Robust Stability and Stabilization via Lyapunov's Methods

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Class of systems

$$\begin{aligned} \dot{x}(t) &= f(x(t), u(t), w(t)) \qquad (SYS) \\ y(t) &= g(x(t)) \end{aligned}$$

$$\begin{aligned} w(t) &\in W \\ \dot{w}(t) &\in \dot{W} \end{aligned}$$

Assumptions

$$\begin{aligned} 0 &= f(0,0,0) \\ 0 &= g(0) \end{aligned}$$

Definition 0.1 *The system*

$$\dot{x}(t) = f(x(t), w(t))$$

is Globally Uniformly Asymptotically Stable (GUAS) if there exists a strictly decreasing function $\phi(t)$ *such that* $\phi(t) \to 0$ *as* $t \to 0$ *and a non-decreasing function* $\psi : \mathbb{R}^+ \to \mathbb{R}^+$, *with* $\psi(0) = 0$ $\psi(\lambda) > 0$ *for* $\lambda > 0$, *such that for all* $w(t) \in W$

$$\|x(t)\| \leq \phi(t)\psi(\|x(0)\|)$$

Definition 0.2 *The system is Uniformly Ultimately Bounded (UUB) (practically stable) within the compact set* S, $0 \in int[S]$, *if for all* $k > 0$ *there exists* T *such that for all* $w(t) \in W$ *and* $\|x(0)\| \leq k$

$$x(t) \in S, \quad \text{for all} \;\; t \geq T.$$

Analysis problem:

Check if (S) is GUAS (UUB).

Synthesis problem:

Given the system

$$\begin{aligned} \dot{x}(t) &= f(x(t), u(t), w(t)) \\ y(t) &= g(x(t)) \end{aligned}$$

find a control (in a given class) such that the closed loop system is GUAS (UUB).

Robustness

Definition 0.3 *A property $\mathcal{P}$ is said* robust *for the family $\mathcal{F}$ of dynamic systems if any member of $\mathcal{F}$ satisfies $\mathcal{P}$*

The family $\mathcal{F}$ and the property $\mathcal{P}$ must be properly specified. For instance if $\mathcal{P}$ is "stability" and $\mathcal{F}$ is a family of systems with uncertain parameters ranging in a set, we have to specify if these parameters are constant or time–varying.

Time-varying parameters

Parameter variation may have a crucial effect on stability. Consider the system

$$\dot{x}(t) = A(w(t))x(t)$$

$$A(w) = \begin{bmatrix} 0 & 1 \\ -1+w & -a \end{bmatrix} \qquad |w| \leq \bar{w},$$

where

$a > 0$ is a damping parameter

$\bar{w} < 1$ uncertainty bound.

For any constant $\bar{w} \leq 1$ and $a > 0$, the corresponding time-invariant system is stable. However, there exist $\bar{w} < 1$ and a (small enough) such that for suitable time-varying w, with $|w(t)| \leq \bar{w}$, (without derivative bounds) the system is unstable.

Lyapunov functions

Consider the system

$$\dot{x}(t) = f(x(t), w(t))$$

$$0 = f(0, w), \quad w \in \mathcal{W}$$

with f Loc. Lipschitz.

A function $V : \mathbb{R}^n \to R$, $V \in \mathcal{C}^1$ such that

$$\begin{aligned} V(0) &= 0 \\ V(x) &> 0, \quad x \neq 0 \\ \lim_{||x|| \to \infty} V(x) &= +\infty \end{aligned}$$

and

$$\dot{V}(x, w) = \nabla V(x)^T f(x, w) < 0, \quad x \neq 0, \quad \forall \;\; w \in \mathcal{W}$$

is a Lyapunov function.

Theorem

If the system

$$\dot{x}(t) = f(x(t), w(t))$$

$$0 = f(0, w), \quad w \in \mathcal{W}$$

f Loc. Lipschitz, admits a Lyapunov function, then $x(t) \to 0$ as $t \to \infty$.

Control-Lyapunov functions

$$\dot{x}(t) = f(x(t), u(t), w(t))$$

$$0 = f(0, 0, w), \qquad w \in \mathcal{W}$$

Consider the function $V \in \mathcal{C}^1$ such that

$$\begin{aligned} V(0) &= 0 \\ V(x) &> 0, \quad x \neq 0 \\ \lim_{||x|| \to \infty} V(x) &= +\infty \end{aligned}$$

If there exists a control $u = \phi(x)$ such that

$$\dot{V}(x, w) = \nabla V(x)^T f(x, \phi(x), w) < -\beta V(x), \quad x \to 0,$$

for some $\beta > 0$, then V is a control-Lyapunov function.

