline{\mathcal{R}}\left(\left\{\mathcal{R}_\Pi(k|k) \times \overline{\Pi}\mathcal{R}(k|k-1)\right\} \bigcap \mathcal{R}(k|k-1)\right) \qquad \overline{\Pi} = [0_2 \ \ I_2]
\end{aligned}$$

Result: The above recursion guarantees that

$$\mathcal{L}(k) \subseteq \mathcal{R}_\Pi(k|k) \quad \forall\ k$$

Set membership localization algorithms

- Choice of the approximating regions (boxes, parallelotopes,...)
- Suboptimal solution $\rightarrow$ Recursive minimum area approximation (e.g.: sequential approximation of intersection box/ring)
- Low computational complexity $\Rightarrow$ good for real-time!
- For n landmarks: $O(n^2)$ operations at each time update (generally n not very high, landmarks are "expensive")

Simultaneous Localization and Mapping (SLAM)

⇒ A more challenging problem in autonomous navigation

Two tasks: *self localization* and *map building*

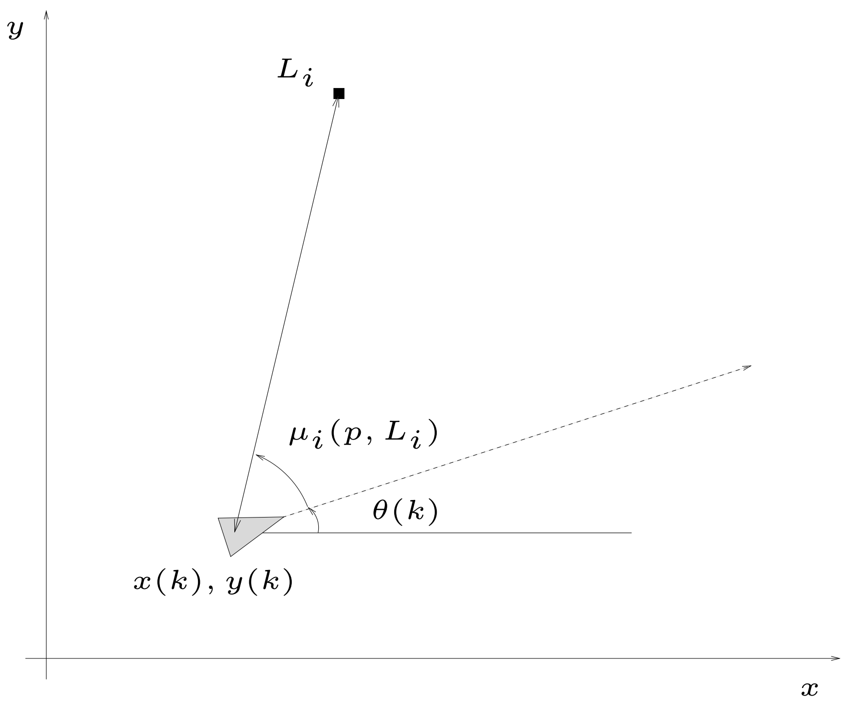
Problem setting

✓ a *landmark-based* map of the environment $L_i = (x_{L_i}, y_{L_i})$

✓ an *uncertain dynamic model* of the vehicle pose $p(k) = [x(k)\ y(k)\ \theta(k)]'$

✓ *exteroceptive measurements* with respect to the landmarks:

- distance measurements
- relative angular measurements



Formulation of SLAM problem

Landmark location *and* robot pose must be estimated simultaneously!
State vector: $X(k) = [p'(k) \;\; L'_1(k) \;\; \cdots \;\; L'_n(k)]'$

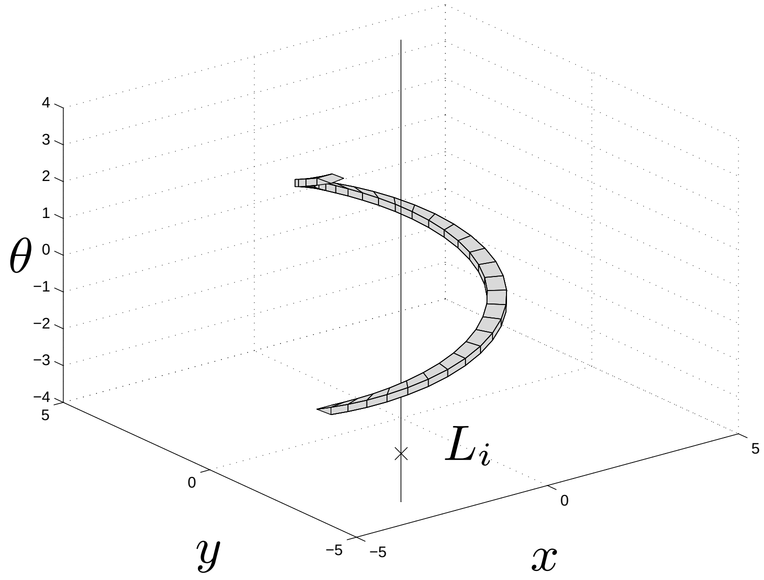
State estimation problem usually tackled in statistical framework
(EKF, Markov estimation, iconic matching...)

