A novel family of pursuit strategies for the lion and man problem

Marco Casini, Andrea Garulli

Abstract

In this paper, a new family of pursuit strategies is presented for a fundamental pursuit-evasion game: the lion and man problem. The classic formulation due to David Gale is considered: the players move alternatively in a discrete-time setting and are constrained to remain in the non-negative quadrant of the plane. The proposed solution relies on the stepwise optimization of a suitable functional, whose generality provides an appealing degree of freedom for customization within specific pursuit problems and bounded environments. Numerical simulations show the benefits of the new approach with respect to existing pursuit strategies.

Index Terms

Lion and man problems, pursuit-evasion games, mobile robotics

I. Introduction

Pursuit-evasion is a popular research topic which is relevant to a number of applications in mobile robotics (see, e.g., [1], [2], [3] and the surveys [4], [5]). Among the many alternative formulations, the main differences concern the considered environment (limited or unlimited, with or without obstacles, convex or not, etc.) and the time evolution, which can be continuous or discrete.

A fundamental discrete-time formulation within a limited environment is the classic *lion and man problem* [6]. A lion (pursuer) and a man (evader) move one at a time in a two-dimensional environment. The distance they can travel each time is bounded by a fixed constant and they are constrained to remain in the positive quadrant of the plane. The lion wins the game when it is able to reach exactly the position of the man. In the continuous-time version of the game, it has been proven that the man is able to indefinitely escape the lion even in closed environments like a circle [7]. On the contrary, in the considered discrete-time framework, it is known that a winning strategy for the lion always exists, if the lion starts on the upper right side with respect to the man. An ingenious lion strategy which guarantees capture in finite time has been proposed in [8]. Recently, an improved version has been presented in [9], whose upper bound on the capture time dominates the one given in [8]. However, the lion strategy minimizing the capture time has not been devised yet. It is worth stressing that reducing the capture time is important, because these basic pursuit algorithms are used as building blocks within more complex strategies, such as multiple pursuer schemes [10] or pursuit with line-of-sight visibility [11].

In this paper, we present a new family of pursuit strategies for the lion and man problem. The approach is based on the choice of a suitable functional which is minimized by the lion at each move. As a byproduct, the strictly convex level curves of the functional iteratively limit the feasible space where the man can move without being captured. The possibility of choosing among an entire family of admissible functionals, provides a remarkable degree of freedom which can be exploited within complex pursuit schemes, depending on the objective or on the structure of the environment. Numerical simulations show that the proposed method provides a significant reduction of the capture time with respect to the approaches in [8], [9].

The paper is organized as follows. In Section II, the considered problem is formulated, while the proposed lion strategy and its main properties are presented in Section III and IV, respectively. An algorithm implementing the lion strategy is described in Section V. Numerical simulations are reported in Section VI, while conclusions are drawn in Section VII.

II. Problem formulation

The notation adopted in the paper is standard. Let $\mathbb{R}^n_+$ denote the n-dimensional Euclidean space of non-negative numbers. A row vector with elements $v_1, \ldots, v_n$ is denoted by $V = [v_1, \ldots, v_n]$, while V' is the transpose of V. Denote by $\boldsymbol{\mathcal{C}}(C, r)$ the circle of center C and radius r. Notation $V \perp W$ means that two vectors V and W are orthogonal, while $V \succ W$ denotes the componentwise strict inequality.

In this paper, we consider the version of the *lion and man problem* formulated by David Gale [6]. Two players move in the non-negative quadrant of the Cartesian plane. Time is assumed discrete, while space is continuous. At each round (hereafter called *time*) both players are allowed to travel a distance less or equal to a given radius r from their current position (w.l.o.g., $r = 1$ is assumed). The man moves first at each round. Let us denote by $M_t \in \mathbb{R}^2_+$ and $L_t \in \mathbb{R}^2_+$ the man and lion position at time t, respectively. Hence, $\|M_{t+1} - M_t\| \leq 1$, $\|L_{t+1} - L_t\| \leq 1$. The game ends (lion wins) if the lion moves exactly to the man position.

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The man wins if he is able to escape indefinitely from the lion. It is assumed that the initial man coordinates are strictly smaller than the corresponding lion coordinates, otherwise it is straightforward to observe that the man wins the game by moving straight upwards or to the right.

The proposed lion strategy aims at minimizing a cost functional defined as follows.

Definition 1: A function $f : \mathbb{R}_+^2 \to \mathbb{R}_+$ is called a *clearing function (C-function)* if it satisfies the following properties:

- $f(P) = 0$ if and only if $P = [0, 0]'$;
- $\forall P_2 \succ P_1$, one has $f(P_2) > f(P_1)$;
- $f([x, 0]') < \infty$ and $f([0, y]') < \infty,\ \forall x, y \in \mathbb{R}_+^2$;
- let $F_\beta = \{P \in \mathbb{R}_+^2 : f(P) = \beta\}$ be the β-level curve of f; for any fixed $\beta > 0$, there exists a strictly convex function g_β such that for any $P = [x_P, y_P]' \in F_\beta$, $y_P = g_\beta(x_P)$ and $c^f(P) = \left.\dfrac{d\,g_\beta}{dx}\right|_{x_P} \leq 0$.

We recall that a function g is strictly convex if it is twice differentiable and $\dfrac{d^2 g}{dx^2} > 0$.

As it will become clear in the next section, the level curves F_β separate the portion of the non-negative quadrant in which the man can move (*contaminated region*) from the *cleared region* [5], which is the zone where the man cannot enter (otherwise he is captured in the next move). Notice that F_β collapses to the origin for $\beta = 0$.

Due to the symmetry of the game w.r.t. the coordinate axis, a natural choice is to deal with clearing functions which are symmetric w.r.t. the bisector of the non-negative quadrant. In the following, two examples of such functions are reported.