Dynamic feedback

$$\begin{aligned}\dot{x}(t) &= f(x(t), u(t)) \\ y(t) &= g(x(t))\end{aligned}$$

$$\begin{aligned}\dot{z}(t) &= h(z(t), y(t)) \\ u(t) &= k(z(t), y(t))\end{aligned}$$

is equivalent to the static feedback for the following augmented system

$$\begin{aligned}\dot{x}(t) &= f(x(t), u(t)) \\ \dot{z}(t) &= v(t)\end{aligned}$$

$$\begin{aligned}u(t) &= k(z(t), y(t)) \\ v(t) &= h(z(t), y(t))\end{aligned}$$

To work with L.F. we must consider the extended state-space

Quadratic robust stability

The most popular Lyapunov functions are the quadratic ones

$$V(x) = x^T P x$$

having gradient

$$\nabla V(x)^T = 2x^T P$$

Given the system

$$\dot{x}(t) = f(x(t), w(t)), \qquad 0 = f(0, w), \quad w \in \mathcal{W}$$

its Lyapunov derivative is

$$2x^T P f(x, w) < 0$$

If such a condition holds, the system is said quadratically stable.

Linear systems

$$\dot{x}(t) = A(w(t))x(t) + B(w(t))u(t)$$

Parametric uncertainty

$$A(w), B(w), \quad w \in \mathcal{W}$$

$$\begin{aligned} e.g. \quad A(w) &= \sum_{i=1}^{s} w_i A_i, \qquad \sum_{i=1}^{s} w_i = 1, \quad w_i \geq 0, \\ B(w) &= \sum_{i=1}^{s} w_i B_i, \qquad \sum_{i=1}^{s} w_i = 1, \quad w_i \geq 0, \end{aligned}$$

Non–parametric uncertainty

$$\begin{aligned} A(\Delta) &= A_0 + D\Delta E, \\ B(\Delta) &= B_0 + D\Delta F, \quad \|\Delta\| \leq 1, \end{aligned}$$

Quadratic stability

Parametric

$$A(w) = \sum_{i=1}^{s} w_i A_i, \qquad \sum_{i=1}^{s} w_i = 1, \quad w_i \geq 0,$$

$$\dot{V}(x, w) = x^T P A(w) x < 0, \quad \text{for all} \;\; w \in \mathcal{W}$$

iff

$$A_i^T P + P A_i < 0 \qquad\qquad (LMI)$$

Non–Parametric

$$A(\Delta) = A_0 + D\Delta E \qquad \|\Delta\| \leq \rho,$$

$$xTPA(\Delta)x < 0, \quad \text{for all} \quad \|\Delta\| \leq 1$$

iff A_0 is stable and

$$\|E(sI - A_0)^{-1} D\|_\infty < \frac{1}{\rho}$$

Quadratic stabilizability

Parametric case

$$\dot{x}(t) = A(w(t))x(t) + B(w(t))u(t)$$

$$\begin{aligned} e.g. \quad A(w) &= \sum_{i=1}^{s} w_i A_i, \qquad \sum_{i=1}^{s} w_i = 1, \quad w_i \geq 0, \\ B(w) &= \sum_{i=1}^{s} w_i B_i, \qquad \sum_{i=1}^{s} w_i = 1, \quad w_i \geq 0, \end{aligned}$$

Linear controller case. Consider the control $u = Kx$

$$\dot{x}(t) = [A(w(t)) + B(w(t))K]x(t)$$

The condition is

$$(A_i + B_i K)^T P + P(A_i + B_i K) < 0$$

Set

$$Q = P^{-1}, \quad KQ = R$$

then we get

$$QA_i^T + A_i Q + R^T B_i^T + B_i R < 0, \qquad (LMI)$$

which is a linear condition in Q and R

Quadratic stabilizability

Non-parametric case

$$
\begin{aligned}
\dot{x}(t) &= [A_0 + D\Delta E]x(t) + [B_0 + D\Delta F]u(t) \\
y(t) &= C_0 x(t), \qquad ||\Delta(t)|| \leq 1,
\end{aligned}
$$

$u(s) = K(s)y(s)$ is quadratically stabilizing iff the d-to-z transfer function of the loop

$$
\begin{aligned}
sx(s) &= A_0 x(s) + Dd(s) + B_0 u(s) \\
z(s) &= Ex(s) + Fu(s) \\
y(s) &= C_0 x(s) \\
u(s) &= K(s)y(s)
\end{aligned}
$$

is s.t.

$$
\|W_{zd}(s)\| \leq 1
$$

Robust stability

The system

$$\dot{x}(t) = Ax(t)$$

is stable iff it admits a quadratic Lyapunov function.

