A discrete-time pursuit-evasion game in convex polygonal environments

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Abstract

This paper studies a discrete-time pursuit-evasion game within a convex polygonal environment. Building on solutions of the classic lion-and-man problem, two strategies are proposed for the pursuer, which guarantee exact capture in finite time and provide upper bounds on the time-to-capture at each move of the game. A numerical procedure for updating the so-called *center* of the game, which is instrumental for computing the lion's move, is devised. Numerical simulations show that optimizing the center position, with respect to a suitable cost function taking into account the structure of the environment, allows one to remarkably reduce the number of moves required to capture the evader.

Keywords: Pursuit-evasion games, autonomous agents, game theory, lion and man problem

1. Introduction

Pursuit-evasion games have been intensively studied in recent years due to both the intriguing theoretical problems they pose and the number of applications in several different fields. The interested reader is referred to the surveys [1, 2] and references therein, for a thorough review of the relevant literature.

When the agents are moving in a limited environment, the complexity of the problem and the techniques for its solution strongly depend on the assumptions on the nature of the game and of the structure of the environment. To mention only a well-known watershed, if the agents' motion model is formulated in continuous time, an evader can indefinitely escape even in the simple case of a circular arena [3], and it is necessary to assume a finite radius of capture to guarantee the success of the pursuer [4, 5, 6]. Conversely, in a discrete-time formulation, there always exist strategies that lead to capture of the evader in finite time. For example, in the David Gale's version of the lion-and-man problem,

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which takes place in the positive quadrant of the plane, several different winning strategies for the lion have been proposed [7, 8]. The approach presented in [9] for the case of multiple pursuers in an n-dimensional environment, can be applied also to the case of a single pursuer within a convex polygon, by considering the presence of virtual pursuers on the polygon sides. All the above approaches rely on the so-called *lion's move*, which amounts to keep the pursuer on the segment connecting the evader to a reference point, called *center* of the game, the main differences being in the way such center is computed and (possibly) updated during the game. An alternative approach, based on a different lion's move, has been recently proposed in [10]. Many other discrete-time settings have been considered in the literature, including multi-pursuer search in simply connected polygons [11], in generic polygons [12, 13] and in topological spaces [14]. Several works have also addressed the presence of sensing limitations, see e.g., [15, 16].

In this paper, a discrete-time pursuit-evasion problem within a convex polygonal environment is addressed. Two lion strategies are proposed, which extend to the considered setting the lion's moves introduced in [7] and [8], respectively. The former is based on a fixed center chosen once and for all at the beginning of the game, while the latter updates the center at every game move, according to a suitable cost function that takes into account the structure of the polygonal environment. The main contributions consist in showing that the two strategies guarantee the capture of the evader in finite time and providing an upper bound on the time-to-capture at each move of the game. An algorithm for computing the optimal game center is also provided. The results of an extensive campaign of numerical simulations are reported, to assess the performance of the proposed strategies. With respect to the solutions presented in [7, 8], the main novelty lies in that the game center (and hence the pursuer strategy) depends explicitly on the structure of the environment, and in particular on its vertices, in a non trivial way.

The rest of the paper is organized as follows. The lion and man problem is reviewed in Section 2, along with the solutions proposed in the literature. The two new lion strategies for convex polygonal environments are presented in Section 3. The algorithm for the computation of the optimal center is devised in Section 4. Numerical simulations are provided in Section 5, while concluding remarks are drawn in Section 6.

1.1. Notation

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space and $\mathbb{R}^n_+$ the n-dimensional Euclidean space of non-negative numbers. We denote by $\lceil x \rceil$ the smallest integer greater or equal to x. A row vector with elements $v_1, \ldots, v_n$ is denoted by $V = [v_1, \ldots, v_n]$, while V' is the transpose of V. Given two vectors V and W, notation $V \succ W$ denotes the componentwise strict inequality. $\boldsymbol{\mathcal{C}}(C, r)$ is the circle of center C and radius r, i.e., $\boldsymbol{\mathcal{C}}(C, r) = \{p \in \mathbb{R}^2 \colon \|C - p\| \leq r\}$.

2. Review of lion and man problem

In this section, we recall two lion strategies devised for the lion and man problem formulated by David Gale (problem 31 in [17]). In this problem, a lion (pursuer) and a man (evader) can move in the first quadrant of the Cartesian plane. Space is continuous while time is discrete, i.e., players move in turn. By convention, the man moves first. Both players can travel the same maximum distance in a single move, set to 1 without loss of generality. The lion wins the game if, at a finite time t, he is able to move *exactly* to the man position. Let $M_t \in \mathbb{R}^2_+$ and $L_t \in \mathbb{R}^2_+$ denote the man and lion position at time t, respectively. Let the initial condition be such that $L_0 \succ M_0$. If this condition is not satisfied, the man easily wins the game by moving straight up or to the right.

2.1. Fixed Center Lion Strategy

If $L_0 \succ M_0$, a lion strategy able to guarantee the lion victory for any possible man strategy has been proposed in [7]. Such a strategy is based on the computation of a point (*center*) which remains fixed throughout the game. For this reason, this strategy will be referred to as *Fixed Center Lion Strategy (FCLS)*.

Let us define the unit vector pointing from M_t to L_t as

$$W_t = \frac{L_t - M_t}{\|L_t - M_t\|} \ . \qquad (1)$$

Definition 1. *Let M_0 and L_0 be given. A point $C_0 = [x_0, y_0]' \in \mathbb{R}^2_+$ is said* center of the FCLS *if*

1. $r_0 = \|C_0 - L_0\| = \max\{x_0, y_0\}$,
2. $C_0 = L_0 + r_0 W_0$.

The quantity r_0 is named the radius of the FCLS.

