An improved lion strategy for the lion and man problem

Marco Casini, Andrea Garulli

***Abstract*—In this paper, a novel lion strategy for David Gale's lion and man problem is proposed. The devised approach enhances a popular strategy proposed by Sgall, which relies on the computation of a suitable "center". The key idea of the new strategy is to update the center at each move, instead of computing it once and for all at the beginning of the game. Convergence of the proposed lion strategy is proven and an upper bound on the game length is derived, which dominates the existing bounds.**

***Index Terms*—Robotics, autonomous systems, game theory**

I. Introduction

PURSUIT-EVASION games have attracted the interest of researchers for long time, both for the intriguing mathematics they require (see [1] for a nice introduction), and for the variety of applications they find in different contexts, ranging from mobile robotics to surveillance, resource harvesting, network security and many others. When the game is played in a limited environment, problems become even more challenging. Among the huge number of different formulations (an extensive survey is provided in [2]), two main classifications can be singled out, concerning respectively time and space being continuous or discrete. If continuous time is assumed, it is well known that an evader can indefinitely escape a single pursuer travelling at the same velocity, even in very simple continuous environments, like a circle [3]. On the other hand, if time is discrete, the pursuer can capture the evader in finite time, in many situations of interest. This has generated a rich literature, considering different assumptions on the number of pursuers, the structure of the environment and the information available to the players (see, e.g., [4]–[9] and references therein).

A fundamental problem at the basis of the above literature is the so-called *lion and man problem*, whose formulation is ascribed to Gale (see problem 31 in [10]). A lion and a man move alternately in the positive quadrant of the plane, travelling a distance of at most one unit at each move. It is known that the lion can catch the man in finite time, provided that his initial coordinates are componentwise larger than those of the man. Nevertheless, the optimal lion strategy is still an open problem. In [11], Sgall has proposed a nice strategy for the lion, which guarantees capture in finite time. Moreover, he has given an upper bound on the capture time which is achieved for some specific initial conditions. Sgall's strategy is based on the definition of a fixed *center*, depending on the initial lion and man positions: then, the lion always keeps on the line connecting the center to the man's position, until capture occurs. This strategy has been used in several mobile robotics application, as reported in the tutorial [12]. In particular, a slight variation of the solution proposed in [11] is adopted iteratively in [5], where it is instrumental to devise a strategy for two pursuers to capture an evader in simply connected polygonal environments. A similar strategy is employed in [13], when dealing with games in monotone polygons with line-of-sight visibility.

In this paper, a new lion strategy is proposed for the lion and man problem, which improves the one proposed in [11]. The main idea is to compute a new center at each move, in order to enhance the advantage gained by the lion in a single step. This turns out to be effective also on the whole, as it allows one to derive an upper bound on the maximum number of moves required to guarantee capture, which dominates the one given in [11]. Simulations of randomly generated games confirm the superiority of the proposed strategy.

The paper is organized as follows. In Section II, the lion and man problem is formulated. The solution proposed in [11] is reviewed in Section III. The new strategy is introduced in Section IV and its convergence properties are derived in Section V. Numerical examples are reported in Section VI, while in Section VII conclusions are drawn.

II. Problem formulation

The notation adopted in the paper is standard. Let $\mathbb{N}$ and $\mathbb{R}^n_+$ denote the set of all natural numbers and the n-dimensional Euclidean space of non-negative numbers, respectively. Let $\lceil x \rceil$ be the smallest integer greater or equal to x. A row vector with elements $v_1, \dots, v_n$ is denoted by $V = [v_1, \dots, v_n]$, while V' is the transpose of V.

In this paper, we consider the version of the *lion and man problem* formulated by David Gale. Two players, a man and a lion, can move in the first quadrant of the Cartesian plane. Time is assumed discrete, while space is continuous. At each round (hereafter called *time*) both players are allowed to move to any point inside the non-negative quadrant, with distance less or equal to a given radius r from their current position. Hereafter, it will be assumed $r = 1$ without loss of generality. The man moves first. Let us denote by $M_t \in \mathbb{R}^2_+$ and $L_t \in \mathbb{R}^2_+$ the man and lion position at time t, respectively. Hence, $\|M_{t+1} - M_t\| \leq 1$, $\|L_{t+1} - L_t\| \leq 1$. The game ends (lion wins) if the lion moves *exactly* to the man position. If the man can escape indefinitely from the lion, the man wins. It is assumed that the initial man coordinates are strictly smaller than the corresponding lion coordinates, otherwise it is easy to see that the man wins the game by moving straight up or right.

M. Casini and A. Garulli are with the Dipartimento di Ingegneria dell'Informazione e Scienze Matematiche, Università di Siena, via Roma 56, 53100 Siena, Italy. E-mail: casini@dii.unisi.it, garulli@dii.unisi.it.

III. Fixed Center Lion Strategy

Before introducing the proposed lion strategy, let us recall the one devised in [11], hereafter referred to as *Fixed Center Lion Strategy (FCLS)*. If the initial man coordinates are strictly smaller than the corresponding lion coordinates, the FCLS allows the lion to capture the man in a finite number of moves, for any man strategy.