Example 1 (equilateral hyperbola)

Let

$$f([x, y]') = c(x + y) + xy = \beta\ , \tag{1}$$

for $c > 0$ given. For fixed β, F_β is a branch of equilateral hyperbola centered in $[-c,\ -c]'$. In fact, f can be written as $f([x, y]') = (x + c)(y + c) - (\beta + c^2) = 0$. Moreover, one has $y = g_\beta(x) = \frac{\beta - cx}{x + c}$, with $x \in [0, \beta/c]$.

Example 2 (quarter of circumference)

Let us consider

$$f([x, y]') = x + y + \sqrt{2xy} = \beta\ . \tag{2}$$

For a given β, F_β is the lower left quarter of the circumference $\mathcal{C}([\beta,\ \beta]', \beta)$. In fact, from $(x - \beta)^2 + (y - \beta)^2 = \beta^2$, solving for β one has $\beta = x + y \pm \sqrt{2xy}$. We are interested in the solution $\beta = x + y + \sqrt{2xy}$. For given β, it follows that $y = g_\beta(x) = \beta - \sqrt{2x\beta - x^2}$, with $x \in [0, \beta]$.

III. Clearing Function Lion Strategy

In this section, the proposed lion strategy, hereafter referred to as *Clearing Function Lion Strategy (CFLS)*, is described. Such a strategy is based on the choice of a C-function and its successive minimization until capture occurs.

Definition 2: At any time t, let $\mathcal{R}_t$ denote the straight line whose points are equidistant to M_t and L_t, i.e.,

$$\mathcal{R}_t = \{P \in \mathbb{R}_+^2 : \|P - M_t\| = \|P - L_t\|\}\ . \tag{3}$$

Such a line is sometime called *line of control* and it provides the Voronoi partition of the environment associated to M_t and L_t [3]. Since both coordinates of the man must be strictly less than those of the lion (otherwise man escapes), the line of control can be written as

$$\mathcal{R}_t = \{(x, y)' \in \mathbb{R}_+^2 : y = a_t x + b_t\} \tag{4}$$

with $a_t < 0$ and $b_t > 0$. As a consequence, since the environment is restricted to the non-negative quadrant, the line of control collapses to a segment and the Voronoi cell associated to the man is a triangle.

Let f denote the C-function chosen once and for all at the beginning of the game. At a given time t, let

$$\beta_t = \sup_{P \in \mathcal{R}_t} f(P) \tag{5}$$

and

$$P_t = \arg \sup_{P \in \mathcal{R}_t} f(P) = F_{\beta_t} \cap \mathcal{R}_t\ . \tag{6}$$

The uniqueness of P_t in (6) is guaranteed by the strict convexity of g_{β_t}.

Let M_{t+1} and L_t be given. The aim of the proposed strategy is to move the lion to a position L^*_{t+1} such that β_{t+1} is minimized, i.e.,

$$\begin{aligned} L^*_{t+1} &= \arg \inf_{\|L_{t+1} - L_t\| \leq 1} \beta_{t+1} \\ &= \arg \inf_{\|L_{t+1} - L_t\| \leq 1} \sup_{P \in \mathcal{R}_{t+1}} f(P)\ . \end{aligned} \tag{7}$$

Moreover, let us define

$$\beta^*_{t+1} = \inf_{\|L_{t+1}-L_t\|\leq 1} \beta_{t+1} = \inf_{\|L_{t+1}-L_t\|\leq 1} \sup_{P\in\mathcal{R}_{t+1}} f(P) \tag{8}$$

and

$$P^*_{t+1} = \arg \sup_{P\in\mathcal{R}_{t+1}} f(P) = F_{\beta^*_{t+1}} \cap \mathcal{R}_{t+1} \ . \tag{9}$$

The rational behind this strategy is to push the current β-level curve as close as possible to the origin, in order to minimize the contaminated region. In fact, if the man crosses F_{β^*}, he also crosses the line of control $\mathcal{R}$ and therefore the lion will catch him in the next move.

For a given C-function f, the computation of L^*_{t+1} in (7) is not straightforward, in general. In the next section, some properties of the CFLS are reported in order to fully characterize the optimal lion move. Fig. 1 shows an example of an optimal lion move, along with the involved level curves.

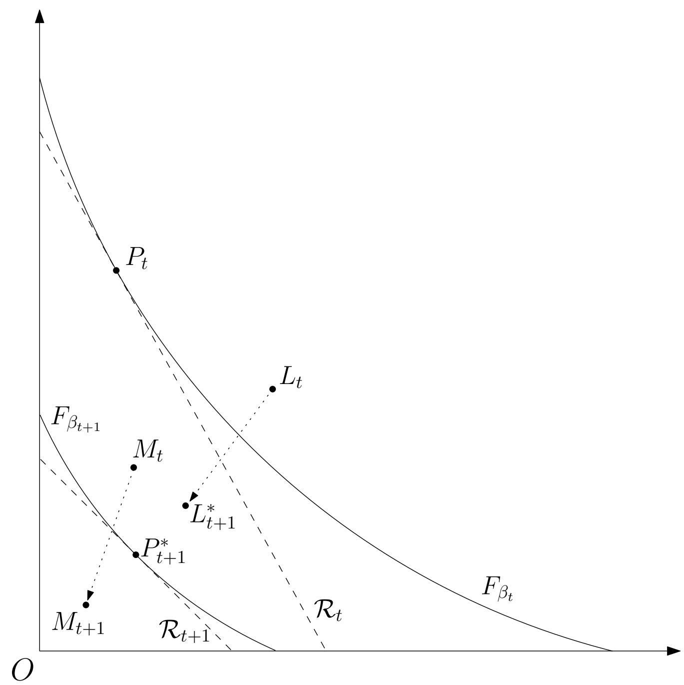

Fig. 1. Example of an optimal lion move.

IV. Properties of the CFLS

At a given time t, let M_{t+1} and L_t be given. We want to compute the optimal lion move L^*_{t+1} satisfying (7). Hereafter, we assume $\|M_{t+1} - L_t\| > 1$, otherwise the lion catches the man in one step.