Definition 2 (Lion's move). *At a given time t, let the man move from M_t to M_{t+1}. The lion moves according to the following rules:*

- *if $\|M_{t+1} - L_t\| \leq 1$, then the lion moves to M_{t+1} and wins the game;*
- *if $\|M_{t+1} - L_t\| > 1$, the lion moves to a point L_{t+1} on the line connecting M_{t+1} to the center, such that $\|L_{t+1} - L_t\| = 1$. Between the two points satisfying such a condition, he chooses the one closer to M_t.*

An example of FCLS lion's moves is illustrated in Fig. 1-a.

2.2. Moving Center Lion Strategy

A variation of the FCLS has been recently proposed in [8]. Contrary to the FCLS, in this strategy the center is updated at each time. For this reason, we refer to this strategy as *Moving Center Lion Strategy (MCLS)*. The center at a generic time t is computed according to the following definition.

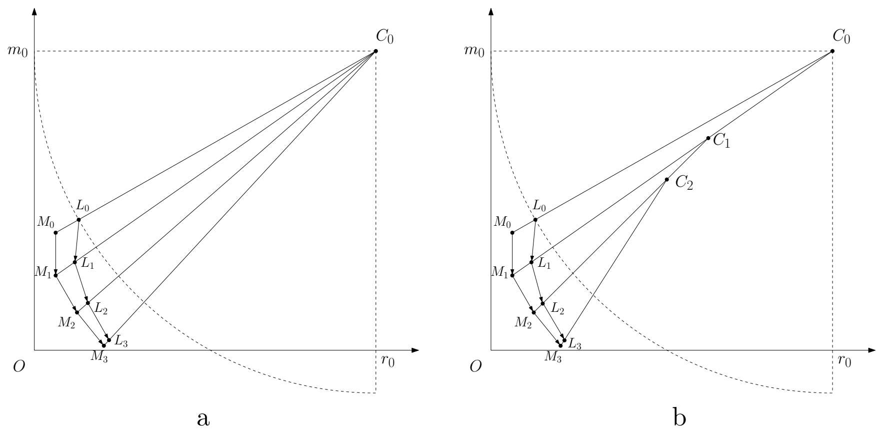

Figure 1: Examples of lion's moves. a: FCLS, b: MCLS.

Definition 3. *Let M_t and L_t be given. A point $C_t = [x_t, y_t]' \in \mathbb{R}^2_+$ is said* center of the MCLS *at time t if*

1. $r_t = \|C_t - L_t\| = \max\{x_t, y_t\}$,
2. $C_t = L_t + r_t W_t$.

The quantity r_t is the radius of the MCLS *at time t.*

The MCLS proceeds as follows (an example is shown in Fig. 1-b). At time $t = 0$, set C_0 as in Definition 1. At a generic time t, let the man move from M_t to M_{t+1}. Then:

- the lion moves from L_t to L_{t+1} according to Definition 2, with center C_t;
- the center is updated from C_t to C_{t+1}, according to Definition 3.

In [8], it has been shown that the MCLS guarantees that the lion wins the game in finite time. Moreover, the following theorem has been proved.

Theorem 1. *Let N^{FCLS}_{max} and N^{MCLS}_{max} denote the maximum number of moves to end the game needed by the FCLS and MCLS, respectively. Then,*

$$N^{MCLS}_{max} \leq N^{FCLS}_{max} \ . \tag{2}$$

Theorem 1 states that the MCLS dominates the FCLS. Moreover, numerical simulations show that the number of moves needed by the MCLS is in general much less than that of FCLS [8]. In the next section, the ideas at the basis of FCLS and MCLS are employed to devise lion strategies able to cope with convex polygonal environments.

3. Lion strategies in convex polygons

In this section, it is assumed that the man and the lion move within a convex polygonal environment. Two lion's strategies based on the choice of a suitable center will be described. The former is based on a fixed center, while the latter relies on a center which changes at each time step. Notice that, since the environment is closed, the assumption $L_0 \succ M_0$ related to the initial conditions is no longer needed.

Let $\mathcal{P} \subset \mathbb{R}^2$ be a convex polygon defining the game environment and let $\mathcal{V} = \{V^{(i)} \in \mathcal{P},\, i = 1, \ldots, n\}$ be the set of vertices of $\mathcal{P}$. An alternative way to define $\mathcal{P}$ is

$$\mathcal{P} = \{x \in \mathbb{R}^2 \colon Ax \leq b\} \tag{3}$$

where $A \in \mathbb{R}^{n\times 2}$ and $b \in \mathbb{R}^n$.

Let us now introduce the concept of center for a convex polygonal environment.

Definition 4. *Let the game environment $\mathcal{P} \subset \mathbb{R}^2$ be a convex polygon. At time t, let W_t be given by* (1) *and set*

$$C_t = \mathbf{C}_t(r_t) \triangleq L_t + r_t W_t \tag{4}$$

with $r_t \geq 0$. If $C_t \in \mathcal{P}$, then C_t is a center *in $\mathcal{P}$ at time t, and $r_t = \|C_t - L_t\|$ is the corresponding* radius.

Notice that, at a given time t, different values of r_t provide different centers.

By (3), the constraint $C_t \in \mathcal{P}$ is equivalent to

$$A\,(L_t + r_t W_t) \leq b\ .$$

Then, one can compute the maximum radius r_t such that $C_t \in \mathcal{P}$ by solving the following Linear Program (LP)

$$\begin{aligned} r_t^{max} = &\sup\ r \\ &s.t.: \\ &A(L_t + rW_t) \leq b \end{aligned} \tag{5}$$

Therefore, the admissible interval for the radius at time t is $r_t \in [0, r_t^{max}]$.

Definition 5. *At a given time t, let $r_t \in [0, r_t^{max}]$ be fixed and set $C_t = \mathbf{C}_t(r_t)$ according to* (4).