Definition 1: Let M_0 and L_0 be the man and lion position at time 0, respectively. Let $C_0 = [x_0,\, y_0]' \in \mathbb{R}^2_+$ be the point satisfying

1) $C_0 = L_0 + \eta(L_0 - M_0)$, with $\eta > 0$;
2) $\|C_0 - L_0\| = \max\{x_0, y_0\}$.

Then C_0 is called the *center* of the FCLS.

Let C_0 be the center of the FCLS. Such a center is fixed and does not change during the game. At a given time t, let the man move from M_t to M_{t+1}. The lion adopts the following strategy (FCLS):

- if $\|M_{t+1} - L_t\| \le 1$, then the lion moves to M_{t+1} and catches the man;
- otherwise, the lion moves to a point on the line connecting M_{t+1} to C_0 with unitary distance from L_t. Between the two points satisfying such a condition, he chooses the one farther from C_0.

Let $C_0 = [x_0,\, y_0]'$ and denote by r_0 and m_0 the greatest and smallest element of C_0, respectively, i.e., $r_0 = \max\{x_0, y_0\}$ and $m_0 = \min\{x_0, y_0\}$. Let us denote by N^{FCLS}_{max} the upper bound derived in [11] on the maximum number of moves needed by the lion to catch the man by using FCLS. The next results, proved in [11], will be useful in the sequel.

Proposition 1: Let the lion play the FCLS. At every t, one has

i) $\|L_t - C_0\|^2 + 1 \le \|L_{t+1} - C_0\|^2 \le \|C_0\|^2$;
ii) both elements of $C_0 - L_t$ are strictly positive.

Proposition 2: Let the lion play the FCLS. Then, the lion captures the man in a number of moves equal at most to

$$N^{FCLS}_{max} = \left\lceil \|C_0\|^2 - \|C_0 - L_0\|^2 \right\rceil = \left\lceil m_0^2 \right\rceil \tag{1}$$

The bound in (1) has been proved to be tight whenever the lion and the man start sufficiently close to each other. In this case, the optimal strategy for the man is to move orthogonally w.r.t. the line connecting him to C_0.

IV. Moving Center Lion Strategy

In this section, the proposed lion strategy, hereafter referred to as *Moving Center Lion Strategy (MCLS)*, is introduced. The main difference between MCLS and FCLN regards the computation of the center. While in FCLS the center in computed once and for all at the beginning of the game, in the proposed strategy it is updated at each move, and then it is used to compute the lion move.

Before describing the devised lion strategy, let us introduce some definitions which will be used throughout the paper.

Definition 2: Let M_t and L_t be the man and lion position at time t, respectively. Let $C_t = [x_t,\, y_t]' \in \mathbb{R}^2_+$ be the point satisfying

1) $C_t = L_t + \eta(L_t - M_t)$, with $\eta > 0$
2) $\|C_t - L_t\| = \max\{x_t, y_t\}$.

Then C_t is called the *center* of the MCLS at time t.

Definition 3: At a given time t, let us define the following quantities:

1) $r_t = \max\{x_t, y_t\} = \|C_t - L_t\|$
2) $m_t = \min\{x_t, y_t\}$
3) $\tilde{r}_{t+1} = \|L_{t+1} - C_t\|$.

At a given time t, let the man move from M_t to M_{t+1}. The lion moves according to the following strategy (MCLS):

- compute the center C_t, based on man and lion position at time t, according to Definition 2;
- if $\|M_{t+1} - L_t\| \le 1$, then the lion moves to M_{t+1} and catches the man;
- otherwise, the lion moves to a point on the line connecting M_{t+1} to C_t with unitary distance from L_t. Between the two points satisfying such a condition, he chooses the one farther from C_t.

The following propositions hold.

Proposition 3: Let the lion play the MCLS. At every time t, one has

i) $\|L_t - C_t\|^2 + 1 \le \|L_{t+1} - C_t\|^2 \le \|C_t\|^2$;
ii) both elements of $C_t - L_{t+1}$ are strictly positive ;
iii) the following inequalities hold

$$r_t^2 + 1 \le \tilde{r}_{t+1}^2 \le r_t^2 + m_t^2 \ . \tag{2}$$

Proof: Items $i) - ii)$ follow directly from Proposition 1, because at each time t, the center C_t is defined in the same way as C_0 in Definition 1. Item $iii)$ stems from Definition 3 and item $i)$. ■

Proposition 4: At a given time t, let $m_t \le 1$ and let the lion play the MCLS. Then, the lion captures the man in one move.

Proof: The proof is a direct consequence of Proposition 2, with $C_0 = C_t$ and $L_0 = L_t$. ■

Remark 1: It is worth stressing the key difference between the FCLS and the MCLS. The FCLS is based on a circle centered in the fixed center C_0, with radius equal to the distance center-lion. Such a circle defines a *cleared area*, i.e. a region the man cannot enter, otherwise he is captured in the next move. Capture is guaranteed by the fact that the radius of the circle steadily increases at each move and will include the origin in finite time. The MCLS proposed in this paper is based on a different invariant. A cleared area is the portion of the plane outside the rectangle whose opposite vertices are the origin and the moving center C_t (i.e., the dashed rectangle in the first quadrant of Fig. 2). In the next section, it will be proven that the center moves in such a way that this rectangle shrinks at every move. Then, capture is achieved by proving that the minimum side of the rectangle goes below 1 in finite time, which guarantees that the lion wins at the next move, according to Proposition 4.