Assume that

$$\dot{x}(t) = A(w(t))x(t), \qquad w \in \mathcal{W}$$

is stable. Does it admit a quadratic Lyapunov function?

NO!

robust stability does not imply robust quadratic stability.

Example

$$A(w) = \begin{bmatrix} 0 & 1 \\ -1 + w(t) & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w \begin{bmatrix} 1 & 0 \end{bmatrix}, \qquad |w| \leq \rho$$

the system is stable iff

$$\rho < \rho_{ST} = 1, \quad \text{(robust stability radius)}$$

However the system is quadratically stable iff

$$\rho < \rho_{Q} = \frac{2}{\sqrt{3}}, \quad \text{(quadratic stability radius)}$$

Theorem

The system

$$\dot{x}(t) = A(w(t))x(t), \quad w \in \mathcal{W}$$

is stable if and only if it admits a piecewise-linear Lyapunov function

Molchanov and Pyatnitskii (1986)

Brayton and Tong (1980)

(also polynomial, piecewise-quadratic, ...)

Theorem

The system

$$\dot{x}(t) = A(w(t))x(t) + B(w(t))u(t), \quad w \in \mathcal{W}$$

is stabilizable if and only if it admits a polyhedral control Lyapunov function.

Blanchini (1995)

Quadratic and non-quadratic margins

$$\dot{x}(t) = A(w(t))x(t) + B(w(t))u(t), \qquad (S)$$

Take

$$w \in \rho\mathcal{W}$$

$\rho \geq 0$, and define the following stabilizability margins

$$\begin{aligned} \rho_{ST} &= \sup\{\rho : \text{(S) is stabilizable}\} \\ \rho_Q &= \sup\{\rho : \text{(S) is quadratically stabilizable}\} \end{aligned}$$

There are systems for which

$$\frac{\rho_{ST}}{\rho_Q} = \infty$$

For instance

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} w(t) \\ 1 \end{bmatrix}$$

$$\rho_{ST} = \infty$$

$$\rho_Q = 1$$

Non-linear versus linear

$$\dot{x}(t) = A(w(t))x(t) + B(w(t))u(t), \qquad (S)$$

stabilizability does not imply linear stabilizability (= stabilizability via linear compensator). For instance, the system

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad B = \begin{bmatrix} w(t) \\ 1 \end{bmatrix}, \qquad |w| \leq 100$$

is stabilizable but it is not stabilizable by means of a linear state static state feedback of the form

$$u = k_1 x_1 + k_2 x_2$$

(in which k_1 and k_2 do not depend on w).

There are examples of stabilizable systems which cannot be stabilized via linear (even dynamic) compensators.

Stabilization of nonlinear uncertain systems

Consider the system

$$\dot{x}(t) = f(x(t), u(t), w(t)), \qquad w \in \mathcal{W}$$

and assume that there exists a control Lyapunov function $V(x)$ namely, there exists $\Phi(x)$ such that for all $x \neq 0$

$$\nabla V(x)^T \, f(x, \Phi(x), w) < -\beta V(x), \quad \text{for all} \;\; w \in \mathcal{W}$$

$\beta > 0$, then the system with the control $u(t) = \phi(x(t))$ is asymptotically stable.

Question 1) Given $V(x)$ how can we determine $\Phi(x)$?

Question 2) How can we determine V(x)?