- *A point $Q \in \mathcal{P}$ is said* reachable *from M_t if there exists a continuous path inside $\mathcal{P}$ from M_t to Q which does not intersect $\boldsymbol{\mathcal{C}}(C_t, r_t)$.*
- *The* contaminated area *at time t is defined as*

$$\widehat{A}_t(C_t) = \{Q \in \mathcal{P} \colon Q \textit{ is reachable from } M_t\}\ . \tag{6}$$

- *The* cleared area *at time t is given by $\widetilde{A}_t(C_t) = \mathcal{P} \backslash \widehat{A}_t(C_t)$.*
- *The set of* reachable vertices *of $\mathcal{P}$ at time t is defined as*

$$\mathcal{V}_t^R(C_t) = \{V \in \mathcal{V} \colon V \in \widehat{A}_t(C_t)\} \ . \tag{7}$$

- *The reachable vertex farthest from C_t is given by*

$$V_t^{max}(C_t) = \arg \max_{V \in \mathcal{V}_t^R(C_t)} \|C_t - V\| \tag{8}$$

Notice that the cleared area is the region that the man cannot enter without being captured by the lion, while the contaminated area is its complement. The reachable vertices are defined as the vertices (potentially) reachable by the man, i.e., belonging to the contaminated area. Clearly, the lion wins whenever the contaminated area collapses to zero. In Fig. 2, two examples showing the above defined regions are reported. In the left figure, the cleared area is given by $\widetilde{A}_t(C_t) = \boldsymbol{\mathcal{C}}(C_t, r_t) \cap \mathcal{P}$, while in the right one $\widetilde{A}_t(C_t) \supset (\boldsymbol{\mathcal{C}}(C_t, r_t) \cap \mathcal{P})$. Notice that, as shown in the latter case, the cleared area may contain points of $\mathcal{P}$ which are outside $\boldsymbol{\mathcal{C}}(C_t, r_t)$.

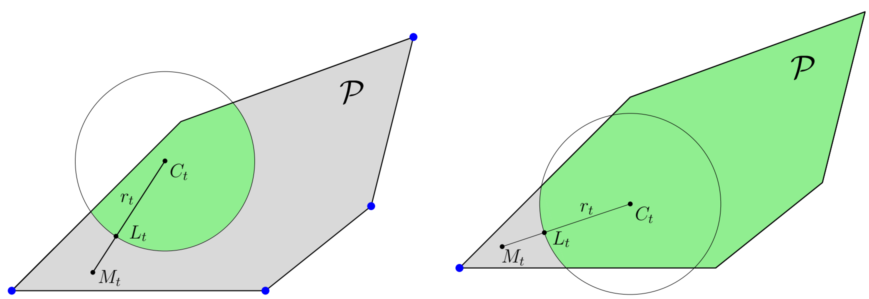

Figure 2: Examples showing the cleared area $\widetilde{A}_t(r_t)$ (green), the contaminated area $\widehat{A}_t(r_t)$ (gray) and the reachable vertices $\mathcal{V}_t^R(C_t)$ (blue dots).

Let us now introduce the cost function which will be used throughout the paper to evaluate the progress of the game:

$$J_t(C_t) = \|C_t - V_t^{max}(C_t)\|^2 - \|C_t - L_t\|^2 = \|C_t - V_t^{max}(C_t)\|^2 - r_t^2 \tag{9}$$

where $V_t^{max}(C_t)$ is given by (8). In particular, the function J_t will allow us to derive upper bounds on the number of game moves.

The two lion strategies considered in the paper are presented next.

3.1. Polygonal Fixed Center Lion Strategy

In this subsection, a lion strategy for convex polygonal environments based on a fixed center is described. At the beginning of the game, let us set C_0 as in Definition 4, i.e., $C_0 = \mathbf{C}_0(r_0)$ for some r_0, and let us keep it fixed throughout the game. At any time step, the lion's move is performed according to Definition 2, with center C_0. We will refer to such a lion strategy as *Polygonal Fixed Center Lion Strategy (P-FCLS)*.

At a given time t, the cost (9) boils down to

$$J_t(C_0) = \|C_0 - V_t^{max}(C_0)\|^2 - \|C_0 - L_t\|^2, \ \forall t \geq 0. \tag{10}$$

The next lemma is instrumental for proving that the P-FCLS leads to a victory of the lion in a finite number of moves, irrespectively of the strategy adopted by the man.

Lemma 1. *Let the lion play the P-FCLS with center C_0. Then,*

$$J_{t+1}(C_0) \leq J_t(C_0) - 1.$$

Proof: According to Definition 2, L_t lies on the segment connecting M_t and C_0. Therefore, at each time t, there exists $r_t \geq 0$ such that $C_0 = L_t + r_t W_t$, with W_t given by (1). By Lemma 1 in [7], for any man's move one has

$$\|C_0 - L_{t+1}\|^2 \geq \|C_0 - L_t\|^2 + 1 \tag{11}$$

i.e., $r_{t+1}^2 \geq r_t^2 + 1$. This means that r_t is a strictly increasing sequence. From Definition 5, one gets that the contaminated area keeps decreasing, i.e., $\widehat{A}_{t+1}(C_0) \subset \widehat{A}_t(C_0)$. In turn, from (7) and (8), one has that $\|C_0 - V_t^{max}(C_0)\|$ is a nonincreasing sequence. Hence, by using (11), one has

$$\begin{aligned} J_{t+1}(C_0) &= \|C_0 - V_{t+1}^{max}(C_0)\|^2 - \|C_0 - L_{t+1}\|^2 \\ &\leq \|C_0 - V_t^{max}(C_0)\|^2 - \|C_0 - L_t\|^2 - 1 = J_t(C_0) - 1, \end{aligned}$$

which proves the claim. □

Lemma 1 states that, for a fixed center, the function J_t decreases at least by 1 at each step. Then, at a generic time, it is possible to exploit this result to derive an upper bound on the number of moves necessary to conclude the game.