Let us define

$$\widetilde{C}_t = \frac{M_{t+1} + L_t}{2} \tag{10}$$

and, for a given L_{t+1},

$$C_{t+1} = \frac{M_{t+1} + L_{t+1}}{2} \tag{11}$$

as the midpoints between M_{t+1} and L_t, and M_{t+1} and L_{t+1}, respectively. Notice that, by (3) one has $C_{t+1} \in \mathcal{R}_{t+1}$ and $\mathcal{R}_{t+1} \perp (C_{t+1} - M_{t+1})$. By (11), one gets

$$L_{t+1} = 2\, C_{t+1} - M_{t+1}. \tag{12}$$

Notice that for any $L_{t+1} \in \boldsymbol{\mathcal{C}}(L_t, 1)$ there exists a unique $C_{t+1} \in \boldsymbol{\mathcal{C}}(\widetilde{C}_t, 1/2)$ and viceversa, see Fig. 2. Therefore, solving problem (7) is equivalent to solving the following problem

$$\begin{aligned} C^*_{t+1} &= \arg \inf_{C_{t+1}\in\boldsymbol{\mathcal{C}}(\widetilde{C}_t,\,1/2)} \beta_{t+1} \\ &= \arg \inf_{C_{t+1}\in\boldsymbol{\mathcal{C}}(\widetilde{C}_t,\,1/2)} \sup_{P\in\mathcal{R}_{t+1}} f(P) \end{aligned} \tag{13}$$

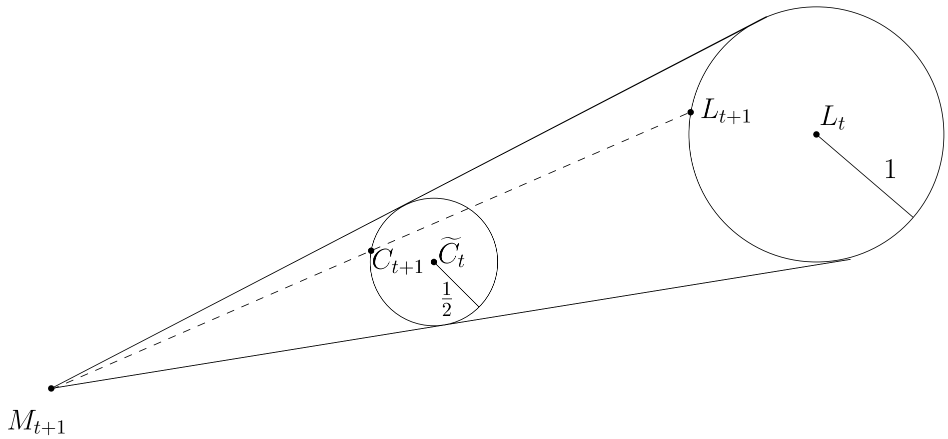

Fig. 2. Definition of $\widetilde{C}_t$.

and then according to (12), set $L^*_{t+1} = 2\,C^*_{t+1} - M_{t+1}$.

The next lemma provides some useful properties of C^*_{t+1}.

Lemma 1: Let M_{t+1} and L_t be such that $\|M_{t+1} - L_t\| > 1$, and P^*_{t+1} and C^*_{t+1} be given by (9) and (13). Let $Z = \frac{M_{t+1}+P^*_{t+1}}{2}$ and $r = \left\|\frac{M_{t+1}-P^*_{t+1}}{2}\right\|$. Then, the following conditions hold:

$$i)\ \|Z - M_{t+1}\| = \|Z - P^*_{t+1}\| = \|Z - C^*_{t+1}\| = r \tag{14}$$

$$ii)\ \mathcal{C}(Z, r) \cap \mathcal{C}(\widetilde{C}_t, 1/2) = C^*_{t+1}\ . \tag{15}$$

Proof: Being $P^*_{t+1}, C^*_{t+1} \in \mathcal{R}_{t+1}$, $(M_{t+1} - C^*_{t+1})$ must be orthogonal to $(P^*_{t+1} - C^*_{t+1})$. Hence, condition i) is a direct consequence of Thales' Theorem (see Fig. 3).

Let us now prove condition ii). First, let us prove that $\mathcal{C}(Z, r)$ and $\mathcal{C}(\widetilde{C}_t, 1/2)$ do not coincide, i.e., $\mathcal{C}(Z, r) \neq \mathcal{C}(\widetilde{C}_t, 1/2)$. By (10) one has that $Z = \widetilde{C}_t$ implies $P^*_{t+1} = L_t$ and $r = 1/2$ implies $\|M_{t+1} - L_t\| = 1$, which contradicts the hypothesis $\|M_{t+1} - L_t\| > 1$. Since C^*_{t+1} must belong both to $\mathcal{C}(Z, r)$ and to $\mathcal{C}(\widetilde{C}_t, 1/2)$, one has $\mathcal{C}(Z, r) \cap \mathcal{C}(\widetilde{C}_t, 1/2) \neq \emptyset$. So, $\mathcal{C}(Z, r)$ and $\mathcal{C}(\widetilde{C}_t, 1/2)$ intersect in one or two points. By contradiction, let us assume $\mathcal{C}(Z, r)$ and $\mathcal{C}(\widetilde{C}_t, 1/2)$ intersect in two points. Let $\beta^*_{t+1} = f(P^*_{t+1})$ and $F_{\beta^*_{t+1}}$ be the corresponding C-function level curve. By (9), $P^*_{t+1} = \mathcal{R}_{t+1} \cap F_{\beta^*_{t+1}}$. For each candidate $\overline{C}_{t+1} \in \mathcal{C}(\widetilde{C}_t, 1/2)$, let $\overline{\mathcal{R}}_{t+1}$ be the line of control passing through $\overline{C}_{t+1}$, i.e., the line of control associated to M_{t+1} and $\overline{L}_{t+1} = 2\,\overline{C}_{t+1} - M_{t+1}$, according to (12). By simple geometric arguments, it can be observed that it is always possible to find a $\overline{C}_{t+1} \in \mathcal{C}(\widetilde{C}_t, 1/2)$ such that $\|Z - \overline{C}_{t+1}\| < r$ and the corresponding line of control $\overline{\mathcal{R}}_{t+1}$ does not intersect $\mathcal{R}_{t+1}$ in the first quadrant (an example is shown in Fig. 4). Therefore, due the strict convexity of the C-functions, $\overline{\mathcal{R}}_{t+1}$ does not intersect also the level curve $F_{\beta^*_{t+1}}$. From the definition of C-function, this means that $\sup_{P\in\overline{\mathcal{R}}_{t+1}} f(P) < \beta^*_{t+1}$, which contradicts the definition of C^*_{t+1} in (13). ■