Given $V(x)$ determine $\Phi(x)$ - Selection

Consider the control–affine system

$$\dot{x}(t) = F(x(t), w(t)) + G(x(t))u(t)$$

The condition

$$\dot{V}(x, w) = \nabla V(x)^T (F(x, w) + G(x)u) < -\beta V(x)$$

yields the following condition on u

$$\nabla V(x)^T G(x)u < -\nabla V(x)^T F(x, w) - \beta V(x)$$

Take $u = \Phi(x)$ of the form

$$\Phi(x) = -\gamma(x)G(x)^T \nabla V(x)$$

γ must be such that for all x and $w \in \mathcal{W}$

$$-\gamma(x)\|G(x)^T \nabla V(x)\|^2 < -V(x)^T F(x, w) - \beta V(x)$$

How can we find V? The special case of matched uncertainties

The uncertainties satisfy the "matching conditions" if

$$\dot{x}(t) = f(x(t)) + Bg(x(t), w(t)) + Bu(t)$$

$$\|g(x, w)\| \leq \rho(x)$$

Assume that the nominal systems

$$\dot{x}(t) = f(x(t))$$

is asymptotically stable and it admits a Lyapunov function $V \in \mathcal{C}^1$:

$$\dot{V}(x) = \nabla V(x)^T f(x) \leq -\beta V(x)$$

then there exists a "practically stabilizing" control.

One of such controls is the following u as follows

$$u = -\frac{B^T \nabla V(x) \rho^2(x)}{\|B^T \nabla V(x) \rho(x)\| + \epsilon}$$

$$\begin{aligned}
\dot{v}(x,w) &= \nabla V(x)^T [f(x) + Bg(x,w) + Bu] \leq \\
&\leq \nabla V(x)^T f(x) + \|\nabla V^T(x) B \rho(x)\| - \frac{\|\nabla V^T(x) B\|^2 \rho^2(x)}{\|B^T \nabla V(x) \rho(x)\| + \epsilon} \\
&\leq -\beta V(x) + \epsilon \frac{\|\nabla V(x)^T(x) B\| \rho(x)}{\|B^T \nabla V(x) \rho(x)\| + \epsilon} \\
&\leq -\beta V(x) + \epsilon
\end{aligned}$$

The state is confined in the set $\mathcal{S} = \{x : \quad V(x) \leq \epsilon/\beta\}$.

The above control is not unique !!

Backstepping: beyond matching conditions

$$\dot{x}(t) = F(x(t), w(t)) + G(x(t), w(t))u(t)$$

Strict feedback form

$$\begin{aligned}
F(x,w) &= \begin{bmatrix} f_{11} & f_{12} & 0 & \dots & 0 \\ f_{21} & f_{22} & f_{23} & \dots & 0 \\ \dots & \dots & \dots & \dots & 0 \\ f_{n-1,1} & f_{n-1,2} & f_{n-1,3} & \dots & f_{n-1,n} \\ f_{n,1} & f_{n,2} & f_{n,3} & \dots & f_{n,n} \end{bmatrix} x + F(0,w) \\
f_{i,j} &= f_{i,j}(x_1, x_2, \dots, x_i, w) \\
G(x,w) &= \begin{bmatrix} 0 \\ 0 \\ : \\ 0 \\ f_{n,n+1} \end{bmatrix} \\
f_{i,i+1} &\neq 0.
\end{aligned}$$

Example:

$$
\begin{aligned}
\dot{x}_1(t) &= x_1(t)F(x_1(t))w(t) + x_2(t) \\
\dot{x}_2(t) &= u(t)
\end{aligned}
$$

$$|w| \leq 1, \qquad |F(x_1)| \leq m$$

Consider the first equation with the "virtual" control x_2

$$
\begin{aligned}
x_2 &= S(x_1)x_1 \\
\dot{x}_1 &= x1[F(x_1)w(t) + S(x_1)]
\end{aligned}
$$

where $S(x_1)$ is smooth and bounded with bounded derivative. If

$$[F(x_1)w(t) + S(x_1)] < 0$$

this system is stable. But x_2 is **not** a control variable !!