Theorem 2. *Let the lion play the P-FCLS with center C_0. Then, $\lceil J_t(C_0) \rceil$ is an upper bound on the number of moves needed by the lion to win the game after time t.*

Proof: Let us set $N = \lceil J_t(C_0) \rceil$. By Lemma 1, J_t is a function which decreases at least by 1 at each step, so one has $J_{t+N}(C_0) \leq 0$. By (10), this means that $\|C_0 - V_{t+N}^{max}(C_0)\| \leq \|C_0 - L_{t+N}\|$. By Definition 2, L_{t+N} belongs to the segment with extremal points M_{t+N} and C_0, and then

$$\|C_0 - M_{t+N}\| > \|C_0 - L_{t+N}\| \geq \|C_0 - V_{t+N}^{max}(C_0)\| \ .$$

Since $V_{t+N}^{max}(C_0)$ is the reachable vertex farthest from C_0 at time $t+N$, this would imply M_{t+N} is outside the contaminated area $\widehat{A}_t(C_0)$, which means that the man is captured by the lion within time $t+N$. □

For ease of notation, in the rest of the paper we will omit the symbol $\lceil\cdot\rceil$ and refer directly to the function J_t as the upper bound on the number of moves.

According to Theorem 2, at a generic time t, the quantity $J_t(C_0)$ provides an upper bound on the remaining number of moves to end the game. In particular, the P-FCLS guarantees that the lion wins the game in at most $J_0(C_0)$ moves, for every initial condition L_0 and M_0, and for every choice of C_0, no matter of the strategy adopted by the man.

In [7], it has been shown that if the lion and the man are close enough, the optimal man strategy is to move orthogonally to W_t at any time t. Under these conditions, one has

$$\|C_0 - L_{t+1}\|^2 \simeq \|C_0 - L_t\|^2 + 1 \ . \tag{12}$$

So, if the farthest vertex $V_t^{max}(C_0)$ remains the same, $J_t(C_0)$ decreases by (about) 1 at each step, and the bound given by Theorem 2 is almost tight.

An alternative approach, which significantly improves over the P-FCLS, is presented in the next subsection.

3.2. Polygonal Moving Center Lion Strategy

Instead of fixing the center once and for all at the beginning of the game, it may be useful to change it at every move. A meaningful criterion for selecting the center is trying to minimize the remaining number of moves before the end of the game.

By Theorem 2, at time t, the game will last for at most J_t moves if the center is kept fixed for the rest of the game. Hence, a meaningful choice of the center is the one minimizing J_t in (9). Let us state the following definition.

Definition 6. *At a given time t, the* optimal radius r_t^* *is defined as*

$$r_t^* = \arg \min_{r\in[0, r_t^{max}]} J_t(\mathbf{C}_t(r)) \tag{13}$$

and the corresponding optimal center *is*

$$C_t^* = \mathbf{C}_t(r_t^*) = L_t + r_t^* W_t \ . \tag{14}$$

By building on the notion of optimal center, the *Polygonal Moving Center Lion Strategy (P-MCLS)* is defined as follows: at a generic time t, let the man move from M_t to M_{t+1}. Then,

- the lion moves from L_t to L_{t+1} according to Definition 2;
- the center is updated from C_t^* to C_{t+1}^* according to Definition 6.

The following result shows that the P-MCLS is always a winning strategy for the lion.

Theorem 3. *Let the lion play the P-MCLS. For any initial condition M_0 and L_0, the lion wins the game in at most $J_0(C_0^*)$ moves, with C_0^* given by* (14). *Moreover, at every time t, the remaining moves of the game are bounded by $J_t(C_t^*)$.*

Proof: Let C_0^* be given by (14). According to (13)-(14), at $t = 1$ one has

$$J_1(C_1^*) \le J_1(C_0^*) \le J_0(C_0^*) - 1 \tag{15}$$

where the first inequality is due to the fact that both C_0^* and C_1^* are candidate centers at $t = 1$ (with the latter minimizing J_1), while the second inequality stems from Lemma 1. By iterating the same reasoning at each time t, one gets

$$J_t(C_t^*) \le J_t(C_{t-1}^*) \le J_{t-1}(C_{t-1}^*) - 1 \tag{16}$$

which implies that $J_t(C_t^*) \le 0$ for $t \ge J_0(C_0^*)$. By (9), this means that $\|C_t^* - V_t^{max}(C_t^*)\| \le \|C_t^* - L_t\|$. Since $\|C_t^* - M_t\| > \|C_t^* - L_t\|$, one gets

$$\|C_t^* - M_t\| > \|C_t^* - V_t^{max}(C_t^*)\|$$

i.e., $M_t \notin \widehat{A}_t(C_t^*)$, and hence one can conclude that capture must occur in the first $J_0(C_0^*)$ moves. Clearly, the same idea can be applied at each time t, and hence $J_t(C_t^*)$ is an upper bound to the number of remaining moves from time t onwards. □

A byproduct of the proof of Theorem 3 is the fact that at each move t, the upper bound on the number of remaining moves for the P-MCLS decreases more than the corresponding bound for the P-FCLS. Indeed, being both C_{t-1}^* and C_t^* candidate centers at time t, and $J_t(C_t^*) \le J_t(C_{t-1}^*)$, moving the center from C_{t-1}^* to C_t^* reduces the bound more than keeping the center fixed at C_{t-1}^*. It should be remarked that this does not guarantee that the upper bound associated to the P-MCLS is always smaller than that of the P-FCLS, because the actual values of the bounds depend also on the strategy adopted by the man. Nevertheless, as recalled at the end of Section 3.1, if the lion and man are sufficiently close at the beginning of the game, the man can play in such a way that the P-FCLS bound $J_t(C_0)$ decreases approximately by 1 at every move. In such cases, the leftmost inequality in (16) guarantees that the P-MCLS bound dominates the P-FCLS one, i.e., $J_t(C_t^*) \le J_t(C_0)$, $\forall t$. This provides a strong motivation for choosing the moving center approach, as it will be confirmed by the simulation campaign presented in the next section.