Notice that condition i) in Lemma 1 states that M_{t+1}, P^*_{t+1} and C^*_{t+1} belong to the circle $\mathcal{C}(Z, r)$, while condition ii) imposes that the circles $\mathcal{C}(Z, r)$ and $\mathcal{C}(\widetilde{C}_t, 1/2)$ are tangent in C^*_{t+1}, see Fig. 3.

Theorem 1: Let M_{t+1}, L_t and P^*_{t+1} be given. Then, the optimal lion move at time $t+1$ is given by

$$L^*_{t+1} = L_t + V_t \tag{16}$$

where

$$V_t = \frac{P^*_{t+1} - L_t}{\|P^*_{t+1} - L_t\|}\ .$$

Proof: Let

$$Z = \frac{M_{t+1} + P^*_{t+1}}{2}$$

and

$$V_t = \frac{Z - \widetilde{C}_t}{\|Z - \widetilde{C}_t\|} = \frac{P^*_{t+1} - L_t}{\|P^*_{t+1} - L_t\|}\ .$$

By item ii) in Lemma 1, one has

$$C^*_{t+1} = \widetilde{C}_t + \frac{V_t}{2}\ .$$

By (12) and (10), one has

$L^*_{t+1} = 2\,C^*_{t+1} - M_{t+1} = 2\,\widetilde{C}_t + V_t - M_{t+1} = L_t + V_t.$ ■

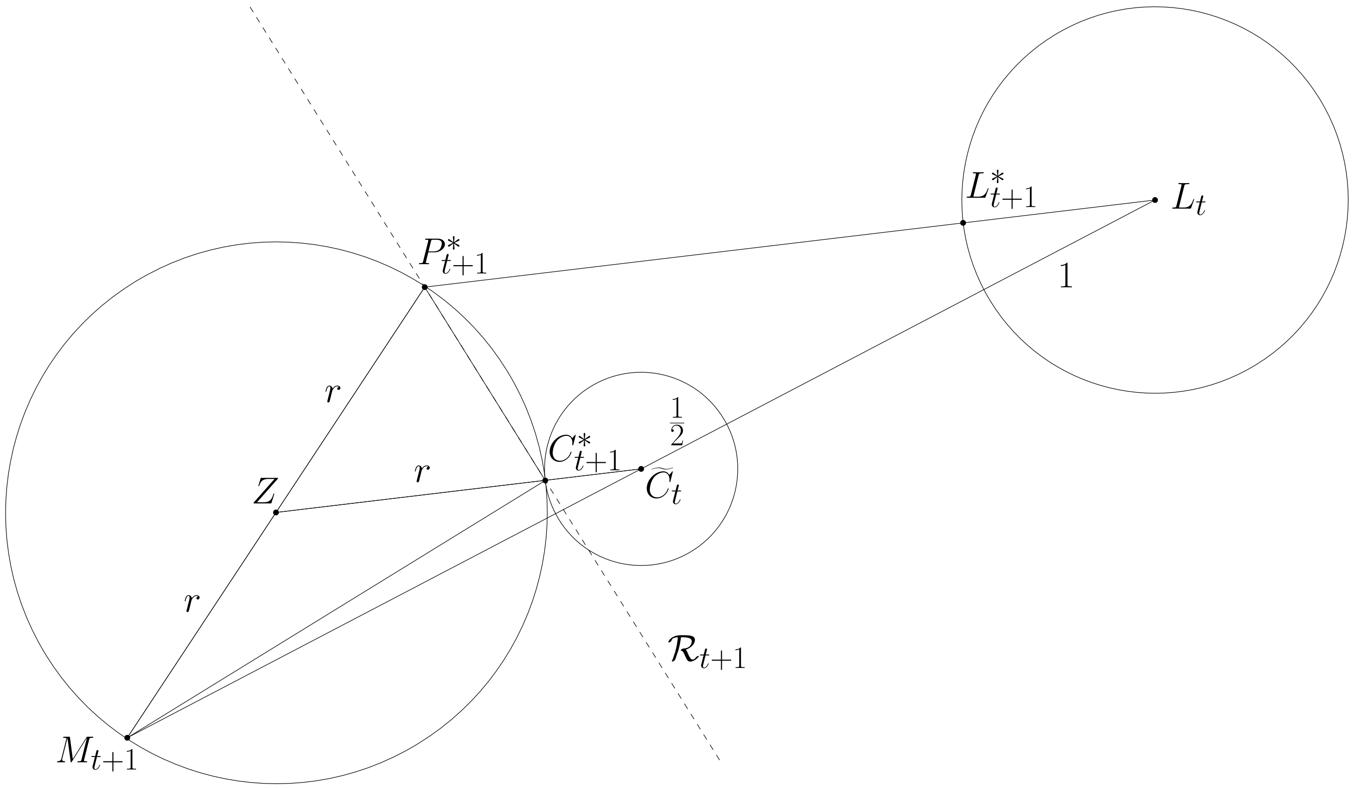

Fig. 3. Proof of Lemma 1: definition of C^*_{t+1}.