V. Convergence Analysis

In this section, an upper bound to the maximum number of moves needed by the lion to catch the man, when using the MCLS, is derived. Moreover, it is shown that such an upper bound is always smaller than the upper bound N^{FCLS}_{max} provided by the FCLS.

Before stating the main results, some technical lemmas are needed. The first one states the key invariant of the proposed strategy: both coordinates of the moving center are strictly decreasing, i.e., the center keeps moving leftwards and downwards, approaching the origin.

Lemma 1: Let $C_t = [x_t, y_t]$. Then, at every time t, $x_{t+1} < x_t$ and $y_{t+1} < y_t$.

Proof: In the MCLS, L_{t+1} lies on the line connecting C_t and M_{t+1}. On the other hand, by Definition 2, C_{t+1} lies on the line joining L_{t+1} and M_{t+1}. Hence, C_t, L_{t+1} and C_{t+1} are collinear, i.e.

$$C_{t+1} = C_t + (C_t - L_{t+1})\alpha \triangleq C_t + [d_x, d_y]' \qquad (3)$$

where $\alpha, d_x, d_y \in \mathbb{R}$. By Proposition 3, both coordinates of $C_t - L_{t+1}$ are strictly positive, i.e., $sign(d_x) = sign(d_y) = sign(\alpha)$. Hence, it is sufficient to show that $\alpha < 0$. To arrive at a contradiction, assume $\alpha \geq 0$ and let us define $d = \sqrt{d_x^2 + d_y^2} \geq 0$, see Fig. 1. Then, being $d_x \geq 0$ and $d_y \geq 0$, from (3) and Definition 3, one has

$$\begin{aligned}\|C_{t+1} - L_{t+1}\| = \tilde{r}_{t+1} + d > r_t + d \geq r_t + \max\{d_x, d_y\} \\ \geq \max\{x_t + d_x, y_t + d_y\} = \max\{x_{t+1}, y_{t+1}\}\end{aligned}$$

where the strict inequality comes from (2). Since $\|C_{t+1} - L_{t+1}\| > \max\{x_{t+1}, y_{t+1}\}$, C_{t+1} cannot be a center, according to Definition 2. ■

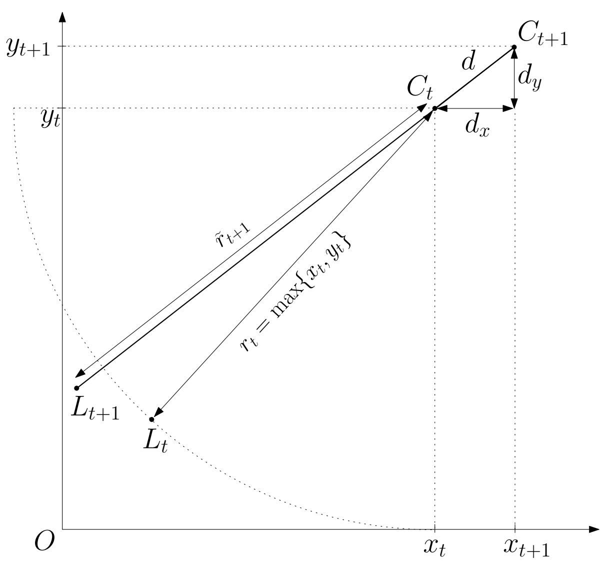

Fig. 1. The contradiction in the proof of Lemma 1.

The next Lemma provides an explicit expression for the worst move of the lion L_{t+1}, in terms of the minimum coordinate m_{t+1} of the updated center C_{t+1}.

Lemma 2: Let r_t and m_t be given according to Definition 3. Let $\tilde{r}_{t+1}$ be a fixed constant, such that $\tilde{r}_{t+1} > r_t$ and assume that L_{t+1} satisfies $\|C_t - L_{t+1}\| = \tilde{r}_{t+1}$. Then,

$$\begin{aligned}\widehat{m}_{t+1} &\triangleq \max_{L_{t+1}:\|C_t - L_{t+1}\| = \tilde{r}_{t+1}} m_{t+1} \\ &= \max\left\{ m_t \frac{r_t - \sqrt{\tilde{r}_{t+1}^2 - m_t^2}}{\tilde{r}_{t+1} - \sqrt{\tilde{r}_{t+1}^2 - m_t^2}},\, r_t \frac{m_t - \sqrt{\tilde{r}_{t+1}^2 - r_t^2}}{\tilde{r}_{t+1} - \sqrt{\tilde{r}_{t+1}^2 - r_t^2}} \right\}.\end{aligned} \qquad (4)$$

Proof: Let us consider the case $r_t = x_t$ (the case $r_t = y_t$ is analogous). Let $C_{t+1} = [x_{t+1}, y_{t+1}]$ be the center at time $t+1$, and θ be the angle between the x axis and the vector $C_t - L_{t+1}$. Notice that, since C_{t+1} lies on the line connecting L_{t+1} and C_t, θ is also the angle between the x axis and the vector $C_{t+1} - L_{t+1}$ (see Fig. 2). It follows that $\theta \in [\overline{\theta}, \underline{\theta}]$ where

$$\underline{\theta} = \arccos\left(\frac{r_t}{\tilde{r}_{t+1}}\right) \quad \text{and} \quad \overline{\theta} = \arcsin\left(\frac{m_t}{\tilde{r}_{t+1}}\right) .$$