Remark 1. *When using the P-FCLS, the center C_0 is fixed once and for all at the beginning of the game. Since $r_t^2 = \|C_0 - L_t\|^2$ increases by at least 1 at each move, the circle $\boldsymbol{\mathcal{C}}(C_0, r_t)$ is steadily growing. According to Definition 5, this implies that the cleared area is increasing by a finite quantity at each move and the game surely ends when the cleared area encompasses the whole $\mathcal{P}$.*

The cleared area related to P-MCLS evolves in a different way. At each time step, the optimal center and radius change according to Definition 6 and then there is no guarantee that $\widetilde{A}_t(C_t^*) \subseteq \widetilde{A}_{t+1}(C_{t+1}^*)$. Even if the cleared area may shrink at some move, the P-MCLS chooses the optimal center to minimize (9)*, which represents an upper bound on the game duration. Of course, also in this case, the cleared area at the end of the game will contain the whole environment, i.e., the contaminated area vanishes so that no safe move can be played by the man.*

Remark 2. *Notice that in Definition 4, it is required that the center C_t belongs to the polygon $\mathcal{P}$. This condition is essential to guarantee a feasible lion's move, i.e., a lion's move inside $\mathcal{P}$. In fact, as a counterexample, let us assume that $C_t \notin \mathcal{P}$, as depicted in Fig. 3. By performing the move according to Definition 2, the lion position at time $t+1$ may lie outside $\mathcal{P}$. On the contrary, since $\mathcal{P}$ is convex, if $C_t \in \mathcal{P}$ then the whole segment with extremal points M_{t+1} and C_t is inside $\mathcal{P}$ and then L_t is inside $\mathcal{P}$ too. Alternative approaches relaxing the constraint $C_t \in \mathcal{P}$ can be devised, but they require minor amendments of the lion's move in Definition 2.*

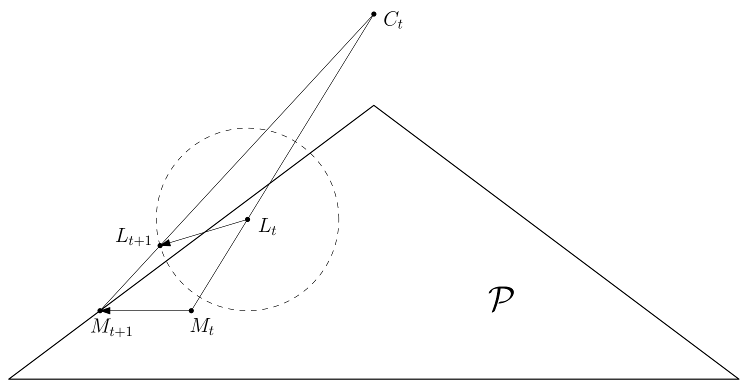

Figure 3: Example of an unfeasible lion's move when $C_t \notin \mathcal{P}$.

Remark 3. *The approach proposed in [9] for the case of multiple pursuers can be adapted to the problem considered in this paper, by placing virtual pursuers on the sides of the polygon. Then, it can be shown that the lion strategy called "spheres", adopted in [9], is equivalent to the P-FLCS, for any choice of the center C_0 consistent with Definition 4.*

4. Computation of the optimal center

In this section, an algorithm to compute the optimal radius and center according to Definition 6 is presented. By (14), once r_t^* is known, C_t^* is directly computed. So, we will focus on the computation of the optimal radius r_t^* in (13), which in general may not be trivial.

By exploiting (8), for a given r_t and $C_t = \mathbf{C}_t(r_t)$, let us rewrite (9) as

$$\begin{aligned} J_t(C_t) &= \|C_t - V_t^{max}(C_t)\|^2 - r_t^2 \\ &= \max_{V \in \mathcal{V}_t^R(C_t)} \|L_t + r_t W_t - V\|^2 - r_t^2 \\ &= \max_{V \in \mathcal{V}_t^R(C_t)} \|L_t - V\|^2 + r_t^2\|W_t\|^2 + 2r_t(L_t - V)'W_t - r_t^2 \\ &= \max_{V \in \mathcal{V}_t^R(C_t)} \|L_t - V\|^2 + 2r_t(L_t - V)'W_t \end{aligned} \tag{17}$$

where the last equality comes from $\|W_t\| = 1$ in (1).

Hence, the optimal radius expression (13) can be rewritten as

$$r_t^* = \arg \min_{r \in [0, r_t^{max}]} \max_{V \in \mathcal{V}_t^R(\mathbf{C}_t(r))} \|L_t - V\|^2 + 2r(L_t - V)' W_t \tag{18}$$

At a given time t, let us define for $i = 1, \ldots, n$

$$N_{i,t} = \|L_t - V^{(i)}\|^2$$

$$m_{i,t} = 2(L_t - V^{(i)})' W_t \ .$$

By substituting, (18) becomes

$$r_t^* = \arg \min_{r \in [0, r_t^{max}]} \max_{i : V^{(i)} \in \mathcal{V}_t^R(\mathbf{C}_t(r))} N_{i,t} + m_{i,t}\, r \ . \tag{19}$$

Let us consider the following LP

$$\begin{aligned} (\overline{J}, \overline{r}) = & \arg \min_{J,r} J \\ & s.t. : \\ & N_{i,t} + m_{i,t}\, r \le J \quad , \quad i = 1, \ldots, n \\ & 0 \le r \le r_t^{max} \end{aligned} \tag{20}$$

Let $\overline{C}$ be the center associated to the solution $\overline{r}$. The following cases can occur:

i) if $\mathcal{V}_t^R(\overline{C}) = \mathcal{V}$, i.e., all the vertices are reachable, then $r_t^* = \overline{r}$ and $J_t^* = \overline{J}$. In fact, under this condition, (19) can be written as

$$r_t^* = \arg \min_{r \in [0, r_t^{max}]} \max_{i = 1, \ldots, n} N_{i,t} + m_{i,t}\, r \tag{21}$$

whose solution is given by (20).

ii) if $\mathcal{V}_t^R(\overline{C}) \subset \mathcal{V}$, not all the vertices in $\mathcal{V}$ are reachable by the man, and so $\overline{J}$ provides an upper bound to J_t^*, i.e., $\overline{J} \ge J_t^*$.