From Theorem 1, it is apparent that the optimal lion move L^*_{t+1} relies on the computation of P^*_{t+1} in (9). Hereafter, a useful characterization of such a point is provided.

Lemma 2: Let M_{t+1}, L_t and P^*_{t+1} be given. Then,

$$\|P^*_{t+1} - L_t\| - \|P^*_{t+1} - M_{t+1}\| = 1 \ . \tag{17}$$

Proof: Let Z and r be defined as in Lemma 1. By item ii) of Lemma 1, $\mathcal{C}(Z, r)$ and $\mathcal{C}(\widetilde{C}_t, 1/2)$ are tangent. So, $\|Z - \widetilde{C}_t\| = r + 1/2$ and by item i) of Lemma 1 one has

$$\|Z - \widetilde{C}_t\| = \|Z - M_{t+1}\| + 1/2 \ .$$

By substituting Z and $\widetilde{C}_t$ with their corresponding expressions, (17) holds. ■

Definition 3: Let M_{t+1} and L_t be given. Let us define

$$\mathcal{I}_{t+1} = \{P \in \mathbb{R}^2_+ \colon \|P - L_t\| - \|P - M_{t+1}\| = 1\}$$

which, according to (17), contains the locus of points P^*_{t+1} for all possible choices of the C-function f.

Notice that $\mathcal{I}_{t+1}$ is the branch of the hyperbola (restricted to the first quadrant) closer to M_{t+1} with focal points M_{t+1} and L_t and major axis equal to 1. Such a curve does not depend on f, but only on M_{t+1} and L_t. The next result gives the key property of P^*_{t+1} which will be exploited for its computation.

Theorem 2: Let M_{t+1} and L_t be given. Let β^*_{t+1} and P^*_{t+1} be defined by (8) and (9), respectively. Then

$$P^*_{t+1} = \mathcal{I}_{t+1} \cap \mathcal{R}_{t+1} = \mathcal{I}_{t+1} \cap F_{\beta^*_{t+1}} \ . \tag{18}$$

Proof: Recall that $\mathcal{R}_{t+1}$ is the line passing through P^*_{t+1} and C^*_{t+1}. By (9) and by Lemma 2 one has $P^*_{t+1} \in \mathcal{R}_{t+1}$ and $P^*_{t+1} \in \mathcal{I}_{t+1}$, respectively. By (16), P^*_{t+1}, L^*_{t+1} and L_t lie on the same line, see Fig. 3. We want to prove that P^*_{t+1} is the unique point in $\mathcal{I}_{t+1} \cap \mathcal{R}_{t+1}$. Let $P \in \mathcal{R}_{t+1}$, $P \neq P^*_{t+1}$. Since P, L^*_{t+1} and L_t are not aligned, by the triangle inequality one has $\|P - L_t\| < \|P - L^*_{t+1}\| + 1$. Moreover, by Definition 2, one has $\|P - M_{t+1}\| = \|P - L^*_{t+1}\|$ and then $\|P - L_t\| - \|P - M_{t+1}\| < 1$. Since (17) is not satisfied, $P \notin \mathcal{I}_{t+1}$. Therefore, $P^*_{t+1} = \mathcal{I}_{t+1} \cap \mathcal{R}_{t+1}$ holds. Hence, $\mathcal{R}_{t+1}$ is tangent to $\mathcal{I}_{t+1}$ in P^*_{t+1} and it is also tangent to $F_{\beta^*_{t+1}}$ in the same point, due to (9). Since $\mathcal{I}_{t+1}$ is the branch of hyperbola closer to M_{t+1}, it is clear that $\mathcal{I}_{t+1}$ and $F_{\beta^*_{t+1}}$ lie on opposite sides w.r.t. $\mathcal{R}_{t+1}$ and therefore we have also $P^*_{t+1} = \mathcal{I}_{t+1} \cap F_{\beta^*_{t+1}}$. ■

The following result provides a sufficient condition for the termination of the pursuit-evasion game.