Then control the first equation

$$\dot{x}_2(t) = u(t)$$

in such a way that x_2 is "close" to $S(x_1)x_1$

$$\dot{x}_2 = u = -k[x_2 - S(x_1)x_1]$$

Consider the change of variables

$$\begin{cases} z_1 = x_1 \\ z_2 = x_2 - S(x_1)x_1 \end{cases} \qquad \begin{cases} x_1 = z_1 \\ x_2 = z_2 - S(z_1)z_1 \end{cases}$$

and the candidate Lyapunov function

$$V(z_1, z_2) = z_1^2 + z_2^2$$

we have

$$\begin{aligned}\dot{z}_1(t) &= z_1 F(x_1) w + z_2 + S(z_1) z_1 \\ \dot{z}_2(t) &= S(z_1)[z_1 F(x_1) w + z_2 + S(z_1) z_1] + S'(z_1) + u\end{aligned}$$

the control considered above is

$$u = -k z_2$$

$$\dot{V} = [F(x_1)w + S(z_1)]z_1^2 + [1 + S'(z_1) + S(z_1)F(x_1)w + S(z_1)2]z_1 z_2 + [S(z_1) - k]z_2^2$$

If k is sufficiently large then

$$\dot{V} \leq -a z_1^2 + b|z1||z2| - [k - c]z_2^2 < 0, \quad \text{for} \quad (z_1, z_2) \neq 0$$

State estimation for uncertain systems

$$\begin{aligned} \dot{x}(t) &= A(w(t))x(t) + B(w(t))u(t) \\ y(t) &= Cx(t) \end{aligned}$$

Observer

$$\begin{aligned} \dot{z}(t) &= (A_0 - LC)z(t) + B_0u(t) + Ly(t) \\ y(t) &= Cx(t) \end{aligned}$$

Define the error $e = z - x$

$$\dot{e}(t) = (A_0 + LC)e(t) + (A_0 - A(w(t)))x(t) + (B_0 - B(w(t)))u(t)$$

By its nature, an observer must replicate the system dynamics. If the dynamics is not known exactly, error convergence depends upon u, w and x. Observer principle is fragile...

Controlling nonlinear systems via robust control methods

Given

$$\dot{x}(t) = f(x(t)) + Bu(t) \qquad (NL)$$

assume that

$$f(x) \in conv\{A_i\}\ x$$

that is

$$f(x) = A(w)x, \qquad w = w(x)$$

for some

$$A(w) = \sum_{k=1}^{r} w_i A_i, \qquad \sum_{k=1}^{r} w_i = 1, \quad w_i \geq 0,$$

Then, if the control $u = \Phi(x)$ stabilizes

$$\dot{x}(t) = A(w(t))x(t) + Bu(t)$$

then it stabilizes (NL).

Observer design for nonlinear systems by means of robust control algorithms

$$\begin{aligned} \dot{x}(t) &= f(x(t)) + Bu(t) \\ y(t) &= Cx(t) \end{aligned}$$

$$\begin{aligned} \dot{z} &= f(z(t)) + L(y - Cz) + Bu \\ \dot{e} &= [f(e+x) - f(x)] + LCe \end{aligned}$$

where $e(t) = z(t) - x(t)$. If for all x

$$[f(e+x) - f(x)] = Ae, \quad A \in \mathcal{A},$$

e.g.

$$\frac{\partial f}{\partial x} \in \mathcal{A}$$

and

$$\dot{e} = (A(t) - LC)e, \qquad A(t) \in \mathcal{A}$$

is robustly stable, then $e(t) \to 0$.

Bounded parameter variation

$$\dot{x}(t) = f(x(t), w(t))$$

$$w(t) \in \mathcal{W}, \qquad \dot{w} \in \dot{\mathcal{W}}$$

Lyapunov function candidate

$$V(x(t), w(t)) \geq \nu(x)$$

where $\nu(x)$ is positive definite. The Lyapunov derivative is

$$\dot{V}(x, w) = \nabla V_x f(x, w) + \nabla V_w \dot{w}$$

that must be negative for all $w \in \mathcal{W}$ and $\dot{w} \in \dot{\mathcal{W}}$ and $x \neq 0$.

Gain scheduling control

$$\dot{x}(t) = F(x(t), w(t)) + G(x(t), w(t))u(t)$$

$$u = \Phi(x, w), \qquad \text{gain-scheduling controller}$$

$$u = \Phi(x), \qquad \text{robust controller}$$

In the linear case

$$u = Kx$$

$$u = K(w)x$$

References

- Aubin Cellina, "Differential Inclusions", Spinger-Verlag.
- Boyd S., El Ghaoui L., Feron E. and Balakrishnan V., "Linear matrix inequality in system and control theory", SIAM Studies in Applied Mathematics, Philadelphia, 1994.
- Z. Qu, "Robust Control of Nonlinear Uncertain Systems", John Wiley.
- R.A. Freeman and P.V. Kokotovic, "Robust nonlinear control design", Birkauser.