Computing r_t^* in the latter case is more involved than in the former one: it requires a numerical procedure, which will be devised next.

Remark 4. *Notice that the optimal radius r_t^* (and hence the optimal center) may not be unique. An example of such a situation is depicted in Fig. 4, where all the points belonging to the segment $\mathcal{R}$ are optimal centers. In fact, for any r_t such that $C_t = \mathbf{C}_t(r_t) \in \mathcal{R}$, the reachable vertex farthest from C_t is V_t^{max}, and hence $J_t(C_t) = \|C_t - V_t^{max}\|^2 - r_t^2 = \|L_t - V_t^{max}\|^2$ which does not depend on r_t. In such cases, any $r_t^* \in \mathcal{R}$ can be chosen.*

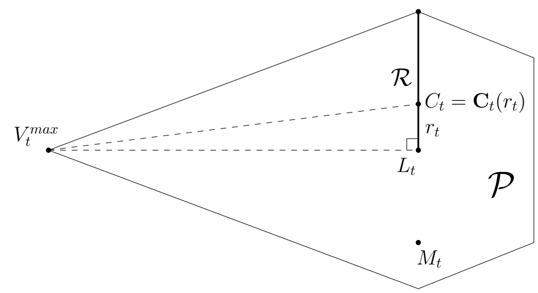

Figure 4: All the centers belonging to the segment $\mathcal{R}$ are optimal.

Let us define the following quantities

$$\begin{aligned}\widehat{\mathcal{V}}_t &= \{V \in \mathcal{V} \colon (L_t - V)' W_t \geq 0\}\\ \widetilde{\mathcal{V}}_t &= \{V \in \mathcal{V} \colon (L_t - V)' W_t \leq 0\}\ .\end{aligned}$$

Assuming the lion faces the man, $\widehat{\mathcal{V}}_t$ and $\widetilde{\mathcal{V}}_t$ contain the vertices of $\mathcal{V}$ in front of and behind the lion, respectively.

The following lemmas provide useful properties of the set of reachable vertices.

Lemma 2. *At a given time t, let $r_t \in [0, r_t^{max}]$ be given and let C_t be the corresponding center. Then $\widehat{\mathcal{V}}_t \subseteq \mathcal{V}_t^R(C_t)$.*

Proof: Let S be the line orthogonal to W_t passing through L_t. Since M_t, L_t and C_t are collinear, then $\boldsymbol{\mathcal{C}}(C_t, r_t) \cap S = \{L_t\}$. So, $\widehat{\mathcal{V}}_t \cap \widetilde{A}_t(C_t) = \emptyset$ and hence $\widehat{\mathcal{V}}_t \subseteq \widehat{A}_t(C_t)$. The result follows from the definition of $\mathcal{V}_t^R(C_t)$ in (7). □

Lemma 3. *At a given time t, let $\tilde{r}_t \geq 0$ be given, such that $\tilde{r}_t < r_t \leq r_t^{max}$. Let $\widetilde{C}_t = \mathbf{C}_t(\tilde{r}_t)$ and $C_t = \mathbf{C}_t(r_t)$. Then $\mathcal{V}_t^R(\widetilde{C}_t) \supseteq \mathcal{V}_t^R(C_t)$.*

Proof: Since $\widetilde{C}_t$, C_t and L_t are collinear, one has $\boldsymbol{\mathcal{C}}(\widetilde{C}_t, \tilde{r}_t) \subset \boldsymbol{\mathcal{C}}(C_t, r_t)$. Then, by the definition of contaminated area in (6), one has $\widehat{A}_t(\widetilde{C}_t) \supseteq \widehat{A}_t(C_t)$ and the result follows directly. □

It is now possible to state the theorem which will be used to design an algorithm for the computation of r_t^*.

Theorem 4. *At a given time t, let $r_t \in [0, r_t^{max}]$, C_t be the corresponding center and*

$$\mathcal{V}_t^{max}(C_t) = \{V \in \mathcal{V}_t^R(C_t) \colon \|C_t - V\| \geq \|C_t - \overline{V}\|,\ \forall \overline{V} \in \mathcal{V}_t^R(C_t)\}. \tag{22}$$

Then,

i) if $\mathcal{V}_t^{max}(C_t) \cap \widehat{\mathcal{V}}_t \neq \emptyset$, then $\exists r_t^* \colon r_t^* \leq r_t$;

ii) if $\mathcal{V}_t^{max}(C_t) \cap \widetilde{\mathcal{V}}_t \neq \emptyset$, then $\exists r_t^* \colon r_t^* \geq r_t$.