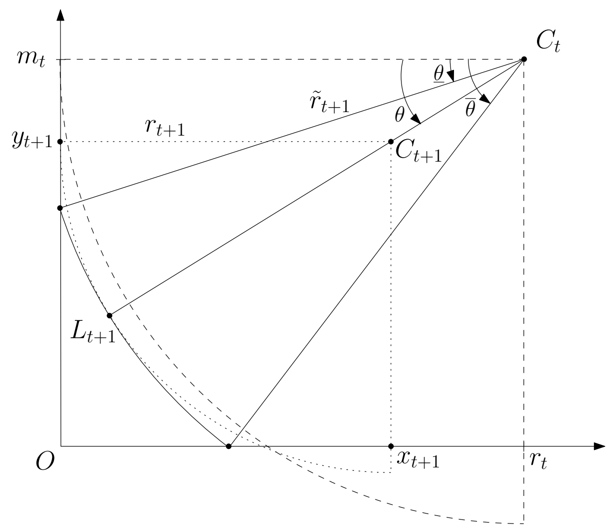

Fig. 2. Sketch of the proof of Lemma 2.

Let us define $\delta = \tilde{r}_{t+1} - r_{t+1}$. By Definition 2 and Lemma 1, it turns out $\delta > 0$. Moreover, one has

$$x_{t+1} = \quad r_t - \delta\cos\theta \quad \triangleq f_x(\theta) \qquad (5)$$

$$y_{t+1} = \quad m_t - \delta\sin\theta \quad \triangleq f_y(\theta) \; . \qquad (6)$$

Let us define $\mathcal{M}(\theta) = \min\{f_x(\theta), f_y(\theta)\}$ and $\mathcal{R}(\theta) = \max\{f_x(\theta), f_y(\theta)\}$. Then, finding $\widehat{m}_{t+1}$ defined in (4), boils down to

$$\begin{aligned}\widehat{m}_{t+1} &= \max_{\theta \in [\overline{\theta}, \underline{\theta}]} \min\{x_{t+1}, y_{t+1}\} \\ &= \max_{\theta \in [\overline{\theta}, \underline{\theta}]} \min\{f_x(\theta), f_y(\theta)\} = \max_{\theta \in [\overline{\theta}, \underline{\theta}]} \mathcal{M}(\theta) \; . \end{aligned} \qquad (7)$$

Let us analyze the case $\mathcal{M}(\theta) = f_x(\theta)$, $\mathcal{R}(\theta) = f_y(\theta)$. Notice that there exists at least one value of θ such that this condition is satisfied. In fact, for $\theta = \underline{\theta}$ one has

$$\begin{aligned}\mathcal{M}(\underline{\theta}) &= f_x(\underline{\theta}) = r_t - (\tilde{r}_{t+1} - r_{t+1})\cos(\underline{\theta}) \\ &= \frac{r_{t+1}\, r_t}{\tilde{r}_{t+1}} < r_{t+1} = f_y(\underline{\theta}) = \mathcal{R}(\underline{\theta}) \; .\end{aligned}$$

Since $\mathcal{R}(\theta) = f_y(\theta) = r_{t+1}$, by (6) one has $r_{t+1} = m_t - (\tilde{r}_{t+1} - r_{t+1})\sin(\theta)$ which leads to

$$r_{t+1} = \frac{m_t - \tilde{r}_{t+1}\sin(\theta)}{1 - \sin(\theta)} \; . \qquad (8)$$

By substituting (8) into (5), after some algebra one gets

$$\begin{aligned}\mathcal{M}(\theta) &= f_x(\theta) = r_t - (\tilde{r}_{t+1} - r_{t+1})\cos(\theta) \\ &= r_t - \frac{(\tilde{r}_{t+1} - m_t)\cos(\theta)}{1 - \sin(\theta)} \; .\end{aligned} \qquad (9)$$

Taking derivatives w.r.t. θ one obtains

$$\frac{\partial \mathcal{M}(\theta)}{\partial \theta} = \frac{m_t - \tilde{r}_{t+1}}{1-\sin(\theta)} < 0$$

since $\tilde{r}_{t+1} > r_t \geq m_t$. Since the minimum feasible value of θ is $\underline{\theta}$ and by using (8), one has

$$\max_{\theta:\mathcal{M}(\theta)=f_x(\theta)} \mathcal{M}(\theta) = \mathcal{M}(\underline{\theta}) = \frac{r_{t+1}\, r_t}{\tilde{r}_{t+1}}$$

$$= \frac{r_t}{\tilde{r}_{t+1}} \frac{m_t - \tilde{r}_{t+1}\sin(\underline{\theta})}{1-\sin(\underline{\theta})} = r_t \frac{m_t - \sqrt{\tilde{r}_{t+1}^2 - r_t^2}}{\tilde{r}_{t+1} - \sqrt{\tilde{r}_{t+1}^2 - r_t^2}} \ . \quad (10)$$