Assumption: UBB unmodeled dynamics and measurement errors
$\rightarrow$ SLAM problem can be formulated in the set membership setting

SM SLAM Problem: Let $\Xi(0) \subset \mathbb{R}^{2n+3}$ be a set containing the initial pose of the vehicle and the position of the landmarks, $X(0)$. Given the dynamic model and measurement equations, find at each time $k = 1, 2, \cdots$, the set $\Xi(k|k)$ of state vectors $X(k)$ compatible with model, measurements and UBB noise assumptions.

Main problem: nonlinear measurements equations lead to nonlinear and nonconvex feasible sets

$\rightarrow$ efficient approximations required (accurate and fast)



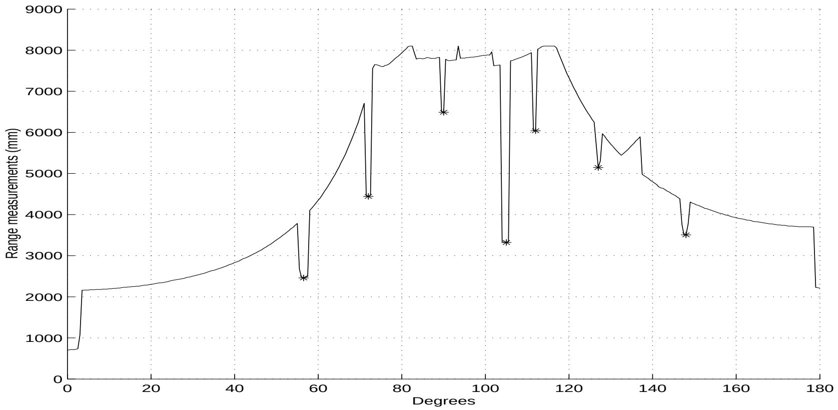
set of feasible poses for a pair of measures (distance and orientation) w.r.t. L_i

Towards efficient SM SLAM algorithms

- ✓ state decomposition into subsets of state variables (robot pose, landmark positions)
- ✓ guaranteed set membership estimation of each subset
- ✓ set approximations via simple regions (boxes, parallelotopes)
- ✓ low computational complexity $\Rightarrow$ suitable for real-time applications

Experimental testing: setup

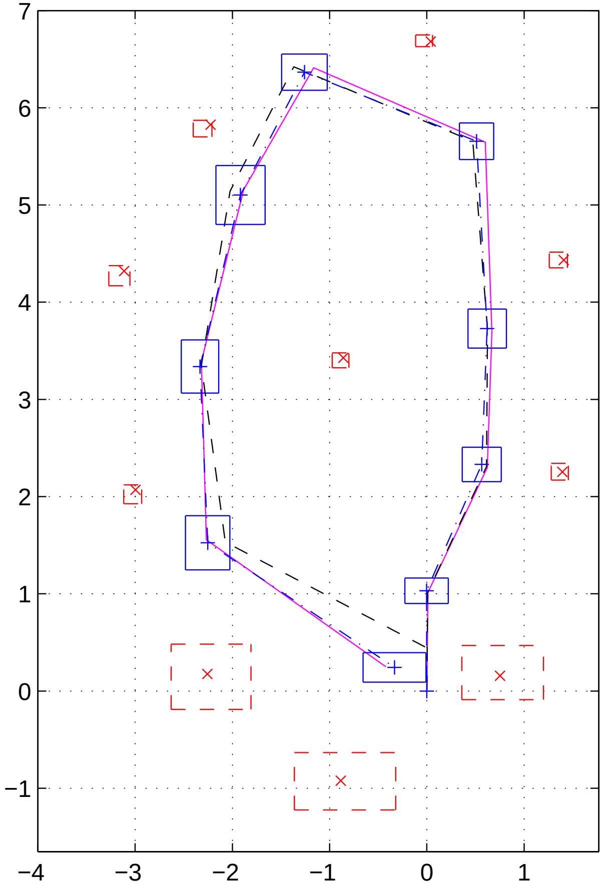
- mobile robot Nomad XR4000
- 10 artificial landmarks
- angular and distance measures from laser rangefinder
- measurement error bounds from landmark extraction



planar laser rangefinder:

180° scanning angle and 0.5° resolution

Experimental testing: results



nominal trajectory (odometry): dashed
true trajectory: solid
estimated trajectory: dash-dotted

- ▷ average robot position and heading errors: 4 cm, 0.67°
- ▷ average robot position and orientation uncertainty: 0.17 m^2, 10.3°
- ▷ average landmark position error: 3 cm
- ▷ average landmark uncertainty region: 0.18 m^2

- ▷ average number of landmarks detected at each measurement step: $\sim$ 6
- ▷ time required at each measurement step: $\sim$ 0.1 sec.