Proof: i) Let $\widehat{V} \in \mathcal{V}_t^{max}(C_t) \cap \widehat{\mathcal{V}}_t$. By (17), one has

$$J_t(C_t) = \|C_t - V_t^{max}(C_t)\|^2 - r_t^2 = \|L_t - \widehat{V}\|^2 + 2r_t(L_t - \widehat{V})'W_t \ . \tag{23}$$

Let $\hat{r}_t \in (r_t, r_t^{max}]$ and $\widehat{C}_t = \mathbf{C}_t(\hat{r}_t)$. Since $\widehat{V} \in \widehat{\mathcal{V}}_t$, then by Lemma 2, $\widehat{V} \in \mathcal{V}_t^R(\widehat{C}_t)$. By (17), one gets

$$\begin{aligned} J_t(\widehat{C}_t) &= \max_{V \in \mathcal{V}_t^R(\widehat{C}_t)} \|L_t - V\|^2 + 2\hat{r}_t(L_t - V)'W_t \\ &\geq \|L_t - \widehat{V}\|^2 + 2\hat{r}_t(L_t - \widehat{V})'W_t \geq J_t(C_t) \end{aligned}$$

where the last inequality follows by (23) and by $\widehat{V} \in \widehat{\mathcal{V}}_t$, i.e., $(L_t - \widehat{V})'W_t \geq 0$. So, for any $\hat{r}_t > r_t$, one has $J_t(\widehat{C}_t) \geq J_t(C_t)$ and then item i) is proved.
ii) Let $\widetilde{V} \in \mathcal{V}_t^{max}(C_t) \cap \widetilde{\mathcal{V}}_t$. By (17), one has

$$J_t(C_t) = \|C_t - V_t^{max}(C_t)\|^2 - r_t^2 = \|L_t - \widetilde{V}\|^2 + 2r_t(L_t - \widetilde{V})'W_t \ . \tag{24}$$

Let $\tilde{r}_t \in [0, r_t)$ and $\widetilde{C}_t = \mathbf{C}_t(\tilde{r}_t)$. Since $\widetilde{V} \in \mathcal{V}_t^{max}(C_t)$, by (22) one has $\widetilde{V} \in \mathcal{V}_t^R(C_t)$, and by Lemma 3, $\widetilde{V} \in \mathcal{V}_t^R(\widetilde{C}_t)$. By (17), one gets

$$\begin{aligned} J_t(\widetilde{C}_t) &= \max_{V \in \mathcal{V}_t^R(\widetilde{C}_t)} \|L_t - V\|^2 + 2\tilde{r}_t(L_t - V)'W_t \\ &\geq \|L_t - \widetilde{V}\|^2 + 2\tilde{r}_t(L_t - \widetilde{V})'W_t \geq J_t(C_t) \end{aligned}$$

where the last inequality follows by (24) and by $\widetilde{V} \in \widetilde{\mathcal{V}}_t$, i.e., $(L_t - \widetilde{V})'W_t \leq 0$. So, for any $\tilde{r}_t < r_t$, one has $J_t(\widetilde{C}_t) \geq J_t(C_t)$ which proves ii). □

The intuition at the basis of Theorem 4 is as follows. Let a center C_t and the corresponding radius r_t be given. Assume for simplicity that $V_t^{max}(C_t)$ is the unique vertex farthest from C_t. Let the lion face the man. Theorem 4 states that, if $V_t^{max}(C_t)$ is in front of the lion, i.e., $V_t^{max}(C_t) \in \widehat{\mathcal{V}}_t$, then there exists at least one optimal radius r_t^* such that $r_t^* \leq r_t$ (recall that by Remark 4, different values of the radius may be optimal). An opposite result holds if $V_t^{max}(C_t)$ is behind the lion.

Theorem 4 allows one to devise a numerical procedure for computing r_t^*, reported in Algorithm 1. If the condition on line 7 is satisfied, according to (21), r_t^* is the solution of the LP (20) and the procedure ends. Otherwise, a bisection on the interval $[0, r_t^{max}]$ is performed, by exploiting the result of Theorem 4.

Remark 5. *The computational burden of Algorithm 1 is mainly dictated by the solution of the two LPs in* (5) *and* (20) *and by the chosen tolerance* ϵ_r*. It is worth remarking that there is not a strict precision requirement in the computation of the optimal center* C_t^**. In fact, the result in Theorem 3 holds for any choice of the center* C_t^* *which guarantees* $J_t(C_t^*) \leq J_t(C_{t-1}^*)$ *in the left hand side of* (16).

Algorithm 1 Algorithm for the computation of r_t^*, C_t^* and J_t^*.

Data: $\mathcal{V}$, L_t, M_t
Result: r_t^*, C_t^*, J_t^*
$W_t = (L_t - M_t)/\|L_t - M_t\|$
$r^{max} = solve\_LP\_in\_(5)$
$(\overline{J}, \overline{r}) = solve\_LP\_in\_(20)$
$\overline{C} = L_t + \overline{r} W_t$
if $\mathcal{V}_t^R(\overline{C}) == \mathcal{V}$ **then**
  $r_t^* = \overline{r}$, $C_t^* = \overline{C}$, $J_t^* = \overline{J}$
else
  $r = 0$, $R = r^{max}$
  while $(R - r) > \varepsilon_r$ **do**
    $\hat{r} = (r + R)/2$, $\widehat{C} = L_t + \hat{r} W_t$
    $V_t^{max}(\widehat{C}) = \arg \max\limits_{V \in \mathcal{V}_t^R(\widehat{C})} \|\widehat{C} - V\|$
    if $(V_t^{max}(\widehat{C}) - L_t)' \, W_t \leq 0$ **then**
      $R = \hat{r}$
    else
      $r = \hat{r}$
    end if
  end while
  $r_t^* = R$, $C_t^* = L_t + r_t^* W_t$
  $V_t^{max}(C_t^*) = \arg \max\limits_{V \in \mathcal{V}_t^R(C_t^*)} \|C_t^* - V\|$
  $J_t^* = \|C_t^* - V_t^{max}(C_t^*)\|^2 - {r_t^*}^2$
end if

5. Numerical simulations

In this section, the results of a campaign of numerical simulations are reported, to show the effectiveness of the P-MCLS. In particular, a comparison between P-FCLS and P-MCLS is reported. When playing the P-FCLS, the fixed center C_0^* is chosen according to (13)-(14).