Let us now repeat the same reasoning for the case in which $\mathcal{M}(\theta) = f_y(\theta)$. Notice that $\theta = \overline{\theta}$ satisfies such a condition, yielding

$$\begin{aligned} \mathcal{M}(\overline{\theta}) &= f_y(\overline{\theta}) = m_t - (\tilde{r}_{t+1} - r_{t+1})\sin(\overline{\theta}) \\ &= \frac{r_{t+1}\, m_t}{\tilde{r}_{t+1}} < r_{t+1} = f_x(\overline{\theta}) = \mathcal{R}(\overline{\theta}) \ . \end{aligned}$$

Since $\mathcal{R}(\theta) = f_x(\theta) = r_{t+1}$, by (5) one has $r_{t+1} = r_t - (\tilde{r}_{t+1} - r_{t+1})\cos(\theta)$ which leads to

$$r_{t+1} = \frac{r_t - \tilde{r}_{t+1}\cos(\theta)}{1-\cos(\theta)} \ . \quad (11)$$

Substituting (11) into (6), after some algebra one gets

$$\begin{aligned} \mathcal{M}(\theta) &= f_y(\theta) = r_t - (\tilde{r}_{t+1} - r_{t+1})\sin(\theta) \\ &= m_t - \frac{(\tilde{r}_{t+1} - r_t)\sin(\theta)}{1-\cos(\theta)} \ . \end{aligned} \quad (12)$$

Taking derivatives w.r.t. θ one obtains

$$\frac{\partial \mathcal{M}(\theta)}{\partial \theta} = \frac{\tilde{r}_{t+1} - r_t}{1-\cos(\theta)} > 0 \ .$$

Thus,

$$\max_{\theta:\mathcal{M}(\theta)=f_y(\theta)} \mathcal{M}(\theta) = \mathcal{M}(\overline{\theta}) = \frac{r_{t+1}\, m_t}{\tilde{r}_{t+1}}$$

$$= \frac{m_t}{\tilde{r}_{t+1}} \frac{r_t - \tilde{r}_{t+1}\cos(\overline{\theta})}{1-\cos(\overline{\theta})} = m_t \frac{r_t - \sqrt{\tilde{r}_{t+1}^2 - m_t^2}}{\tilde{r}_{t+1} - \sqrt{\tilde{r}_{t+1}^2 - m_t^2}} \ . \quad (13)$$

The result follows directly by (10) and (13). ■

In order to show that the MCLS leads to capture of the man in a finite number of moves, it is sufficient to prove that the strategy leads to $m_t \leq 1$ for some finite t (recall Proposition 4). In this respect, the worst situation for the lion is the one in which m_t is maximized at each step. Lemma 2 states that for given C_t and $\tilde{r}_{t+1}$, the lion location L_{t+1} which maximizes the smallest element of the next center C_{t+1}, is one of the two points on the coordinate axes with distance $\tilde{r}_{t+1}$ from C_t. These points correspond to the extreme angles $\underline{\theta}$ and $\overline{\theta}$ for the direction of $C_t - L_{t+1}$ in Fig. 2.

Now, in order to establish an upper bound on the capture time, it is useful to find the worst-case position of the center C_t, resulting in the maximum possible value of m_{t+1}. This gives the worst possible center move from C_t to C_{t+1}.

Lemma 3: At a given time t, let $\widehat{m}_{t+1}$ be defined as in (4). Then,

$$r_t^* \triangleq \arg \max_{m_t \leq r_t \leq \tilde{r}_{t+1}} \widehat{m}_{t+1} = m_t \quad (14)$$

and

$$m_{t+1}^* \triangleq \max_{m_t \leq r_t \leq \tilde{r}_{t+1}} \widehat{m}_{t+1} = m_t \frac{m_t - \sqrt{\tilde{r}_{t+1}^2 - m_t^2}}{\tilde{r}_{t+1} - \sqrt{\tilde{r}_{t+1}^2 - m_t^2}}. \quad (15)$$

Proof: Let $\beta \triangleq r_t/m_t \geq 1$. Then, (14) is equivalent to find

$$\beta^* = \arg \max_{\beta \geq 1} \widehat{m}_{t+1} \ .$$

Recalling (2), let $\gamma \triangleq \tilde{r}_{t+1}/r_t > 1$. For given r_t, m_t and $\tilde{r}_{t+1}$, Lemma 2 states that

$$\widehat{m}_{t+1} = \max \left\{ m_t \frac{\beta - \sqrt{\beta^2\gamma^2 - 1}}{\beta\gamma - \sqrt{\beta^2\gamma^2 - 1}},\ m_t \frac{1 - \beta\sqrt{\gamma^2 - 1}}{\gamma - \sqrt{\gamma^2 - 1}} \right\}.$$

Let us first consider the case $\widehat{m}_{t+1} = m_t \frac{\beta - \sqrt{\beta^2\gamma^2-1}}{\beta\gamma - \sqrt{\beta^2\gamma^2-1}}$.

$$\begin{aligned} \frac{\partial \widehat{m}_{t+1}}{\partial \beta} &= m_t \frac{(\gamma - 1)\left(\sqrt{\beta^2\gamma^2 - 1} - \frac{\beta^2\gamma^2}{\sqrt{\beta^2\gamma^2-1}}\right)}{\left(\beta\gamma - \sqrt{\beta^2\gamma^2 - 1}\right)^2} \\ &= \frac{m_t(1-\gamma)}{\sqrt{\beta^2\gamma^2 - 1}\left(\beta\gamma - \sqrt{\beta^2\gamma^2 - 1}\right)^2} < 0 \ . \end{aligned}$$

Since $\frac{\partial \widehat{m}_{t+1}}{\partial \beta} < 0$, β^* corresponds to its minimum feasible value, i.e., $\beta^* = 1$, leading to $r_t^* = m_t$.