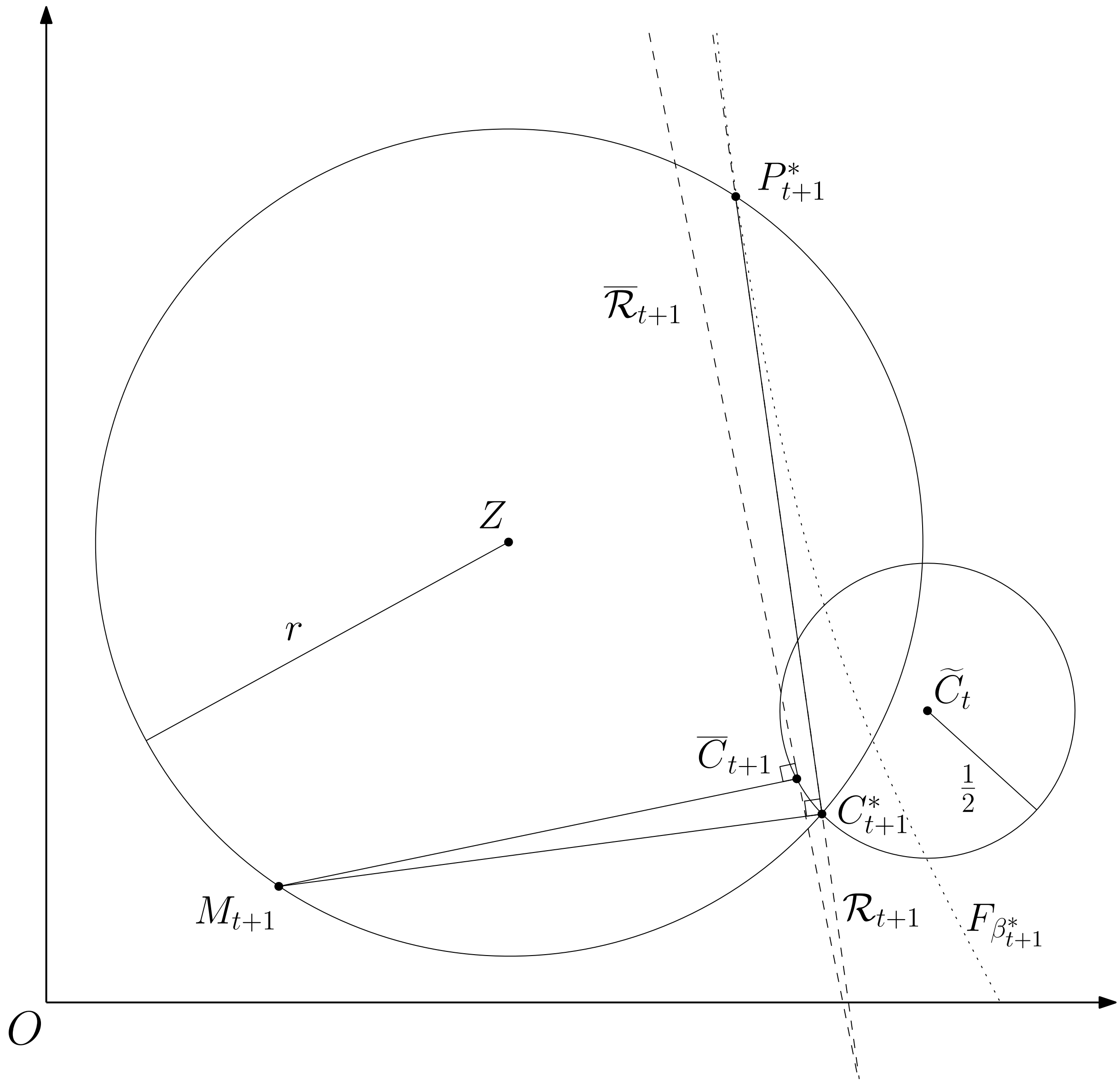

Fig. 4. Proof of Lemma 1. The line of control $\overline{\mathcal{R}}_{t+1}$ does not intersect $F_{\beta^*_{t+1}}$ in the first quadrant, leading to a contradiction.

Proposition 1: Given L_t and M_t, let $R_x = [x_R,\, 0]'$ and $R_y = [0,\, y_R]'$ denote the intersections of $\mathcal{R}_t$ with the coordinate axes. If $l = \max\{x_R,\, y_R\} \leq \frac{1}{\sqrt{2}}$, then the game ends in one move.

Proof: Let $\mathcal{R}^l$ be the segment with endpoints $Q_x = [l,\, 0]'$ and $Q_y = [0,\, l]'$. Let $\mathcal{T}$ be the triangle with vertices Q_x, Q_y and the origin, and $\mathcal{S}$ be the square with diagonal $\mathcal{R}^l$, see Fig. 5. If the man moves to a point M_{t+1} above $\mathcal{R}_t$, the lion catches him in one move. If M_{t+1} lies below $\mathcal{R}_t$, surely $M_{t+1} \in \mathcal{T}$. Since M_t is below $\mathcal{R}_t$, by the definition of line of control in (3), it follows that $L_t \in \mathcal{S}$. Since $l \leq \frac{1}{\sqrt{2}}$, the length of $\mathcal{R}^l$ is ≤ 1 and for any $C \in \mathcal{S}$ one has $\boldsymbol{\mathcal{C}}(C, 1) \supseteq \mathcal{T}$. Therefore, the lion is able to catch the man in one move for any $M_{t+1} \in \mathcal{T}$. ■

V. Computation of the Optimal Lion Move

In this section, the properties of the CFLS presented in the previous section are exploited to devise an algorithm for computing the lion move.

At a given time t, let $L_t = [x_l,\, y_l]'$ and $M_{t+1} = [x_m,\, y_m]'$ be given. Theorem 1 states that the knowledge of P^*_{t+1} is sufficient to compute the optimal lion move L^*_{t+1}. Since $\mathcal{I}_{t+1}$ can be easily computed, the idea is to exploit the relation $P^*_{t+1} = \mathcal{I}_{t+1} \cap F_{\beta^*_{t+1}}$ in (18) in order to compute P^*_{t+1}.

Let $c^{\mathcal{I}_{t+1}}(P)$ denote the angular coefficient of the line tangent to $\mathcal{I}_{t+1}$ in $P \in \mathcal{I}_{t+1}$. Moreover, recall that $c^f(P)$ is the angular coefficient of the tangent to the level curve F_β at P (remind that given P, such F_β is unique). Hence, P^*_{t+1} is the