At present, the optimal strategy for the man, allowing him to survive as long as possible, has not been devised yet. However, if the two players are close enough, it can be shown that the man's move which maximizes $J_{t+1}(C_t)$ is orthogonal to W_t (see [7]). Hence, in general, the man can choose between two feasible moves. Depending on which one he chooses, the following strategies can be defined:

- MSL: looking at the lion, between the two feasible moves, the man chooses the left one.
- MSR: similar to MSL, but the man selects the right move.
- MSV$^{(i)}$: between the two feasible moves, the man goes to the point closest to a given vertex $V^{(i)}$. By adopting this strategy, the man will likely be captured in the surroundings of $V^{(i)}$.
- MSJ: the man moves to the point which maximizes $J_{t+1}(C^*_{t+1})$. Notice that maximizing $J_{t+1}(C^*_{t+1})$ does not necessarily imply that this move is globally optimal. Anyway, this strategy is locally optimal in the sense that it maximizes the same function the lion is minimizing, when he plays the P-MCLS.

To evaluate the performance of the two lion strategies, Monte Carlo simulations have been performed. Ten different polygons have been considered. For each game environment, 100 initial conditions have been randomly generated.

All the man strategies introduced above have been played against each lion algorithm[1]. For each game, the man strategy which allows him to survive longer has been considered. Hereafter, all the reported results refer to such games. Notice that, although the optimal man strategy is unknown, the obtained results can be seen as worst-case performances of the lion strategies, since they have been obtained for the best man strategy among all the considered ones.

For a given polygon $p = 1, \ldots, 10$ and game $g = 1, \ldots, 100$, let $N^F_p(g)$ and $N^M_p(g)$ be the number of moves needed by the lion to capture the man for the P-FCLS and P-MCLS, respectively.

To assess the performance of the lion algorithms, for a given environment p and game g, the ratio between $N^M_p(g)$ and $N^F_p(g)$ is considered, i.e.,

$$R_p(g) = \frac{N^M_p(g)}{N^F_p(g)} \; .$$

In all the 1000 games played, the average value of $R_p(g)$ turns out to be 0.65 with standard deviation 0.24. For each specific polygon p, the average $R_p(g)$ ranges from 0.22 to 0.81. Analyzing the shape of the ten environments, it turns out that the smallest values of R_p occur for "squeezed" polygons while largest values are related to "roundish" ones. This suggests that the benefits of the P-MCLS are more significant when the evader can choose among qualitatively different escape directions.

In Fig. 5, the value of J_t averaged over 100 games is reported, for one of the considered environment. The man strategy which allows him to survive longer is considered for each game. One may notice that when playing the P-FCLS $J_t(C^*_0)$ decreases by approximately 1 at each step, as expected. On

[1] Regarding MSV$^{(i)}$, one game for each vertex $V^{(i)}$ has been played.

the contrary, when the P-MCLS is adopted, $J_t(C_t^*)$ provides a faster decrease, especially during the first moves, providing a much smaller bound on the capture time.

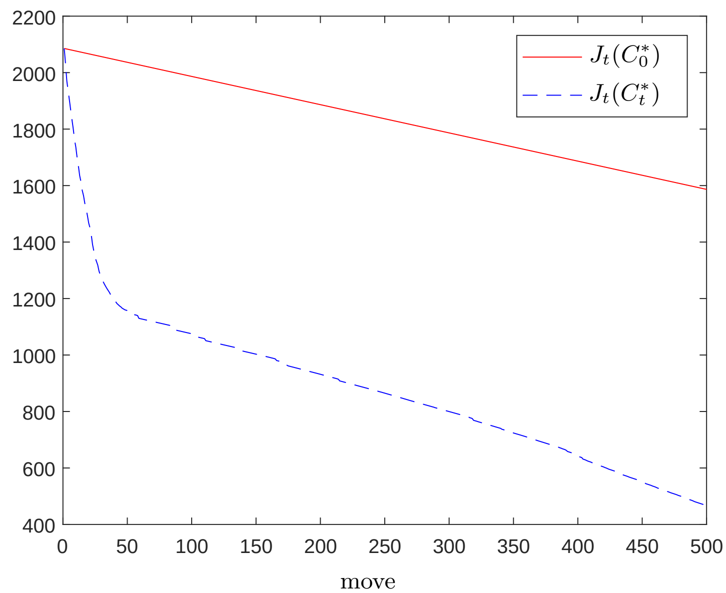

Figure 5: Average value of J_t over the first 500 moves for the P-FCLS (solid red) and the P-MCLS (dashed blue).

Concerning the strategies adopted by the man, the simulations do not reveal a strategy which is significantly better than the others. In fact, among all the simulated games, the MSV$^{(i)}$ strategy turned out to be the most effective in the 59.5% of the games. However, recall that this is rather a set of strategies depending on the specific vertex $V^{(i)}$ and in most cases there was not a preferred vertex prevailing over the others. This confirms that the optimal strategy for the man remains an open problem.

6. Conclusions

Two new pursuit strategies have been proposed for a formulation of the lion and man problem within convex polygonal environments. The solution based on a fixed center has the benefit of being extremely simple and needs only the selection of a suitable center at the beginning of the game. The moving center strategy requires a numerical procedure for updating the center at each move. On the other hand, numerical simulations have shown that it outperforms the fixed center strategy in terms of both upper bound on the game duration and actual number of moves necessary for capturing the evader.

The moving center approach looks very promising and its application to more complex settings is the subject of ongoing research. For instance, one may consider the case of multiple pursuers chasing a single evader. Another interesting scenario is that of environments described by simply connected polygons. In [11], a pursuer strategy similar to the P-FCLS has been adopted for a game with two pursuers against one evader. The use of a moving center strategy is expected to remarkably reduce the capture time, as observed in the case of convex polygons.

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