Let us now consider the case $\widehat{m}_{t+1} = m_t \frac{1-\beta\sqrt{\gamma^2-1}}{\gamma - \sqrt{\gamma^2-1}}$.

By following the same reasoning, one gets once again $\frac{\partial \widehat{m}_{t+1}}{\partial \beta} < 0$, and then $r_t^* = m_t$. Expression (15) is obtained by direct substitution into (4). ■

Lemma 3 states that, for a given m_t, the center C_t which (potentially) leads to the maximum m_{t+1} at the subsequent step is $C_t = [m_t, m_t]'$. This is instrumental to define an upper bound to the evolution of m_t.

Theorem 1: Let $m_t > 1$ and let the lion play the MCLS. Then, for any possible man strategy, one has

$$m_{t+1} \leq \frac{m_t(m_t - 1)}{\sqrt{1 + m_t^2} - 1} \ .$$

Proof: By Lemma 2 and Lemma 3, one has

$$m_{t+1}^* = \max_{r_t \geq m_t} \widehat{m}_{t+1} = m_t \frac{m_t - \sqrt{\tilde{r}_{t+1}^2 - m_t^2}}{\tilde{r}_{t+1} - \sqrt{\tilde{r}_{t+1}^2 - m_t^2}} \ . \quad (16)$$

Let us define $\hat{r} = \tilde{r}_{t+1}/m_t$. By substituting into (16), one has

$$m_{t+1}^* = m_t \frac{1 - \sqrt{\hat{r}^2 - 1}}{\hat{r} - \sqrt{\hat{r}^2 - 1}} \ . \quad (17)$$

Taking derivatives w.r.t. $\hat{r}$, one has

$$\frac{\partial m_{t+1}^*}{\partial \hat{r}} = m_t \frac{\hat{r} - 1 - \sqrt{\hat{r}^2 - 1}}{\left(\hat{r} - \sqrt{\hat{r}^2 - 1}\right)^2}$$

which vanishes for $\hat{r} = 1$. It can be easily checked that $\hat{r} = 1$ corresponds to a maximum and $\frac{\partial m_{t+1}^*}{\partial \hat{r}} < 0$, $\forall \hat{r} > 1$, i.e. $\forall \tilde{r}_{t+1} > m_t$.

Since by (2) one has that $\tilde{r}_{t+1}^2 \geq r_t^2 + 1 = m_t^2 + 1$, the maximum value for m_{t+1}^* is achieved for $\tilde{r}_{t+1} = \sqrt{m_t^2+1}$. By substituting in (17) one has

$$m_{t+1}^* = m_t \frac{m_t - \sqrt{m_t^2+1-m_t^2}}{\sqrt{m_t^2+1} - \sqrt{m_t^2+1-m_t^2}} = m_t \frac{m_t - 1}{\sqrt{m_t^2+1} - 1}$$

which concludes the proof. ∎

A direct consequence of Theorem 1 is that MCLS leads to capture of the man in a finite number of moves. An upper bound to such a number is now derived.

Let us consider the recursion

$$b_{t+1} = \frac{b_t(b_t-1)}{\sqrt{1+b_t^2}-1} \triangleq g(b_t) \ . \tag{18}$$

Let us fix $b_0 = m_0 > 1$. Since the function $g(b_t)$ is monotone increasing for $b_t > 1$, by Theorem 1 one has that, if $m_t \leq b_t$, then

$$m_{t+1} \leq g(m_t) \leq g(b_t) = b_{t+1} \ .$$

Therefore, recursion (18) returns an upper bound of m_t, for all t. By Proposition 4, if $m_t \leq 1$ then the game ends at the next move. Since $g(b_t) < 0$ when $b_t < 1$, an upper bound to the maximum number of moves before the game ends can be computed as follows

$$N_{max}^{MCLS} = \min\{t \in \mathbb{N} : \ b_t < 0\} \ . \tag{19}$$

Notice that N_{max}^{MCLS} is a function of m_0, although it is difficult to express this dependence explicitly. Clearly, N_{max}^{MCLS} can be computed by recursively evaluating b_t in (18) until $b_t < 0$. In Fig. 3, N_{max}^{MCLS} and $N_{max}^{FCLS} = \lceil m_0^2 \rceil$ are compared for $m_0 \in [1, \ 10]$.

From Fig. 3 it is apparent that $N_{max}^{MCLS} \leq N_{max}^{FCLS}$, and equality holds only for small values of m_0 (due to the discretization introduced by the fact that the number of moves must be integer). This is proved in the next theorem for every $m_0 > 1$.