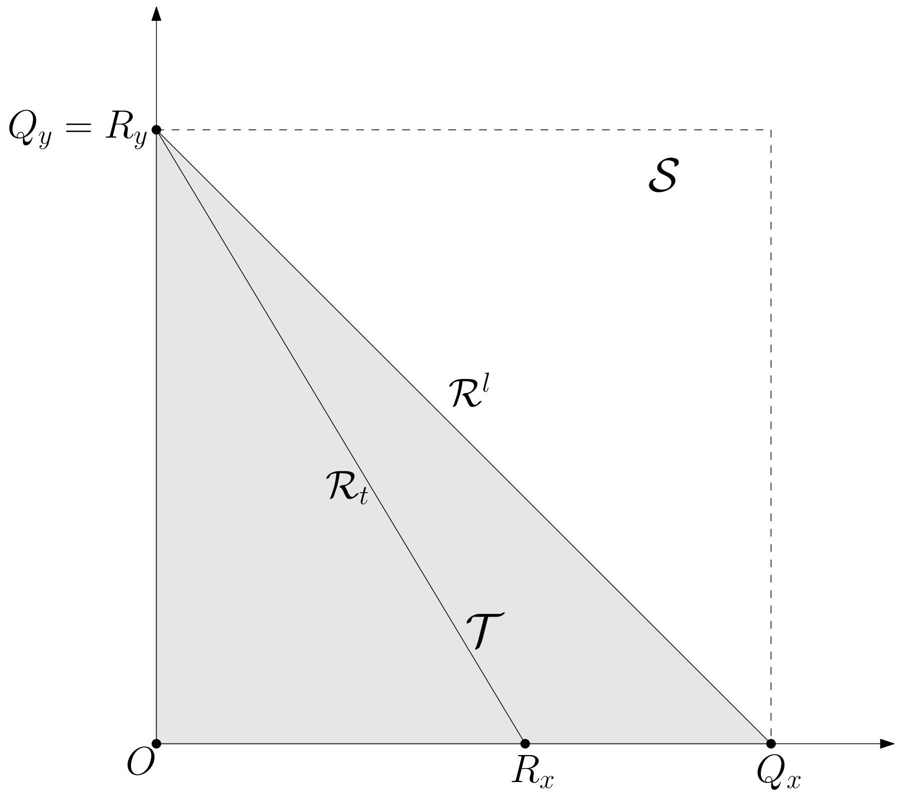

Fig. 5. Sketch of the proof of Proposition 1.

unique point such that $c^{\mathcal{I}_{t+1}}(P^*_{t+1}) = c^f(P^*_{t+1})$. From the strict convexity of the C-function and the definition of $\mathcal{I}_{t+1}$, it can be shown that $c^{\mathcal{I}_{t+1}}(P) - c^f(P)$ is a strictly decreasing function of P and P^*_{t+1} is its unique zero. Hence, P^*_{t+1} can be computed via a simple bisection search.

Let us describe the branch of hyperbola $\mathcal{I}_{t+1}$ by the equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

with

$$\begin{aligned}
A &= 4 - 4(x_l - x_m)^2 \\
B &= -8(x_l - x_m)(y_l - y_m) \\
C &= 4 - 4(y_l - y_m)^2 \\
D &= 4(x_l - x_m)(x_l^2 - x_m^2 + y_l^2 - y_m^2) - 4(x_l + x_m) \\
E &= 4(y_l - y_m)(x_l^2 - x_m^2 + y_l^2 - y_m^2) - 4(y_l + y_m) \\
F &= 4(x_l^2 + y_l^2)(x_m^2 + y_m^2) - (x_l^2 + x_m^2 + y_l^2 + y_m^2 - 1)^2 \ .
\end{aligned}$$

Let us define

$$c^{\mathcal{I}_{t+1}}_{min} = \inf_{P \in \mathcal{I}_{t+1}} c^{\mathcal{I}_{t+1}}(P) \quad , \quad c^{\mathcal{I}_{t+1}}_{max} = \sup_{P \in \mathcal{I}_{t+1}} c^{\mathcal{I}_{t+1}}(P)$$

and let their negative components be defined as

$$\begin{aligned}
\underline{c}^{\mathcal{I}_{t+1}} &= \begin{cases} c^{\mathcal{I}_{t+1}}_{min} & \text{, if } c^{\mathcal{I}_{t+1}}_{min} \leq 0 \\ 0 & \text{, if } c^{\mathcal{I}_{t+1}}_{min} > 0 \end{cases} \\
\overline{c}^{\mathcal{I}_{t+1}} &= \begin{cases} c^{\mathcal{I}_{t+1}}_{max} & \text{, if } c^{\mathcal{I}_{t+1}}_{max} \leq 0 \\ 0 & \text{, if } c^{\mathcal{I}_{t+1}}_{max} > 0 \end{cases} \ .
\end{aligned} \tag{19}$$

Let $P_i = [x_{P_i},\, y_{P_i}]'$, $i = 1, 2$ denote the intersection of $\mathcal{I}_{t+1}$ with the coordinate axes. Notice that P_1 and P_2 may lie on the same axis. Now, $\underline{c}^{\mathcal{I}_{t+1}}$ and $\overline{c}^{\mathcal{I}_{t+1}}$ can be easily derived from the derivatives of $\mathcal{I}_{t+1}$ in P_1 and P_2.

The pseudocode to compute L^*_{t+1} given L_t and M_{t+1} is reported in Algorithm 1. To avoid a bisection search in the semi-infinite interval $c \in (-\infty, 0]$, we exploit the increasing mapping $w = h(c) = \frac{c}{1-c}$, which leads to a bisection search in $w \in (-1, 0]$.

```
Data: L_t, M_{t+1}
Result: L*_{t+1}
Compute c_^{I_{t+1}}, c¯^{I_{t+1}};
w_ = h(c_^{I_{t+1}}); w¯ = h(c¯^{I_{t+1}});
while (w¯ − w_) > tolerance do
    ŵ = (w¯ + w_)/2;
    ĉ = h^{-1}(ŵ);
    P = compute_tangent_point(L_t, M_{t+1}, ĉ);
    c^f = angular_coefficient(P);
    if c^f < ĉ then
        w¯ = ŵ;
    else
        w_ = ŵ;
    end
end
V = (P − L_t)/‖P − L_t‖;
L*_{t+1} = L_t + V;
```

Algorithm 1: Algorithm for the computation of the optimal lion move.

Initially, $\underline{c}^{\mathcal{I}_{t+1}}$ and $\overline{c}^{\mathcal{I}_{t+1}}$ are computed according to (19). At each iteration, the candidate angular coefficient $\hat{c}$ is selected. The function `compute_tangent_point` computes $\mathcal{I}_{t+1}$ for given L_t and M_{t+1}, and then returns the point $P \in \mathcal{I}_{t+1}$ such that $c^{\mathcal{I}_{t+1}}(P) = \hat{c}$. For the chosen C-function f, the function `angular_coefficient` returns $c^f(P)$. Then, a bisection step is performed by comparing c^f and $\hat{c}$. Once a predefined precision is reached, L^*_{t+1} is computed by using (16).