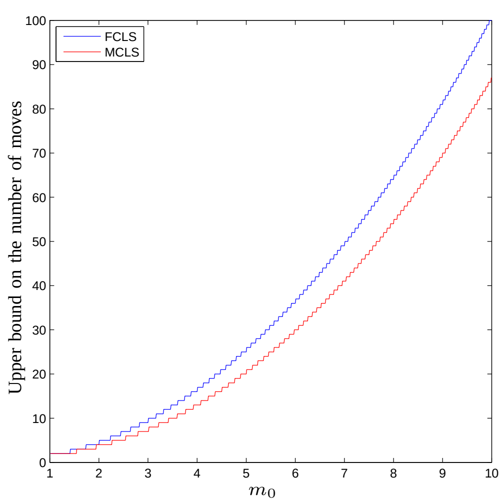

Fig. 3. Upper bounds on the number of moves for different values of m_0. Comparison between N_{max}^{FCLS} (blue) and N_{max}^{MCLS} (red).

Theorem 2: Let N_{max}^{MCLS} and N_{max}^{FCLS} be defined as in (19) and (1), respectively. Then,

$$N_{max}^{MCLS} \leq N_{max}^{FCLS} \ , \ \forall m_0 > 1. \tag{20}$$

Proof: System (18) can be rewritten as

$$b_{t+1} = \frac{(b_t-1)(\sqrt{1+b_t^2}+1)}{b_t} \ . \tag{21}$$

Let $b_t > 1$. By squaring (21), one gets

$$\begin{aligned} b_{t+1}^2 &= \frac{(b_t-1)^2(b_t^2+2+2\sqrt{1+b_t^2})}{b_t^2} \\ &= b_t^2 - \frac{2b_t^3 - 3b_t^2 + 4b_t - 2 - 2(b_t-1)^2\sqrt{1+b_t^2}}{b_t^2} \\ &= b_t^2 - 1 - \frac{2}{b_t^2}\left[(b_t^3 - b_t^2 + 2b_t - 1) - (b_t-1)^2\sqrt{1+b_t^2}\right] \\ &= b_t^2 - 1 - \frac{2(b_t-1)}{b_t^2}\left[(b_t^2 - b_t + 1) - (b_t-1)\sqrt{1+b_t^2}\right] \\ &\leq b_t^2 - 1 \end{aligned}$$

because $(b_t^2 - b_t + 1) > (b_t - 1)\sqrt{1+b_t^2}$ for all $b_t > 1$. Therefore, system (18) decays always faster than system

$$m_{t+1}^2 = m_t^2 - 1 \ .$$

Since $N_{max}^{FCLS} = \lceil m_0^2 \rceil = \min\{t \in \mathbb{N} : \ m_t < 0\}$ and N_{max}^{MCLS} is defined by (19), then (20) immediately follows. ∎

VI. Numerical Examples

In this section, two numerical examples are reported in order to show the effectiveness of the proposed lion strategy.

Example 1: The behavior of FCLS and MCLS is first compared on a single game. The initial position of the man and the lion are $M_0 = [2, 3]'$ and $L_0 = [3, 4]'$, respectively. The center of the FCLS turns out to be $C_0 = [12.657, 13.657]'$. The man plays the strategy proposed in [11], which works as follows. At each time t, the man moves to a point orthogonal to the line connecting him to the lion. Between the two possible directions, he chooses the one which maximizes the product of his coordinates.

The number of moves needed by the FCLS to capture the man is 98, while 38 moves are sufficient for the proposed strategy. The paths traveled by the lion (black) and by the man (red) are depicted in Fig. 4. For the MCLS, the last part of the trajectory of the center C_t is drawn in blue. Notice that C_t keeps moving towards the origin, as dictated by Lemma 1.

Example 2: A set of 10000 games have been simulated. Let $M_0 = [x_m, y_m]'$ and $L_0 = [x_l, y_l]'$ denote the man and lion initial positions, respectively, which have been randomly generated from a uniform distribution such that $x_m, y_m \in (0, 10)$ and $(x_l - x_m), (y_l - y_m) \in (0, 10)$. The man plays the same strategy as in Example 1.

Three lion algorithms are compared: FCLS, MCLS and the SPHERES algorithm reported in [4], which will be denoted by the superscripts F, M, S, respectively. The SPHERES strategy was devised for multiple pursuer games, but it can be easily adapted to the case of a single pursuer. In particular, it can

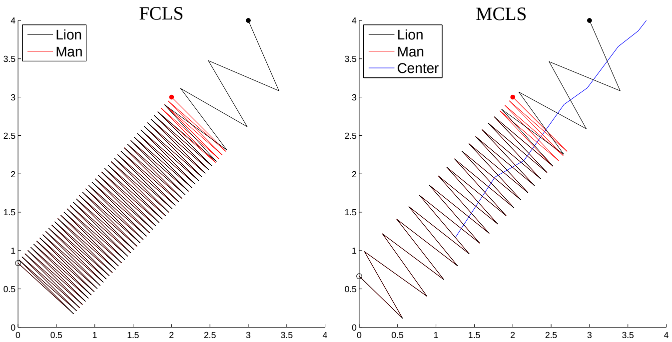

Fig. 4. Example 1. Paths traveled by the lion (black) and by the man (red) for FCLS and MCLS. Dots denote the initial positions while the circle is the capture point. The last part of the path of the MCLS center is drawn in blue.

be seen as a generalized fixed-center strategy, in which the center C_0^S can be any point such that $C_0^S = L_0 + \xi(C_0^F - L_0)$, where $\xi \geq 1$. Notice that, among all the feasible centers of the SPHERES algorithm, the FCLS selects the one closest to the lion ($\xi = 1$).

In the simulated games, $\xi = 3$ has been chosen for the SPHERES algorithm. For a fixed lion strategy a and for a given game k, let $N^a(k)$ denote the number of moves needed by the lion to capture the man. Let us define the performance indexes:

$$r^a(k) = \frac{N^a(k)}{N^F(k)}$$

$$r^a_{max} = \max_k r^a(k)$$

$$g^a = \max_k \frac{N^a(k)}{N^a_{max}(k)}$$

where $N^a_{max}(k)$ is the upper bound on the number of moves for strategy a (which changes with k because it depends on the initial condition). Let $\overline{r}^a$ be the average value of $r^a(k)$ over all played games.

In Table I, the above performance indexes are shown for the different lion strategies. On average, the proposed method allows the lion to catch the man in a number of moves which is less than 20% of that needed by the FCLS, while the SPHERES algorithm provides an even poorer performance. In general, it has been observed that such a performance degrades as ξ increases, thus suggesting that the FCLS choice of the center position is the optimal one for a fixed-center strategy. For all the simulated games, the MCLS is able to end the game in a number of moves which is less than 69% w.r.t. the FCLS, as testified by r^M_{max}. Moreover, g^M shows that the number of moves of the MCLS never approaches the upper bound, being the maximum ratio about 0.37. The much higher value of g^F, close to 0.9, is not surprising because the bound for the FCLS has been proved to be tight in [11]. On the whole, it is apparent that the proposed technique based on a moving center outperforms the pursuer strategies based on a fixed center, and the gap between the actual performances is much larger than that foreseen by the upper bounds.

TABLE I
PERFORMANCE INDEXES FOR THE CONSIDERED LION STRATEGIES

	$\overline{r}$	r_{max}	g
FCLS	1	1	0.894
MCLS	0.197	0.689	0.367
SPHERES	2.806	3.013	0.468

VII. CONCLUSIONS

A new lion strategy has been devised for the discrete-time version of the lion and man problem. This solution dominates the one proposed by Sgall in [11] in terms of maximum number of moves required to guarantee man capture.

An interesting feature of the proposed approach is that the upper bound on the number of moves does not seem to be tight. Indeed, for randomly chosen initial conditions of lion and man, numerical simulations show that the actual number of steps in which the lion reaches the man turns out to be much smaller than that predicted by the bound. Unfortunately, the optimal man strategy for counteracting the proposed lion algorithm, for generic initial conditions, is still an open problem. It is expected that such a result would allow one to significantly improve the upper bound on the number of moves.

We believe that the proposed result is helpful in all the contexts in which the lion and man problem solution is used as a building block within more complex strategies for pursuit-evasion games. The application of the new lion algorithm in these problems and the evaluation of its benefits is the subject of ongoing research.

REFERENCES

[1] P. J. Nahin, *Chases and escapes: the mathematics of pursuit and evasion*. Princeton University Press, 2012.

[2] T. H. Chung, G. A. Hollinger, and V. Isler, "Search and pursuit-evasion in mobile robotics," *Autonomous robots*, vol. 31, pp. 299–316, 2011.

[3] J. E. Littlewood, *Littlewood's miscellany*. Cambridge University Press, 1986.

[4] S. Kopparty and C. V. Ravishankar, "A framework for pursuit evasion games in $\mathbb{R}^n$," *Information Processing Letters*, vol. 96, no. 3, pp. 114–122, 2005.

[5] V. Isler, S. Kannan, and S. Khanna, "Randomized pursuit-evasion in a polygonal environment," *IEEE Transactions on Robotics*, vol. 21, no. 5, pp. 875–884, 2005.

[6] S. D. Bopardikar, F. Bullo, and J. P. Hespanha, "On discrete-time pursuit-evasion games with sensing limitations," *IEEE Transactions on Robotics*, vol. 24, no. 6, pp. 1429–1439, 2008.

[7] D. Bhadauria, K. Klein, V. Isler, and S. Suri, "Capturing an evader in polygonal environments with obstacles: The full visibility case," *The International Journal of Robotics Research*, vol. 31, no. 10, pp. 1176–1189, 2012.

[8] B. Ames, A. Beveridge, R. Carlson, C. Djang, V. Isler, S. Ragain, and M. Savage, "A leapfrog strategy for pursuit-evasion in a polygonal environment," *International Journal of Computational Geometry & Applications*, vol. 25, no. 02, pp. 77–100, 2015.

[9] S. A. Aleem, C. Nowzari, and G. J. Pappas, "Self-triggered pursuit of a single evader," in *2015 54th IEEE Conference on Decision and Control (CDC)*, 2015, pp. 1433–1440.

[10] R. K. Guy, "Unsolved problems in combinatorial games," in *Combinatorics Advances*. Springer, 1995, pp. 161–179.

[11] J. Sgall, "Solution of David Gale's lion and man problem," *Theoretical Computer Science*, vol. 259, no. 1, pp. 663–670, 2001.

[12] N. Noori, A. Beveridge, and V. Isler, "Pursuit-evasion: A toolkit to make applications more accessible," *IEEE Robotics & Automation Magazine*, vol. 23, no. 4, pp. 138–149, 2016.

[13] N. Noori and V. Isler, "Lion and man with visibility in monotone polygons," *The International Journal of Robotics Research*, vol. 33, no. 1, pp. 155–181, 2014.