VI. Comparison with other strategies

In this section, the proposed lion strategy is compared with other strategies available in the literature. In particular, we consider the lion strategy proposed in [8], hereafter referred to as *Lion Strategy 1 (LS1)*, and a variation of it denoted as *Lion Strategy 2 (LS2)*, presented in [9]. Concerning the CFLS, the chosen C-function is the quarter of circumference given in (2).

For computational purposes, we assume that the lion wins the game if, at a given time t, the distance between the two players becomes less than a given capture radius δ, i.e., $\|L_t - M_t\| < \delta$. Notice that when dealing with pursuit-evasion problems in real world applications, it is customary to set a finite capture radius $\delta > 0$.

In order to simulate the considered pursuit strategies, one has to define also the evasion strategy adopted by the man. It is worth remarking that the "best" man strategy (i.e., the worst-case man behaviour for the lion) has not been devised yet for any of the considered approaches. Therefore, two alternative man strategies are used in the simulations, denoted by *MS1* and *MS2*. At any time t, such strategies set the man position $M_{t+1}\colon \|M_{t+1} - M_t\| = 1$ for known M_t and L_t. They are derived as follows.

MS1: M_{t+1} is set such that $(M_{t+1} - M_t) \perp (L_t - M_t)$. Between the two candidate points, the one with the greatest product of the two coordinates is chosen. If the previous points are unfeasible, i.e., their distance from L_t is less than $1 + \delta$, then the man moves to a point M_{t+1} such that $\|M_{t+1} - L_t\| = 1 + \delta + \varepsilon$, where ε is a given tolerance. Loosely speaking, the man aims at moving in a direction which is "as orthogonal as possible" to the line connecting the man and the lion at time t. The choice of the orthogonal direction is motivated by a sort of trade-off between going towards the lion (which is not a good choice!) and moving in the opposite direction, which results in approaching the region close to the origin, where capture occurs (see Proposition 1).

MS2: This strategy has been explicitly designed to play against the CFLS. At any time t, the man moves to a point aimed at maximizing the cost β. Firstly, the man moves towards P_t as long as $\|M_{t+1} - L_t\| > 1 + \delta$; during these moves the functional cost cannot decrease w.r.t. the initial value β_0. When such moves are not feasible anymore, the man moves to a "safe" point satisfying

$$M_{t+1} = \arg \sup_{\substack{\|M_{t+1} - M_t\| = 1 \\ \|M_{t+1} - L_t\| \geq 1 + \delta + \varepsilon}} f(M_{t+1})$$

where the tolerance ε is set as in MS1.

TABLE I

AVERAGE GAME LENGTH FOR DIFFERENT STRATEGIES

	LS1	LS2	CFLS
MS1	151.2	98.8	51.3
MS2	24.9	24.1	30.6

TABLE II

AVERAGE GAME LENGTH IN WORST-CASE SENSE AND NORMALIZED RELATIVE REDUCTION

	LS1	LS2	CFLS
$\overline{N}_{wc}$	151.2	98.8	52.3
$\overline{r}_{wc}$	1	0.680	0.448

A simulation campaign consisting of 10000 games has been performed with the following data. Let $M_0 = [x_m, y_m]'$ and $L_0 = [x_l, y_l]'$ denote the man and lion initial positions, respectively. Such positions have been randomly generated with uniform distribution such that $L_0 \succ M_0$; in particular, the initial positions have been generated to obtain $x_m, y_m \in (0, 10)$ and $(x_l - x_m), (y_l - y_m) \in (0, 10)$. The capture radius is set to $\delta = 10^{-5}$, and the tolerance $\varepsilon = 10^{-6}$.

For a fixed lion strategy and for a given game k, let us denote by $N_{MS1}(k)$ and $N_{MS2}(k)$ the number of moves needed by the lion to catch the man for the two man strategies. Let $\overline{N}_{MS1}$ and $\overline{N}_{MS2}$ be the average values of N_{MS1} and N_{MS2} over all the simulated games.

These figures are reported in Table I for the considered lion strategies. It is evident that the CFLS outperforms the other two strategies when the man plays MS1. On the contrary, the CFLS exhibits a slightly worse performance when playing against MS2. This fact is not surprising since MS2 was explicitly designed to compete against the CFLS. However, $\overline{N}_{MS1}$ is much greater than $\overline{N}_{MS2}$ for all the lion strategies, showing that MS2 is in general a worse man strategy than MS1.

To provide a fair comparison, for a given lion strategy the worst-case game length is computed as

$$N_{wc}(k) = \max\{N_{MS2}(k), N_{MS1}(k)\}, \ \forall k \ .$$

The relative reduction in terms of moves w.r.t. the strategy LS1 is defined as follows

$$r_{wc}(k) = \frac{N_{wc}(k)}{N_{wc}^{LS1}(k)} \ .$$

Let $\overline{N}_{wc}$ and $\overline{r}_{wc}$ denote the average values of N_{wc} and r_{wc}, respectively. Both $\overline{N}_{wc}$ and $\overline{r}_{wc}$ are shown in Table II for the considered lion strategies. It can be observed that, on average, playing the CFLS allows the lion to catch the man in a number of moves which is about 45% of those needed by the LS1.

VII. CONCLUSIONS

A new class of pursuit strategies has been introduced for the classic lion and man problem. The approach is based on the optimization of a suitable functional, which can be performed efficiently, irrespectively of the specific choice of the functional itself. A formal proof that capture occurs in finite time for any choice of the clearing function will be presented in a forthcoming work.

A main feature of the proposed approach is its flexibility: the choice of the C-function is a degree of freedom that can be exploited within different pursuit-evasion settings, according to the specific problem objective. For example, one can imagine to adapt the function to the local shape of the environment portion to be cleared, or to the location of different evaders in multi-agent games. The application of the proposed pursuit scheme to the above scenarios will be the subject of future investigations.

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